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Magnetic monopoles vs. Hopf defects in the Laplacian (Abelian) gauge

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We investigate the Laplacian Abelian gauge on the sphere S^4 in the background of a single 't Hooft instanton. To this end we solve the eigenvalue problem of the covariant Laplace operator in the adjoint representation. The ground state wave function serves as an auxiliary Higgs field. We find that the ground state is always degenerate and has nodes. Upon diagonalisation, these zeros induce toplogical defects in the gauge potentials. The nature of the defects crucially depends on the order of the zeros. For first-order zeros one obtains magnetic monopoles. The generic defects, however, arise from zeros of second order and are pointlike. Their topological invariant is the Hopf index $S^3 \to S^2$. These findings are corroborated by an analysis of the Laplacian gauge in the fundamental representation where similar defects occur. Possible implications for the confinement scenario are discussed.

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1. Introduction

Although not derived from first principles, the dual superconductor scenario [1, 2, 3] is widely believed to explain color confinement in QCD. To realise this idea, ‘t Hooft suggested to use Abelian projections [4] which allow for a straightforward identification of magnetic monopoles in pure Yang Mills theories. In this approach one fixes the gauge group up to its maximal Abelian subgroup. This partial gauge fixing can be characterised by a Higgs field ϕ in the adjoint representation, which becomes diagonal in the Abelian gauge (AG)[1]. Magnetic monopoles arise as gauge fixing defects whenever ϕ vanishes. At these points, the gauge transformation diagonalising ϕ becomes ambiguous. In the low temperature phase of QCD these defects should condense and play the role of Cooper pairs.

This picture is strongly supported by lattice calculations (for recent reviews, see [5, 6]). In the continuum, however, Abelian gauges are not that well understood. Considerable progress has only been made for the Polyakov Abelian gauge (PAG) [7, 8, 9, 10, 11, 12]. The defects occuring in the PAG are characterised by a winding number $S^2 \to S^2$ of the (normalised) Higgs field, $n \equiv \phi/|\phi|$, or equivalently by the magnetic charge q of the Abelian gauge field. A relation between monopole charge q and instanton number $\nu[A]$ has been established which enforces the presence of monopoles in any non-trivial instanton sector ($\nu \neq 0$).

For the maximally Abelian gauge (MAG) [4, 13, 14], there are only few analytical results. It is known that configurations with monopole lines [15] and monopole loops [16] are in this gauge. They are, however, strongly suppressed by the gauge fixing functional, at least in the backgound of single instantons. Recently, it has been explicitly shown that the continuum MAG suffers from a Gribov problem [17] as expected from Singer's theorem [18]: the ‘t Hooft instanton in the singular gauge is located on the Gribov horizon of the MAG [19].

In order to circumvent the Gribov (‘spin glass’) problem of the MAG on the lattice, the Laplacian Abelian gauge (LAG) has been proposed as a superior alternative [20, 21, 22]. Some first applications of this idea in the context of lattice gauge theory have appeared only recently [23, 24]. Analytically, however, it seems that only one result has been obtained so far: by comparing the behaviour of the gauge fixing functionals one finds [20] that in the LAG magnetic degrees of freedom are less suppressed than in the MAG. Some deeper understanding of this Abelian gauge is obviously desirable.

This paper presents our first investigation of the continuum Laplacian Abelian gauge. In order to have a large amount of symmetry, we consider (the orbit of) a single 't Hooft instanton. As we shall see, the LAG is somewhat ill-defined on infinite-volume manifolds, and thus we compactify space to a sphere S^4. For a special choice of the compactification radius, the symmetry is enhanced to SO(5) so that the (gauge fixing) problem can be exactly solved. For other radii, symmetry arguments still provide some insights. Going back to infinite volume, we find that the singular gauge instanton and global SU(2)

[1]In the following we will distinguish between the Abelian gauge, which is a partial gauge fixing, and the Abelian projections, where one neglects the off-diagonal part of the gauge field after gauge fixing.

rotations thereof lie in the LAG. Accordingly, the instanton in the singular gauge is a horizon configuration, as was the case for the MAG.

Of particular interest is the question how the submanifolds of vanishing Higgs field look like. It has been argued that, generically, these are loops, i.e. closed monopole worldlines, having codimension three[2]. The instanton number can be recovered from these loops for general Higgs fields [25, 26]. It is related to the winding number $S^2 \to S^2$. While the same is true for the LAG, we find that the Higgs field associated with a single 't Hooft instanton in addition induces *pointlike* defects, i.e. events localised in space-time. The corresponding topological invariant is the Hopf index $S^3 \to S^2$.

