# Symmetry restoration of scalar models at finite temperature

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The symmetry restoration of scalar models at finite temperature and in less than four dimensions is investigated. For that purpose a series of approximations to the constraint effective lattice potential is introduced. The continuum limit of these mean-field-like potentials is discussed and it is shown that the symmetry is always restored at finite temperature. As an application we derive an estimate for the critical temperature.

## I. INTRODUCTION

A convenient and widely used method to study the temperature effects in a (continuum or lattice) quantum field theory is to use the finite-temperature effective potential. It is generated by the same one-particle-irreducible Feynman diagrams as the zero-temperature effective potential.[1] The $T$ dependence comes solely from the boundary conditions: For $T>0$ one imposes periodic boundary conditions for imaginary time which results in a $T$-dependent free propagator.

However, in certain theories perturbative calculations are plagued with severe infrared divergences.[2,3] This prevents one, for example, from calculating a reliable value for the critical temperature in less than four dimensions. Even worse, the loop expansion predicts that certain scalar models with two fields show no symmetry restoration,[2] contrary to one's expectation and to the improved method of summing over all daisy graphs.[4] Clearly, to study the transition from the broken to the symmetric phase a nonperturbative treatment is required.

The lattice formulation provides such an alternative to explore the phase diagrams at finite temperature. Recently one-component models have been studied by means of Monte Carlo simulations.[5] The calculations indicate that in three dimensions these models exhibit a symmetry restoration. However, because of finite-size effects, the prediction for the critical temperature is inaccurate.

In view of what has been said it is desirable to find an alternative method. The ordinary mean-field (MF) approximation, which is such a useful tool to investigate the phase structure at $T=0$ (Ref. 6), is volume independent and therefore temperature independent. To incorporate the temperature effects one must keep the finite-size effects due to the finite length in the imaginary-time direction. In this work we combine the loop expansion with the lattice formulation to develop an approximation which is analytic and infrared (IR) convergent.

The development of the paper is as follows. In Sec. II we recall the relevant properties of finite-temperature effective potentials. In particular we introduce the constraint effective potential. We will derive our results starting with this potential rather than the conventional one. A thorough discussion of its properties can be found in Refs. 7 and 8. Then we show that scalar theories on lattices with fixed lattice constant may remain broken at all (unphysical) "temperature." Mean-field-like potentials incorporating the temperature effects are introduced and discussed in Sec. III. Of particular interest is the approximation of the $d$-dimensional systems by quantum-mechanical ones on an interval of length $\beta=1/kT$. In Sec. IV we find the renormalization flow in these MF-like theories. Actually, we are able to determine the asymptotic renormalization of the bare parameters by reinterpreting the lattice constant as $\hbar$. This key observation allows for an application of the ordinary loop expansion to construct the continuum limit. The last section is devoted to the problem of symmetry restoration of nonsymmetric two-component models. These models are interesting because they show no symmetry restoration in the conventional loop expansion and for suitable chosen quartic coupling constants. We find that in three dimensions these models find themselves always in the symmetric phase at sufficiently high temperature. This result agrees with the prediction of the self-consistent perturbation theory[4] but not with the one-loop result. Finally, as an application of the developed methods, we give an estimate for the critical temperature of one- and two-component models. It may be worth repeating that, contrary to the conventional and the self-consistent loop expansion, our approximations are *infrared finite*.

## II. LATTICE POTENTIALS AT FINITE TEMPERATURE

Consider a field theory described by a Lagrangian density $L[\phi(x)]$, with a scalar field on $\Omega=\beta V$. To study the combined quantum and finite-temperature corrections to the classical potential in the action

$$S[\phi]=\int_\Omega [\tfrac{1}{2}(\nabla\phi)^2+V(\phi)]d^dx \tag{2.1}$$

one conveniently introduces effective potentials (EP's). Most approaches to this subject begin with the Schwinger function

 

$$W^{\beta}(j)=\frac{1}{\Omega}\ln\int D\phi\exp\left[-S[\phi]+j\int\phi(x)d^dx\right] \tag{2.2}$$

in the presence of a constant external current $j$. The temperature dependence is hidden in the boundary conditions on the allowed configurations in (2.2). At finite temperature one sums over fields periodic in the (imaginary) time $\tau$ with period $\beta$: $\phi(\tau,\mathbf{x})=\phi(\tau+\beta,\mathbf{x})$.

Conventionally one defines the *effective potential* as the Legendre transform of the Schwinger function

$$\Gamma^{\beta}(\Phi)=\sup_j[j\Phi-W^{\beta}(j)]=(LW)^{\beta}(\Phi)\ . \tag{2.3}$$

Clearly, being a Legendre transform, $\Gamma^{\beta}$ is manifestly convex.[8] However, for our purposes it is more convenient to use the so-called *constraint effective potential*

$$U^{\beta}(\Phi)=-\frac{1}{\Omega}\ln\int D\phi\,\delta\left[\frac{1}{\Omega}\int\phi(x)-\Phi\right]e^{-S[\phi]}\ . \tag{2.4}$$

This alternative potential has been introduced by Fukuda and Kyriakopoulos.[7] It can be shown that these potentials approach each other when the volume tends to infinite.[8] This zero-temperature result generalizes immediately to the finite-temperature case. For finite volumes it follows immediately from (2.2) and (2.4) that

$$e^{\Omega W^{\beta}(j)}=\int d\Phi\, e^{\Omega[j\Phi-U^{\beta}(\Phi)]}\ . \tag{2.5}$$

Thus $W^{\beta}$ and $\Gamma^{\beta}$ can always be recovered from $U^{\beta}$ (and conversely) and nothing is lost by considering $U^{\beta}$ instead of $\Gamma^{\beta}$. Note however, that contrary to $\Gamma^{\beta}$ the constraint EP is not necessarily convex for finite volumes.[8] Only in the limit when the spatial volume $V$ tends to infinity does it coincide with the convex $\Gamma^{\beta}$.

One way to *regularize* the above formal expressions is to discretize the space-time region $\Omega$ by a $d$-dimensional lattice with lattice spacing $a$. By rescaling the field, current, masses and coupling constants according to their dimensions (e.g., $m_L=a^2m$) the lattice action reads

$$S[\phi]=\frac{1}{2}\sum_{\langle ij\rangle}(\phi_i-\phi_j)^2+\sum_i V(\phi_i)\ . \tag{2.6}$$

Here $i$ runs from 1 to $\Lambda=lN^{d-1}$, where $l$ is the number of time slices and $N^{d-1}$ the number of sites in a given time slice. The first sum is over all nearest-neighbor pairs and $V(\phi_i)$ is the classical potential with rescaled parameters. At finite temperature we must impose periodic boundary conditions in the time direction, while in the remaining spatial directions we, for convenience, assume periodicity as well. Then the regularized Schwinger function and effective potentials are (up to a factor $a^{-d}$ and $a$-dependent constants) equal to their lattice counterparts

$$W^{l}(j)=\frac{1}{\Lambda}\ln\int\prod d\phi_i\exp\left[j\sum\phi_i-S[\phi]\right] \tag{2.7}$$

and

$$U^{l}(\Phi)=-\frac{1}{\Lambda}\ln\int\prod d\phi_i\delta\left[\frac{1}{\Lambda}\sum\phi_i-\Phi\right]e^{-S[\phi]}\ . \tag{2.8}$$

In what follows we shall be interested in the dependence of the various potentials on the "temperature" $l^{-1}$. We use quotation marks to indicate that $l$ and $N^{d-1}$ are both dimensionless, unphysical numbers. They are related to their physical counterparts by a dimensionful scale parameter (see Sec. IV).

Before studying the temperature dependence of the various potentials one lets the "spatial volume" of the lattice approach infinity to eliminate finite (spatial) size effects. Thus, from now on we shall assume that the limit $N\to\infty$ has been taken. We shall assume that our scalar models are spontaneously broken at low "temperature," i.e., that for sufficiently large $l$'s the expectation value of the scalar field

$$\langle\phi_i\rangle^l=\lim_{j\to 0}\frac{dW^l(j)}{dj}\ , \tag{2.9}$$

is positive. Then, for large $l$'s, the Schwinger function exhibits a bend at the origin

$$(W^l)''(j=0)=\sum_i\langle\phi_0\phi_i\rangle^l=\infty\ . \tag{2.10}$$

Some words about the order of limits are necessary. In (2.10) the current was first set to zero and afterwards the thermodynamic limit $N\to\infty$ has been taken. Whenever the limits are taken in the reverse order we indicate this, like in (2.9). The order of limits is crucial since in the broken phase $\langle\phi_i\rangle_j$ in

$$(W^l)''(j)=\sum_i(\langle\phi_0\phi_i\rangle_j-\langle\phi_i\rangle_j^2)$$

is zero or not zero, depending on the order of limits. We see that in the broken phase (2.10) differs from the susceptibility $\lim_{j\to 0}W''(j)$, which diverges at $T=T_c$ only, by an infinite constant. Therefore $(W^l)''(0)$ as defined in (2.10) is infinite for all temperature below $T_c$ and not only at the critical temperature.

To see whether the model exhibits a symmetry restoration, i.e., has a critical "temperature" $l_c^{-1}$ such that $\langle\phi_i\rangle^l>0$ for $l>l_c$ and $\langle\phi_i\rangle^l=0$ for $l<l_c$, one squeezes the lattice in the time direction. When the Schwinger function becomes smooth at high "temperature" then the *symmetry is restored.*

So far we did not bother removing the lattice regularization. We shall discuss this continuum limit carefully in Sec. IV. However, at this point it seems worth noting that for any *fixed* lattice constant the system may exhibit properties which are artifacts of the regularization only. We shall now discuss such a lattice result which is qualitatively different from its corresponding continuum result (see Sec. IV), partly to see how the arguments break down in the continuum limit. The lattice result we can show to hold in three and more dimensions is the following.

For any *fixed* lattice constant and for sufficiently negative masses (how negative depends on the chosen lattice constant) the symmetry is broken for *all* values of $l$.

In a moment we shall see that this follows from the inequality

$$(W_d^l)''(m,j=0)\geq(W_{d-1})''(m+1,j=0)\ , \tag{2.11}$$

which compares the curvatures of the Schwinger functions of two different models: the $d$-dimensional finite "temperature" model with the $(d-1)$-dimensional zero "temperature" model, but with a shifted mass $m \to m+1$, where the mass term in $V$ is defined by $m\phi^2$ in our notation. We shall prove (2.11) in Appendix A. The point is that zero "temperature" $\phi^4$ lattice models have been studied extensively and are known to be spontaneously broken when $d \geq 2$ and $m < m_c < 0$ (Ref. 9). In those cases their Schwinger functions [which appear on the right-hand side of (2.11)] develop a singularity in the thermodynamic limit. This, together with the inequality (2.11), tells us that

$$(W^l)''(j=0)=\infty$$

when $d \geq 3$ and $m < m_c = 1$, irrespective of the "temperature" $l^{-1}$. Thus the lattice model is broken for all values of $l$. To sum up, therefore, for a *fixed lattice constant a* the squeezing of the lattice does not necessarily force the system into a symmetric state [concerning this aspect, a careful investigation has been done in the SU(2) lattice Higgs model at finite temperature[10]].

On a first sight this shift of the bare mass may look rather insignificant. However, the bare parameters have to be related to physical quantities by *renormalization* and we have to consider whether this renormalization affects our conclusions. In order to discuss the system at finite temperature $T=1/(al)$, one must, in the $a \to 0$ limit, keep $al$ fixed, not $l$. Thus the bare parameters are to be regarded as functions $m(a)$ and $g(a)$, determined by the renormalization conditions. Therefore to achieve the high temperature, one must carefully locate a region in the parameter space so that $T \gg$ *typical physical mass* for a fixed $l$, or equivalently $la \ll$ *correlation length.* This consideration would affect the above argument. Indeed in less than four dimensions the parameters $(m(a), g(a))$ both tend to zero when $a$ does. Thus, for small $a$ the "effective mass" $m(a)+1$ on the right-hand side of (2.11) becomes positive and $W(m+1,j)$ stays smooth as $N \to \infty$. We see that in the continuum limit the inequality (2.11) is worthless and allows the symmetry restoration at high temperature. Indeed, as we shall see in Sec. IV, the symmetry is always restored in the continuum model.

## III. MEAN-FIELD-LIKE POTENTIALS

The most crude one-body approximation, i.e., mean-field theory, provides us with a good qualitative picture of the phase structure at zero temperature.[6] However, in the ordinary mean-field approximation one loses the volume dependence and hence the temperature dependence of the theory. In this section we formulate a modified mean-field approximation to the constraint lattice potential which incorporates the finite-temperature effects. In this approximation the scalar theory simplifies to a one-dimensional field theory or a quantum-mechanical system. The main result of this section, namely, the approximating potential (3.2), will serve as starting point for our considerations in the remaining sections.

### A. Approximations to the constraint effective potential

More generally we introduce a series of approximations, labeled by an integer $p$, such that for $p=d$ we recover the exact theory and for $p=0$ the ordinary MF approximation. Let us, for this purpose, consider the $d$-dimensional lattice as product $\Lambda = \Lambda_p \times \Lambda_{d-p}$. Later we shall be mainly concerned with the case $p=1$. Then $\Lambda_{d-1}$ may be thought of as a time slice of the $d$-dimensional lattice and $\Lambda_1$ as the sites with the same spatial coordinates. In analogy to this case we shall call $\Lambda_{d-p}$ a given time slice of the lattice, even in the more general cases when $d>1$.

Instead of replacing the interaction of $\phi_i$ with all of its nearest neighbors by the interaction with the mean field $M=\sum \phi_i/\Lambda$ (as it is done in the ordinary MF approximation), we make this approximation only for nearest neighbors in the same time slice. Thus we replace the action (2.6) by

$$\frac{1}{2} \sum_{\substack{\langle ij \rangle \\ i-j \in \Lambda_p}} (\phi_i - \phi_j)^2 - (d-p)\Lambda M^2 + \sum_i V_p(\phi_i) , \tag{3.1}$$

where $V_p(\phi)=(d-p)\phi^2+V(\phi)$ and the first sum is over neighbors with the same spatial coordinates. Note that, because of the constraint in (2.8), the second term becomes $(d-p)\Lambda\Phi^2$. The action above describes $\Lambda_{d-p}$ noninteracting copies of a $p$-dimensional lattice model and hence we are left with a $p$-dimensional system. In Appendix B we show that the constraint effective potential (2.8) simplifies to

$$U_p(\Phi) = -(d-p)\Phi^2 + \sup_j [j\Phi - W_p(j)] , \tag{3.2}$$

where

$$W_p(j) = \frac{1}{\Lambda_p} \ln \int \prod_{\Lambda_p} d\phi_j \exp\left[ j \sum \phi_i - S_p[\phi] \right] \tag{3.3}$$

is the Schwinger function of a $p$-dimensional model which differs from the original theory by a shifted mass $m \to m+d-p$ (see the definition of $V_p$). With (3.2) we derived the desired approximations to the exact potential (2.8). We are left with the functional integral on the sublattice $\Lambda_p$ instead of the functional integral on the whole lattice $\Lambda$. Clearly with increasing $p$ the MF-like potentials (3.2) become better approximations. Especially for $p=d$ one recovers the potential (2.8). We shall now study the cases $p=0$ and $p=1$ in turn.