This paper is organised as follows: First, in Section 2, we define the LAG and discuss its properties. Single 't Hooft instantons on S^4 are introduced in Section 3. In Section 4 we diagonalize the covariant Laplacian in the adjoint representation. A classification of its ground state wave functions, which serve as auxiliary Higgs fields, is given in Section 5. A brief discussion of the Laplacian gauge (corresponding to the fundamental representation) is added in Section 6. Finally, we conclude with some remarks on the physical implications of our findings.

2. The Laplacian Abelian gauge

The Laplacian Abelian gauge on $\mathbb{R}^4$ is defined by minimising the Higgs kinetic energy [21, 22],

$$F_{\mathrm{LAG}}[A,\phi] = \frac{1}{2}\int \left(\mathrm{D}_\mu\,\phi^a \mathrm{D}_\mu\,\phi^a - E\phi^a\phi^a\right) \mathrm{d}^4x, \qquad \mathrm{D}_\mu = \partial_\mu - i[A_\mu, \ldots]\ , \tag{1}$$

with respect to the auxiliary Higgs field ϕ in the adjoint representation. The energy variable E is a Lagrange multiplier demanding that ϕ is square integrable, $\int \phi^a\phi^a\,\mathrm{d}^4x < \infty$. The field configuration ϕ minimising F_{LAG} can be viewed as the ground state of the covariant Laplacian,

$$-\mathrm{D}^2_\mu[A]\phi = E\,\phi, \tag{2}$$

where E is the ground state energy. Obviously, (2) represents a four dimensional Schrödinger problem with a potential essentially given by A^2.

The gauge transformation Ω diagonalising ϕ puts the gauge field A into the LAG,

$$A_{\mathrm{LAG}} \equiv {}^{\Omega}\!A, \qquad \text{where}\quad {}^{\Omega}\!\phi \equiv \Omega^{-1}\phi\,\Omega \sim \sigma_3. \tag{3}$$

Ω may be ambiguous for two reasons. First, if ϕ has zeros and second, if the groundstate is degenerate. One can use node and uniqueness theorems to analyse these issues [27].

On a space-time with infinite volume, the LAG is not straightforwardly defined for the following reasons. Since $-D^2_\mu[A]$ is a non-negative operator we have $E \geq 0$. Moreover, whenever the gauge field A tends to zero at infinity, there are scattering states

[2] as one has to solve three equations on a four-dimensional manifold

and the continuous spectrum always starts from zero. Scattering states, however, are not normalisable. Thus, for a generic background (including the instantons to be studied), one does not expect that the covariant Laplacian $-D_\mu^2[A]$ will have a normalisable ground state. The situation is quite analogous to the quantum mechanics of the ordinary 'Laplacian' d^2/dx^2 on the real line. We avoid this problem by considering gauge fields on the four-sphere S^4 which leads to a purely discrete spectrum of the associated covariant Laplacian.

3. Single instanton on the sphere

In the following we consider the single 't Hooft instanton (in singular and regular gauge) on a sphere S^4 of radius R. On Euclidean $\mathbb{R}^4$ the configurations read,

$$A_\mu^{\rm sg} = \bar\eta^a_{\mu\nu} x_\nu \frac{\rho^2}{r^2(r^2+\rho^2)}\sigma_a, \qquad A_\mu^{\rm reg} = \eta^a_{\mu\nu} x_\nu \frac{1}{(r^2+\rho^2)}\sigma_a, \qquad r^2 = x_\mu x_\mu \; , \tag{4}$$

using the conventions of [28]. The configurations (4) are related by the gauge transformation

$$h = \hat x_4 \mathbb{1}_2 + i\hat x_a \sigma_a, \qquad \hat x_\mu \equiv x_\mu / r \; , \tag{5}$$

which also relates the solutions ϕ of (2) in these two backgrounds, $\phi^{\rm reg} = h\,\phi^{\rm sg}\,h^\dagger$.

We benefit from the facts that classical Yang-Mills theories are conformally invariant and that the sphere S^4 is conformally equivalent to compactified Euclidean space $\dot{\mathbb{R}}^4$. If we use conformal coordinates x_μ on the sphere, which are simply the Cartesian coordinates of the point stereographically projected onto $\mathbb{R}^4$, the metric is conformally flat,

$$g_{\mu\nu}(x) = e^{\alpha_R(r)}\delta_{\mu\nu} \equiv \frac{4R^4}{(r^2+R^2)^2}\,\delta_{\mu\nu} \; . \tag{6}$$

Field configurations minimising the Yang Mills action on $\mathbb{R}^4$ are also minimising configurations on the sphere, if the Cartesian coordinates are substituted by conformal coordinates. Thus, we can simply use expressions (4) for the instantons on the sphere.