### B. The case $p=0$ (ordinary MF approximation) and symmetry restoration

In the extreme case $p=0$, $W_0(j)$ is the Schwinger function of a zero-dimensional "field theory" with mass $m+d$. We shall discuss this crudest of our approximations explicitly, although our primary interest is in $p=1$. Our main motivation being that the cases $p=0$ and $p>0$ are conceptually very similar but the manipulations are less involved in the first case. For $p=0$ the potential (3.2) becomes

$$U_0(\Phi) = -d\Phi^2 + (LW_0)(\Phi) = -d\Phi^2 + \Gamma_0(\Phi) , \tag{3.4}$$

where

$$W_0(j) = \ln \int d\phi \, e^{j\phi - V_0(\phi)} \tag{3.5}$$

is given by an ordinary integral and is independent of $\Lambda$. Such zero-dimensional models with a shifted mass, so-called *incoherent models,* were used in Ref. 8 to bound the constraint EP from below and above.

In Sec. IV we shall need the minimum $\Phi_0$ of $U_0$ and the curvature at this minimum. Using $j(\Phi)=\Gamma'(\Phi)$, which relates the current to its conjugate field, one sees at once that the minimum condition becomes

$$j_0=j(\Phi_0)=2d\Phi_0 \ . \tag{3.6}$$

Since $\Gamma_0$ is the Legendre transform of $W_0$, the inverse relation reads $\Phi(j)=W_0'(j)$. By inserting the minimum condition into that equation we find the well-known self-consistency equation[6]

$$\Phi_0=\frac{\int d\phi\,\phi e^{j_0\phi-V_0(\phi)}}{\int d\phi\, e^{j_0\phi-V_0(\phi)}}=\langle\phi\rangle_{j_0} \tag{3.7}$$

for the expectation value of the Higgs field. So one recovers the ordinary mean-field approximation.

To compute $U_0''(\Phi_0)$ we use the relation $\Gamma''(\Phi)=W''[j(\Phi)]^{-1}$ between the curvatures of $\Gamma$ and $W$. Together with the minimum condition one obtains

$$m_0=U_0''(\Phi_0)=-2d+\langle(\phi-\Phi_0)^2\rangle_{j_0}^{-1} \tag{3.8}$$

for the Higgs-boson mass in the broken phase. Clearly the incoherent Schwinger function (3.5) is strictly convex and symmetric and hence $j(\Phi)$ vanishes when $\Phi$ does. From (3.8) we conclude that the curvature of $U_0$ at the origin is negative when $\langle\phi^2\rangle_0>1/2d$. Consequently the potential (3.4) is spontaneously broken in cases where

$$\frac{\int\phi^2 e^{-V_0(\phi)}}{\int e^{-V_0(\phi)}}=\langle\phi^2\rangle_0>1/2d \ . \tag{3.9}$$

Suppose, for example, that the mass $m$ in $V(\phi)=m\phi^2+g\phi^4$ is less than $-d$. Since the $m$ derivative of $\langle\phi^2\rangle_0$ is manifestly negative, the expectation value $\langle\phi^2\rangle_0$ decreases with increasing mass and becomes smaller when $M$ is replaced by $-d$. However, for this value of $m$ the effective mass $(m+d)$ in $V_0$ vanishes and the expectation value can be computed explicitly. In this way one finds from (3.9) that the potential (3.4) is spontaneously broken when

$$m\le -d \quad\text{and}\quad g<[2d\Gamma(\tfrac{3}{4})/\Gamma(\tfrac{1}{4})]^2 \ .$$

This result can easily be generalized to the case where the field has several components. Consider for simplicity an even potential $V(\phi_1^2,\ldots,\phi_n^2)$. Then the matrix $\partial_a\partial_b W_0(0)$ and its inverse $\partial_a\partial_b\Gamma_0(0)$ are both diagonal. One sees at once that the condition (3.9) for a spontaneous symmetry breakdown is now replaced by

$$\max\{\langle\phi_1^2\rangle_0,\ldots,\langle\phi_n^2\rangle_0\}>1/2d \ . \tag{3.9$'$}$$

In Fig. 1 the approximation (3.4) to the constraint EP is compared with the results of Monte Carlo simulations on a one-dimensional and a four-dimensional lattice with 160 and $8^4$ lattice sites, respectively.[8] For the chosen parameters the approximation is surprisingly accurate.

### C. The case $p=1$ (modified MF approximation)

As pointed out earlier, the potential $U_0$ is volume independent. However, at finite temperature we must keep the finite-size effects due to a varying lattice length in the time direction. This suggests that we keep the timelike interactions correctly and approximate in the remaining spatial direction(s). So we take $p=1$ and $\Lambda_1=l$ in (3.2) and call

$$U_1^l(\Phi)=-(d-1)\Phi^2+\Gamma_1^l(\Phi) \tag{3.10}$$

the *modified MF effective potential.* Note, that $\Gamma_1$ is now the Legendre transform of a quantum-mechanical Schwinger function

$$W_1^l(j)=\frac{1}{l}\ln\int\prod_1^l d\phi_j\exp\left[j\sum\phi_i-S_1[\phi]\right] , \tag{3.11}$$

and we are left with a *one-dimensional field theory* with a shifted mass $m\to m+d-1$. One sees at once that the generalizations of the self-consistency equation (3.7) and the Higgs-boson mass (3.8) read

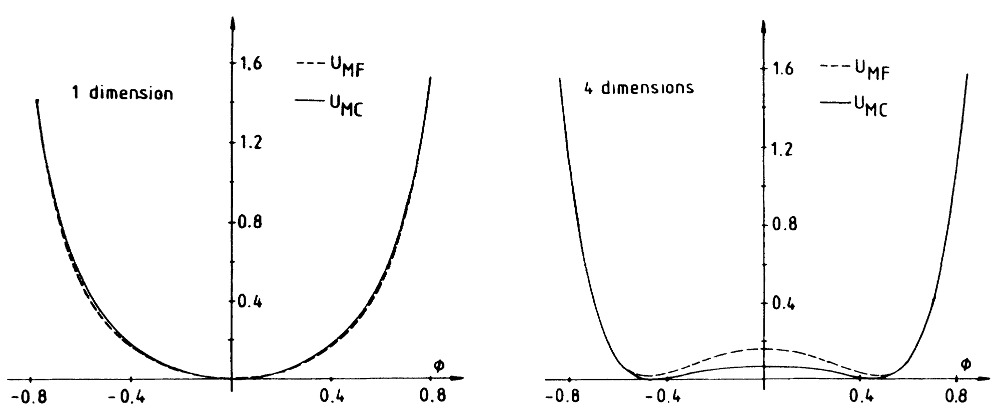


FIG. 1. The mean-field (MF) and Monte Carlo (MC) approximations to the constraint effective potential in one and four dimensions, respectively, for the classical potential $V$ with mass $m=-7.5$ and coupling constant $g=10$.

$$\Phi_0=\langle M\rangle_{j_0}\,, \tag{3.12}$$

$$m_0=U_1''(\Phi_0)=-2(d-1)+\langle(M-\Phi_0)^2\rangle_{j_0}^{-1}\,. \tag{3.13}$$

The expectation values of $M=\sum\phi_i/l$ and $(M-\Phi_0)^2$ are to be computed with the integrand on the right-hand side of (3.11) where the current is $j=j_0=2(d-1)\Phi_0$, i.e., the current conjugate to $\Phi_0$.

By applying the inequality (2.11) with $d=1$, namely, $\langle M^2\rangle_0\geq\langle\phi^2\rangle_0$, where the second expectation value was defined in (3.9), one obtains the upper bound $-2(d-1)+\langle\phi^2\rangle_0^{-1}$ for the curvature of $U_1$ at the origin. We conclude that the potential (3.10) is spontaneously broken at all $l$'s, when

$$\frac{\int\phi^2e^{-V_0(\phi)}}{\int e^{-V_0(\phi)}}>\frac{1}{2(d-1)}\,. \tag{3.14}$$

This implies that there will never be a symmetry restoration at finite "temperature" if $m<-(d-1)$ and $g<[2(d-1)\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})]^2$, similarly to the statement below (3.9). As we have seen in the last section, this is not a peculiarity of the mean-field approximation. In the full lattice theory one has the same situation. Once more we conclude that it is essential to take the continuum limit to study the temperature dependence of $U_1$.

Although we have discussed the special cases $p=0$ and $p=1$ of the series of approximations (3.2), it should be clear how these results apply to the other cases. In particular, whenever the self-consistency equation allows nontrivial solutions, then the approximating potentials $U_p(\Phi)$ are nonconvex. This is true even in the infinite-volume limit, while the exact potential becomes convex.

## IV. THE CONTINUUM LIMIT

In the preceding sections we have not introduced any explicit renormalization. However, the bare quantities $(m,g)$ we have considered have to be related to physical quantities by renormalization. As *physical parameters* we take the expectation value of the Higgs field $\Phi_p$ and the Higgs-boson mass $m_p$ in the broken phase. One conveniently introduces a dimensionless lattice constant $\lambda=a\mu$, where $\mu$ is a scale parameter of mass dimension, and measures the various physical quantities in $\mu$ units.

Let us first consider the *ordinary* MF potential (3.4). To construct the scaling limit one compares the lattices $Z^d$ and $(\lambda Z)^d$ when $\lambda$ is allowed to take values in the interval $0<\lambda<1$. One sees at once that the potential on $(\lambda Z)^d$ becomes

$$\begin{aligned}U_0^\lambda(\Phi)&=\lambda^{-d}U_0(\lambda^{d/2-1}\Phi)\\&=-d\lambda^{-2}\Phi^2+\lambda^{-d}\Gamma_0(m(\lambda),g(\lambda),\lambda^{d/2-1}\Phi)\,,\end{aligned} \tag{4.1}$$

where the scaled bare parameters are to be determined by some renormalization condition.[11] As fixed physical parameters we take the expectation value $\Phi_p$ which minimizes $U_0^\lambda$ and the Higgs-boson mass $m_p=U_0^{\lambda''}(\Phi_p)$.

Obviously, when $\Phi_p$ minimizes $U_0^\lambda$ then $\lambda^{d/2-1}\Phi_p$ minimizes $U_0$ and satisfies the self-consistency equation (3.7). Thus, the first renormalization condition reads

$$\lambda^{d/2-1}\Phi_p=\langle\phi\rangle_{j_p}\,, \tag{4.2}$$

where $j_p=2d\lambda^{d/2-1}\Phi_p$, and the expectation values have been defined in (3.7). In the same way, by using (3.8), one obtains the second renormalization condition

$$\lambda^2m_p=\lambda^2U_0^{\lambda''}(\Phi_p)=-2d+\langle(\phi-\lambda^{d/2-1}\Phi_p)^2\rangle_{j_p}^{-1}\,. \tag{4.3}$$

In Eq. (4.3) the wave-function renormalization constant $Z$ has not been introduced since $Z=1$ in the present case. To see that one needs the nonlocal part in the *effective action*

$$S_{\rm eff}[\bar\phi]=\int[U(\bar\phi)+\tfrac{1}{2}Z(\nabla\bar\phi)^2+\cdots]\,.$$

In the ordinary MF approximation the effective action is easily computed to be[11]

$$\tfrac{1}{2}\sum_{\langle ij\rangle}(\bar\phi_i-\bar\phi_j)^2+\sum_i U_0(\bar\phi_i)$$

which shows that $Z=1$ in this case.

To find the asymptotic form of the bare parameters in the classical potential $V=m\phi^2+g\phi^4$ for small $\lambda$ one expands the expectation values about $\lambda=0$. Since the leading-$\lambda$ behavior of $m$ and $g$ is not known *a priori,* we used the solvable model $V(\phi)=(m+\sqrt{2g})\phi^2-\ln(1+\sqrt{2g}\,\phi^2)\sim m\phi^2+g\phi^4+\cdots$ (Ref. 11) for making a first guess. Next, by inserting the small-$\lambda$ expansion into the above renormalization conditions we determined the coefficients of $\lambda^n$ and found the following renormalization flows in two and three dimensions:

$$\begin{aligned}&d=2:\;\; g(\lambda)\sim\frac{m_p}{8\Phi_p^2}\lambda^2,\;\; m(\lambda)\sim-(\tfrac{3}{2}+2\Phi_p^2)g(\lambda)\,,\\&d=3:\;\; g(\lambda)\sim\frac{m_p}{8\Phi_p^2}\lambda,\;\; m(\lambda)\sim-g(\lambda)\,.\end{aligned} \tag{4.4}$$

For an alternative method for deriving (4.4) one can apply similar arguments to those presented in Appendix C. In order to find the complete mean-field renormalization-group flow we solved Eqs. (4.2) and (4.3) numerically for $\lambda=1,2^{-1},\ldots,2^{-8}$. In Fig. 2 the numerical results are compared with the asymptotic flow (4.4) in three dimensions. One sees that the ratios approach 1 as $\lambda$ tends to 0 rather quickly.

With respect to the continuum limit, it may be worth referring to the triviality property of the theory in various dimensions $d$. As we have seen, in $d<4$ the continuum limit is consistently taken within the broken phase and therefore it leads to an interactive theory. In four dimensions one can show that both $m(\lambda)$ and $g(\lambda)$ approach a constant value when $\lambda\to0$. This contradicts the expectation that the theory is trivial, since if this is the case, the renormalization conditions (4.2) and (4.3) are not compatible with the triviality. In other words, the MF approach fails to predict the triviality in this dimen-

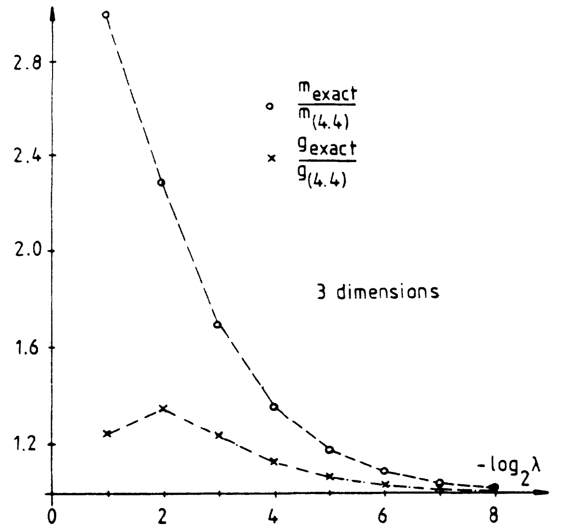


FIG. 2. Ratios of the complete mean-field renormalization flow and the asymptotic flow (4.4) in three dimensions for various values of the scale parameter $\lambda$.

sion. In $d>4$, on the other hand, the conditions (4.2) and (4.3) really fall into the contradiction; one cannot get consistent solutions. This reflects the triviality of the theory, which has been rigorously proven in these dimensions.

Let us now consider the scaling limit of the *modified* MF potential (3.10). For that purpose one compares the lattices $l\times Z^{d-1}$ and $(\lambda l)\times(\lambda Z)^{d-1}$, where the (dimensionless) inverse temperature $\beta=\lambda l$ is to be kept fixed. To determine the renormalization flow of the bare parameters in the scaled potential

$$U_1^\lambda(\beta,\Phi)=-(d-1)\lambda^{-2}\Phi^2 +\lambda^{-d}\Gamma_1(m(\lambda),g(\lambda),\lambda^{d/2-1}\Phi) \quad (4.5)$$

we, like in the $p=0$ approximation, fix the expectation value $\Phi_p$ and the Higgs-boson mass $m_p$. Since it suffices to renormalize the bare parameters at zero temperature to regularize the finite temperature theory, we compute the renormalization flow on the lattice $(\lambda Z)\times(\lambda Z)^{d-1}$, by using the renormalization conditions

$$\lambda^{d/2-1}\Phi_p=\langle M\rangle_{j_p} \quad (4.6)$$

and

$$\lambda^2 m_p Z^{-1}=\lambda^2 U_1^{\lambda''}(\infty,\Phi_p) =-2(d-1)+\langle(M-\lambda^{d/2-1}\Phi_p)^2\rangle_{j_p}^{-1}, \quad (4.7)$$

where $Z$ is the wave-function renormalization of the $\Phi$ field. The expectation values are computed with the integrand in (3.11), wherein $l=\infty$ and $j=j_p=2(d-1)\lambda^{d/2-1}\Phi_p$.

The nonlocal terms in the *effective action* determining $Z$ have two contributions. One is the simple kinetic term coming from the spatial directions and the other, less trivial one, comes from the timelike one-dimensional chain. The latter may yield a wave-function renormalization $Z\neq 1$. Fortunately a wave-function renormalization only rescales the effective potential but does not change the sign of its curvature at the origin. Since the critical temperature is the temperature at which $(U_1^\lambda)''(0)$ changes sign, we see that $T_c$ is independent of the value of $Z$. For that reason we shall not be concerned with the wave-function renormalization in what follows. In particular with $m_p$ we actually mean the combination $m_p Z^{-1}$.

At this point we make the following important observation: since $m(\lambda)$ tends to zero in the continuum limit, the effective mass $m(\lambda)+d-1$ in the potential $V_1$ in (3.11) is positive for small $\lambda$ and thus the ordinary loop expansion is not plagued with infrared divergences. This is true even when the original theory is spontaneously broken. In addition we shall see that the $\lambda$ dependence of the parameters and the field does not ruin the expansion

$$\Gamma_1(m,g,\lambda^{d/2-1}\Phi)=V_{d-1}(\ ,\ ,\ )+V_{1\,\text{loop}}(\ ,\ ,\ ,)+\cdots \quad (4.8)$$

Actually, it turns out that $V_{r\,\text{loop}}$ is of $O(g^{r-1})$ relative to the one-loop contribution and since the bare coupling constant tends to zero in the continuum limit, the $\hbar$ expansion (4.8) becomes an expansion in the lattice constant. Since, contrary to $\hbar$, the lattice constant really tends to zero, the one-loop result (at zero temperature) gives the *correct* asymptotic renormalization flow. So it suffices to take the quantum-mechanical one-loop contribution on the lattice, $V_{1\,\text{loop}}(\Phi)=\frac{1}{2}\text{arccosh}[1+\frac{1}{2}V''_{d-1}(\Phi)]$ (Ref. 12). Part of the classical mass term in (4.8) cancels the first term on the right-hand side in (4.5) and we end up with the expansion

$$U_1^\lambda(\infty,\Phi)\sim m\lambda^{-2}\Phi^2+g\lambda^{d-4}\Phi^4 +\frac{\lambda^{-d}}{2}\text{arccosh}[1+(m+d-1)(1+\delta)], \quad (4.9)$$

where $\delta=6g\lambda^{d-2}\Phi^2/(m+d-1)$. After expanding the one-loop contribution in powers of $\delta$ and by using the renormalization conditions (4.6) and (4.7), one obtains the following asymptotic renormalization flows in two and three dimensions:

$$d=2:\ g(\lambda)\sim\frac{m_p}{8\Phi_p^2}\lambda^2,\quad m(\lambda)\sim-(\sqrt{3}+2\Phi_p^2)g(\lambda),$$
$$d=3:\ g(\lambda)\sim\frac{m_p}{8\Phi_p^2}\lambda,\quad m(\lambda)\sim-\frac{3}{2\sqrt{2}}g(\lambda). \quad (4.10)$$

By comparing (4.10) with (4.4) one sees that the ordinary and the modified MF approximations give rise to the same $\lambda$ dependence of the bare parameters in two and three dimensions. This is not very surprising, since $U_0$ and $U_1$ only differ by a MF approximation in the time direction and since a one-dimensional field theory needs no renormalization. More surprising is the fact that the bare quantities $m(a)$ and $g(a)$ tend to zero as $a\to 0$. In an exact treatment this need not necessarily be so. It is generally only the case for quantum-mechanical systems and in field theories in the tree approximation.

Let us now verify that the $\lambda$ dependence of the parameters and the field in (4.8) does not spoil the above expansion of the effective potential. For that we apply the results in Ref. 13 for the general form for the $r$-loop contribution to the effective potential

$$V_{r\ \mathrm{loop}}=\sqrt{m+d-1}\left[\frac{g}{(m+d-1)^{3/2}}\right]^{r-1}F_r(\Delta)\ ,$$

where $\Delta=1+\delta$ is dimensionless and $F_r$ has the expansion $F_r(\Delta)\sim a+c\delta+c\delta^2+\cdots$. Thus, up to a constant, the $r$-loop contribution in (4.8),

$$V_{r\ \mathrm{loop}}(\ ,\ ,\ )\sim\lambda^{-2}g^r(c_1\phi^2+c_2\lambda^{d-2}g\phi^4+\cdots)\ ,$$

is of $O(g^{r-1})$ relative to the one-loop contribution. It follows that, since the bare coupling constant $g$ approaches zero in the continuum limit, one may neglect the higher-loop contributions to the flows (4.10).

We are now ready to show that the symmetry restoration takes place in three-dimensional Higgs models. To see this, we compute the second derivative of $U_1^\lambda(\beta,\Phi)$ at the origin on a lattice with two time slices, that is for the highest possible temperature. For that we must calculate the curvature of the quantum-mechanical Schwinger function on two sites and with scaled parameters. From (4.10) we take the asymptotic behavior $g(\lambda)\sim\frac{1}{8}m_p\lambda/\Phi_p^2$ and $m(\lambda)\sim-(1+\epsilon)g(\lambda)$ and find the following asymptotic form (see Appendix C, where the computation for a more general model is presented):

$$U_1^{\lambda''}(\beta,0)\sim\alpha\frac{m_p}{32\phi_p^2}\lambda^{-1}\ ,$$

where $\alpha=9-6\sqrt{2}$ is positive for the flow (4.10). With $\beta=l\lambda=2\lambda\ll 1$ it follows that at high temperature the curvature of the potential at the origin,

$$U_1^{\lambda''}(\beta,0)\sim\alpha\frac{m_p}{16\Phi_p^2}T\ , \qquad (4.11)$$

is positive and therefore the *symmetry is restored.* We may use (4.11) to obtain an estimate for the critical temperature. For that we add the high-temperature contribution (4.11) to the zero-temperature EP, which yields

$$U_1^\lambda(\beta,\Phi)\sim-\frac{m_p}{4}\left[1-\frac{1}{8\Phi_p^2}\alpha T\right]\Phi^2+\frac{m_p}{8\phi_p^2}\Phi^4+\cdots\ . \qquad (4.12)$$

The mass term changes sign at

$$T_c=8\frac{\Phi_p^2}{\alpha} \qquad (4.13)$$

which serves as a first guess for the critical temperature of the one-component model. To examine the quality of the estimate (4.13) we compared it with the corresponding MC *result* for the three-dimensional scalar model. From Ref. 5 we took the continuum value $T_c\sim0.62$ for $\Phi_p\sim0.308$. This value is about half of our estimate (4.13), which yields $T_c\sim1.47$.

If we wish to make contact with the *conventional loop expansion,* then we must compare (4.12) with the high-temperature expansion of the conventional one-loop effective potential in three dimensions, namely, with

$$\Gamma_{1\ \mathrm{loop}}(\beta,\Phi)=-\frac{m_p}{4}\left[1-\frac{3T}{2\pi\Phi_p^2}\right]\Phi^2+\frac{m_p}{8\Phi_p^2}\Phi^4-\frac{\zeta(3)}{2\pi}T^3-\frac{V''}{4\pi}T\ln\left[\frac{V''}{4T^2}\right]+O(M^2/T)\ . \qquad (4.14)$$

The remaining terms are positive integer powers of $M^2\beta^2$ times an overall factor $\beta^{-1}M^2$ and are negligible at high temperature. One observes that the high-temperature expansion in three dimensions has a worse infrared behavior than the corresponding expansion in four dimensions. The trouble is that already the leading mass correction (the logarithmic term) shows an *infrared divergence.* If we would discard this singular term in (4.14) then we recover (4.12) and (4.13), wherein $\alpha=12/\pi$. This yields a lower critical temperature as the one we found with our method (actually 0.20 for the above parameters).

As mentioned in the Introduction we never met any infrared divergences in the course of our derivation. The one-loop contribution in (4.9), which may become complex in the conventional loop expansion, stays real for small $\lambda$ for which $m(\lambda)$ is small and therefore the effective mass $m+d-1$ is positive.

## V. SYMMETRY RESTORATION OF THE NONSYMMETRIC TWO-COMPONENT MODEL

In the previous section we introduced methods for discussing the symmetry restoration of scalar theories at finite temperature. We shall now apply the apparatus developed to the subtle and interesting case when there are two interacting fields with classical potential:

$$V(\phi_1,\phi_2)=m_1\phi_1^2+g_1\phi_1^4+m_2\phi_2^2+g_2\phi_2^4-g_{12}\phi_1^2\phi_2^2\ . \qquad (5.1)$$

For the model to be stable we must impose $4g_1g_2>g_{12}^2$. This model is interesting, since it shows no symmetry restoration at finite temperature in the conventional loop expansion when $g_1>2g_{12}>g_2$ (Ref. 2). On the other hand, when one self-consistently solves the equations for the effective masses it shows a transition at some critical temperature.[4]

To see whether the modified MF approximation predicts a symmetry restoration we first compute the renormalization flow for the bare parameters. We apply the same strategy as in the previous section and use the one-loop contribution

$$V_{1\ \mathrm{loop}}(\Phi_1,\Phi_2)=\tfrac{1}{2}\mathrm{tr}\,\mathrm{arccosh}[1+\tfrac{1}{2}V_1''(\Phi_1,\Phi_2)]$$

to the quantum-mechanical effective potential $\Gamma_1$ on the right side of (4.5). After separating the field-independent and the field-dependent contributions to $\frac{1}{2}V_1''$, namely, $(d-1)/Id$ and

$$V''=\begin{bmatrix} 6g_1\Phi_1^2-g_{12}\Phi_2^2 & -2g_{12}\Phi_1\Phi_2 \\ -2g_{12}\Phi_1\Phi_2 & 6g_2\Phi_2^2-g_{12}\Phi_1^2 \end{bmatrix} , \tag{5.2}$$

one easily finds the small-$\lambda$ expansion $V_{1\,\mathrm{loop}}=\frac{1}{2}(d^2-1)^{-1/2}\mathrm{tr}V''+\cdots$. Combining this one-loop result with (4.8) and (4.5) finally yields the desired approximation to $U_1^\lambda$. At this point it is convenient to fix the physical parameters. Again we take the expectation values $(\Phi_{1p},\Phi_{2p})=\Phi_p$ of the two fields and the diagonal elements $(m_{1p},m_{2p})$ of the second derivative of the effective potential at its minimum $\Phi_p$. In order to restrict the number of parameters to four, we furthermore assume that $U_1''(0)$ is proportional to the identity matrix. In terms of these parameters the potential may be written as

$$V(\phi)=\frac{1}{8}\sum_i \frac{m_{ip}}{\Phi_{ip}}(\phi_i^2-\Phi_{ip}^2)^2 + \frac{1}{4}\frac{\Delta m_p}{\Delta\Phi_p^2}(\phi_1^2-\Phi_{p1}^2)(\phi_2^2-\Phi_{p2}^2) , \tag{5.3}$$

where $\Delta m_p=m_{2p}-m_{1p}$ and $\Delta\Phi_p^2=\Phi_{2p}^2-\Phi_{1p}^2$. Note that the two fields decouple when $m_{1p}=m_{2p}$.

By comparing the one-loop result for $U_1^\lambda$ with the parametrization (5.3) one extracts the small-$\lambda$ dependence of the bare parameters in $U_1^\lambda$. In this way one obtains in three dimensions

$$g_i\sim\lambda\frac{m_{ip}}{8\Phi_{ip}^2},\quad g_{12}\sim-\lambda\frac{\Delta m_p}{4\Delta\Phi_p^2} ,$$
$$m_i\sim\frac{1}{4\sqrt{2}}(6g_i-g_{12}) . \tag{5.4}$$

One sees that the interaction term $-g_{12}\Phi_1^2\Phi_2^2$ in (5.1) modifies the $\lambda$ dependence of the bare masses relative to the flow (4.10). Like in the one-component case the higher-loop contributions would not change the asymptotic expansion of the bare parameters.

Again we can use the result (5.4) to obtain information about the high-temperature behavior of the two-component theory. For that we, like in the previous section, calculate the small-$\lambda$ expansion of $U_1^{\lambda''}(\beta,0)$ on a lattice with two lattice sites in the time direction. In Appendix C we show that

$$U_1^{\lambda''}(\beta,0)\sim\frac{\alpha}{96\lambda}\begin{bmatrix} 3\frac{m_{1p}}{\Phi_{1p}^2}+\frac{\Delta m_p}{\Delta\Phi_p^2} & 0 \\ 0 & 3\frac{m_{2p}}{\Phi_{2p}^2}+\frac{\Delta m_p}{\Delta\Phi_p^2}\end{bmatrix} +\mathrm{const} , \tag{5.5}$$

where $\alpha=9-6\sqrt{2}>0$. The stability condition for the parameters in (5.3) and the fact that the geometric mean of two positive numbers is less than or equal to their arithmetic mean, tells us that $m_{ip}/\Phi_{ip}^2$ is greater than $2\,|\,\Delta m_p/\Delta\Phi_p^2\,|$. Thus both entries in the matrix (5.5) are positive, regardless of the sign of $\Delta m_p/\Delta\Phi_p^2$. We see that for a lattice with two time slices, for which $\beta=2\lambda$, the curvature of $U_1^\lambda$ at the origin becomes always positive at high temperature. We conclude, and that is the main result of this section, that the modified MF approximation predicts a *symmetry restoration for all "stable" parameters,* contrary to the conventional loop expansion.

To estimate the critical temperature we again add the high-temperature contribution (5.5) to the zero-temperature EP to see at which temperature the mass terms change signs. In this way one obtains

$$T_c=\max_i\left[\frac{24}{\alpha}\Phi_{ip}^2\frac{m_{2p}\Phi_{2p}^2-m_{1p}\Phi_{1p}^2}{3m_{ip}\Delta\Phi_p^2+\Phi_{ip}^2\Delta m_p}\right] \tag{5.6}$$

for the critical temperature. In particular, for $m_{1p}=m_{2p}$, when the two fields decouple, (5.6) becomes

$$T_c=\max_i\left[\frac{8}{\alpha}\Phi_{ip}^2\right] , \tag{5.7}$$

in agreement with (4.13).

We conclude this section by indicating how one can extend our method to cases where the scalar field $\Phi$ has more than two components. It is not hard to see that the one-loop correction, which determines the renormalization flow, is still given by $V_{1\,\mathrm{loop}}=\frac{1}{2}(d^2-1)^{-1/2}\mathrm{tr}V''$. The only calculational challenge is the computation of the $\lambda$ derivative of $W_1''(0)$ at $\lambda=0$. For its computation it helps to observe that $W_1''(0)$ is diagonal for an even classical potential and its diagonal elements are given by (C4). All one needs for their evaluation is Wick's theorem with respect to a Gaussian measure like the one in (C5). One sees that nothing conceptually new is required in cases where the Higgs field has more components.

It would be very interesting to extend our method to coupled Yang-Mills-Higgs systems. Since the modified MF approximation for pure lattice gauge systems at finite temperature is known,[14] this extension, at least to Abelian Higgs theories, presents itself.

*Note added.* After this work was completed we became aware of another paper[15] addressing similar problems. However the approach is quite different from ours. In Ref. 15 a high-temperature expansion has been used to prove the symmetry restoration for high temperature. Also, the authors took the time continuum limit only, whereas the spatial lattice constant was fixed to be one. We thank J. Jersak for bringing this work to our attention.

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## APPENDIX A

To derive (2.11) we consider the one-parameter family of actions

$$S_\epsilon[\phi]=-\sum_{\mathrm{SL}NN}\phi_i\phi_j-\epsilon\sum_{\mathrm{TL}NN}\phi_i\phi_j+\sum_i V_0(\phi_i) \tag{A1}$$

which interpolates between $l$ copies of a $(d-1)$-dimensional model and the original theory (2.6). The first (second) sum in (A1) is over spacelike (timelike) nearest-neighbor pairs and $V_0(\phi)=d\phi^2+V(\phi)$. Later we show that for any $\epsilon>0$ and any finite $N$ (we assume that the lattice has a finite spatial length)

$$\frac{d}{d\epsilon}\langle\phi_0\phi_i\rangle_\epsilon^l\geq 0\ , \tag{A2}$$

where $\langle O[\phi]\rangle_\epsilon$ denotes the expectation value of $O[\phi]$ computed with $S_\epsilon[\phi]$: namely,

$$\langle O[\phi]\rangle_\epsilon=\frac{\int\prod d\phi_i O[\phi]e^{-S_\epsilon[\phi]}}{\int\prod d\phi_i e^{-S_\epsilon[\phi]}}\ . \tag{A3}$$

To get the desired inequality (2.11) we integrate the "differential inequality" (A2) from $\epsilon=0$ to $\epsilon=1$ and sum over $i$. For $\epsilon=1$ the action (A1) becomes (2.6) and the sum $\sum_i\langle\phi_0\phi_i\rangle_{\epsilon=1}^l$ is, according to (2.10), the left-hand side of the inequality (2.11). In order to recover the right-hand side of (2.11) when $\epsilon=0$ we observe that in this case the action (A1) belongs to $l$ noninteracting copies of a $(d-1)$-dimensional model. Thus $\langle\phi_0\phi_i\rangle_{\epsilon=0}^l$ vanishes if the sites 0 and $i$ lie in different time slices of the lattice. When they belong to the same slice then the integrals over fields on the other $l-1$ slices cancel in the fraction (A3). One sees at once that the remaining $(d-1)$-dimensional action (on the slice defined by $\phi_0$ and $\phi_i$) has a shifted mass $m\to m+1$, and therefore the sum $\sum_i\langle\phi_0\phi_i\rangle_{\epsilon=0}^l$ coincides with the right-hand side of the inequality (2.11). This proves (2.11) for any finite spatial "volume." Now, letting $N\to\infty$ this inequality holds also in the infinite (spatial) "volume" limit. It remains to prove (A2), which the experts will recognize as one of Griffith's inequalities. For the sake of completeness we give the arguments leading to (A2). For that we introduce the abbreviations

$$H_s=\sum_{\mathrm{SL}NN}\phi_i\phi_j\ \text{ and }\ H_t=\sum_{\mathrm{TL}NN}\phi_i\phi_j\ , \tag{A4}$$

where the first sum is over all spacelike (SL) separated nearest neighbors and the second over all timelike (TL) separated nearest neighbors. Then the actions $S_\epsilon$ in (A1) read $S_\epsilon=-H_\epsilon+V_0=-H_s-\epsilon H_t+V_0$. One easily derives

$$\frac{d}{d\epsilon}\langle\phi_0\phi_i\rangle_\epsilon=\langle\phi_0\phi_iH_t\rangle_\epsilon-\langle\phi_0\phi_i\rangle_\epsilon\langle H_t\rangle_\epsilon\ . \tag{A5}$$

By doubling the fields, the right-hand side can be written as

$$Z^{-2}\int\{\phi_0\phi_i(H_t[\phi]-H_t[\chi])\}e^{-S_\epsilon[\phi]-S_\epsilon[\chi]}\prod d\phi_j d\chi_j\ . \tag{A6}$$

To see that this expression is positive we change variables by an orthogonal transformation

$$\phi_i=\frac{1}{\sqrt2}(t_i+q_i)\ \text{ and }\ \chi_i=\frac{1}{\sqrt2}(t_i-q_i)\ ,$$

in terms of which

$$\begin{aligned}\phi_0\phi_i(H_t[\phi]-H_t[\chi])&=\tfrac12(t_0+q_0)(t_i+q_i)\\&\quad\times\sum_{\mathrm{TL}NN}(t_kq_l+q_kt_l)\ ,\\ H_\epsilon[\phi]+H_\epsilon[\chi]&=\sum_{\mathrm{SL}NN}(t_it_j+q_iq_j)\\&\quad+\epsilon\sum_{\mathrm{TL}NN}(t_it_j+q_iq_j)\ ,\\ V_0[\phi]+V_0[\chi]&=\sum_i[(m+d)(t_i^2+q_i^2)\\&\quad+12g(t_i^4+6t_i^2q_i^2+q_i^4)]\ .\end{aligned} \tag{A7}$$

Next one expands $\phi_0\phi_i(H_t[\phi]-H_t[\chi])\exp(H_\epsilon[\phi]+H_\epsilon[\chi])$ in a power series and observes that all monomials in the $t$'s and $q$'s have positive coefficients. However, $V_0[\phi]+V_0[\chi]$ is an even function in both the $q$ and $t$ variables, and hence only even powers in the expansion contribute after integration. Therefore, as a sum of manifestly positive terms, the right side in (A5) is positive.

## APPENDIX B

In this appendix we show that the action (3.1) gives rise to the approximating effective potential (3.2). For that purpose we insert the identity $(q=d-p)$

$$\delta\left[\frac{1}{\Lambda}\sum_{i\in\Lambda}\phi_i-\Phi\right]=\int\prod_{i=1}^{\Lambda_q}d\Phi_i\delta\left[\frac{1}{\Lambda_p}\sum_{j\in\Lambda_p}\phi_j-\Phi_i\right]\times\delta\left[\frac{1}{\Lambda_q}\sum_{i\in\Lambda_q}\Phi_i-\Phi\right]$$

for the constraint. By observing that the Boltzmann factor in (2.8) factorizes, one obtains, after integration over the fields $\phi_j$, the approximating potential

$$U_p(\Phi)=-(d-p)\Phi^2-\frac{1}{\Lambda}\ln\int\prod_i d\Phi_i\delta\left[\frac{1}{\Lambda_q}\sum\Phi_i-\Phi\right]\times e^{-\Lambda_pU_p(\Phi_i)}\ .$$

Here $U_p(\Phi_i)$ denotes the constraint effective potential for a $p$-dimensional lattice theory on $\Lambda_p$ with a shifted mass $m\to m+d-p$. Next we insert the Fourier representation

$$\delta\left[\frac{1}{\Lambda_q}\sum\Phi_i-\Phi\right]=\Lambda\int dk\exp\left[i\Lambda k\Phi-ik\Lambda_p\sum\Phi_i\right]$$

for the $\delta$ distribution and end with

$$U_p(\Phi)=-(d-p)\Phi^2-\frac{1}{\Lambda}\ln\left[\Lambda\int dk\,e^{\Lambda[ik\Phi+W_p(-ik)]}\right]\ ,$$

where

$$\exp[\Lambda_p W_p(-ik)] = \int d\Phi \exp\{-\Lambda_p[ik\Phi + U_p(\Phi)]\} .$$

In the limit when the volume tends to infinity the integral coincides with its value at the saddle point on the imaginary axis. Setting $k = ij$ we finally obtain the potential (3.2).

## APPENDIX C

In three dimensions the bare coupling constants and bare masses scale linearly with $\lambda$ and thus the Schwinger function for a two-component model on two time slices reads

$$W_1(j) = \tfrac{1}{2}\ln \int (\exp[j_1(\phi_1+\phi_2)+j_2(\chi_1+\chi_2)] \times \exp\{2(\phi_1\phi_2+\chi_1\chi_2) - d(\phi_1^2+\phi_2^2+\chi_1^2+\chi_2^2) - \lambda V[\phi,\chi]\}) , \quad \text{(C1)}$$

where $j = (j_1, j_2)$ and the parameters in $V$ are determined by the coefficients in the flow (4.10). For $\lambda = 0$ one finds $W = (4d-4)^{-1}(j_1^2+j_2^2)$ and hence $\lambda^{-d}\Gamma$ in (4.5) has the leading term

$$\lambda^{-d}\Gamma_1(\lambda^{d/2-1}\Phi) \sim (d-1)\lambda^{-2}(\Phi_1^2+\Phi_2^2) , \quad \text{(C2)}$$

which cancels the first term on the right-hand side of (4.5).

Next we evaluate the $\lambda$ derivative of

$$2W_1''(0) = \begin{bmatrix} \langle(\phi_1+\phi_2)^2\rangle & \langle(\phi_1+\phi_2)(\chi_1+\chi_2)\rangle \\ \langle(\phi_1+\phi_2)(\chi_1+\chi_2)\rangle & \langle(\chi_1+\chi_2)^2\rangle \end{bmatrix} \quad \text{(C3)}$$

at $\lambda = 0$. For example, the derivative of the 1-1 component is

$$\langle(\phi_1+\phi_2)^2\rangle_0 \langle V[\phi,\chi]\rangle_0 - \langle(\phi_1+\phi_2)^2 V[\phi,\chi]\rangle_0 . \quad \text{(C4)}$$

Here $\langle\ \rangle_0$ is computed with the Gaussian measure

$$d\mu(\Phi) = \frac{1}{N}e^{-(\Psi, A\Psi)}, \quad A = \begin{bmatrix} d & -1 & 0 & 0 \\ -1 & d & 0 & 0 \\ 0 & 0 & d & -1 \\ 0 & 0 & -1 & d \end{bmatrix} , \quad \text{(C5)}$$

where $\Psi$ is the four-component vector $(\phi_1,\phi_2,\chi_1,\chi_2)$. The expectation values in (C4) can easily be calculated by applying Wick's theorem. For the model (5.1) and its flow (5.4), we found, after some arithmetic,

$$\frac{d}{d\lambda} 2W_1''(0)\bigg|_{\lambda=0} = -\frac{\alpha}{8\times 96}\begin{bmatrix} 3\dfrac{m_{1p}}{\Phi_{1p}^2} + \dfrac{\Delta m_p}{\Delta\Phi_p} & 0 \\ 0 & 3\dfrac{m_{1p}}{\Phi_{1p}^2} + \dfrac{\Delta m_p}{\Delta\Phi_p} \end{bmatrix} .$$

With $2W_1''(j=0,\lambda=0) = 1/2Id$ and $\Gamma''(0) = W''(0)^{-1}$, one finds that the $\lambda$ derivative at $\lambda = 0$ of $\Gamma''(0)$ is, up to a factor $-16$, equal to the $\lambda$ derivative at $\lambda = 0$ of $W''(0)$. Together with (C2) and (4.5) this establishes the asymptotic expansion (5.5) for the modified MF potential.

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