What about the symmetry of these configurations? It is known [29] that on $\mathbb{R}^4$ they are invariant under SO(4) rotations and a combination of translations and special conformal transformations,

$$\delta x_\mu = \omega_{\mu\nu}x^\nu + 2\,c\cdot x\,x_\mu/\rho - c_\mu(x^2+\rho^2)/\rho \; , \tag{7}$$

up to a compensating gauge transformation, $\delta A_\mu = \mathrm{D}_\mu\theta$, with

$$\theta^a = \frac{1}{2}\omega^{\mu\nu}\eta^a_{\mu\nu} - 2c^\mu\eta^a_{\mu\nu}x^\nu \qquad \text{(reg. gauge)}. \tag{8}$$

Together these transformations form (a non-linear representation of) the group SO(5). This symmetry is preserved on S^4 when the radius R of S^4 coincides with the instanton size ρ [30]. To illustrate this point, we note that the gauge invariant Lagrangian density,

$$\mathcal{L} \sim g^{\mu\rho} g^{\nu\sigma}\, F^a_{\mu\nu} F^a_{\rho\sigma} = \frac{12\rho^4}{R^8} \frac{(r^2+R^2)^4}{(r^2+\rho^2)^4}, \tag{9}$$

is constant on S^4 (and thus SO(5)-invariant) only if $R = \rho$. For $R \neq \rho$ the explicit appearance of r, which is only SO(4)-invariant, breaks SO(5) down to SO(4).

Since compensating gauge transformations do not spoil equation (2), the eigenfunctions ϕ furnish representations of SO(5) and SO(4), respectively.

4. Solutions of the covariant Laplacian

Generalising equations (1) and (2) to curvilinear coordinates, we define the LAG on the sphere S^4 via the functional

$$F_{\mathrm{LAG}}[A,\phi] = \frac{1}{2} \int_{S^4} \left(\mathrm{D}_\mu \phi^a \mathrm{D}_\nu \phi^a g^{\mu\nu} - E\phi^a\phi^a\right) \sqrt{g}\, \mathrm{d}^4 x \ , \tag{10}$$

where $g = \exp(2\alpha_R)$ denotes the determinant of the metric (6). The equation of motion is given in terms of the (gauge covariant) Laplace-Beltrami operator,

$$-\frac{1}{\sqrt{g}} \mathrm{D}_\mu \sqrt{g} g^{\mu\nu} \mathrm{D}_\nu \phi = E\phi. \tag{11}$$

To proceed we make use of the symmetry and separate into angular and radial equations. The angular part is expressed in terms of angular momenta derived from the decomposition $so(4) \cong su(2) \oplus su(2)$,

$$M_a = -\frac{i}{2} \bar{\eta}^a_{\mu\nu} x_\mu \partial_\nu,\ \ \vec{M}^2 \to m(m+1), \quad N_a = -\frac{i}{2} \eta^a_{\mu\nu} x_\mu \partial_\nu,\ \ \vec{N}^2 \to n(n+1)\ . \tag{12}$$

In this representation, the two Casimirs coincide, $\vec{M}^2 = \vec{N}^2$. Their eigenvalues are half-integer, $m = n \in \{0,\, 1/2,\, 1,\, 3/2, \ldots\}$. The generators for isospin $t = 1$ are

$$(T_a)_{bc} = i\epsilon_{bac}, \qquad \vec{T}^2 \to t(t+1) = 2\ . \tag{13}$$

The radial equations on the sphere differ from those in Euclidean space by a metric factor, $\exp(-\alpha_R)$, and a dilatation term, $r\partial_r$,

$$e^{-\alpha_R(r)} \left[-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4\vec{M}^2}{r^2} + \frac{4\rho^2(\vec{J}^2 - \vec{M}^2)}{r^2(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} + \frac{4r}{r^2+R^2}\partial_r \right] \phi^{\mathrm{sg}} = E\phi^{\mathrm{sg}} \tag{14}$$

$$e^{-\alpha_R(r)} \left[-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4\vec{N}^2}{r^2} + \frac{4(\vec{J}^2 - \vec{N}^2)}{(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} + \frac{4r}{r^2+R^2}\partial_r \right] \phi^{\mathrm{reg}} = E\phi^{\mathrm{reg}} \tag{15}$$

In the above, we have introduced the conserved angular momentum $\vec{J}$ ('spin from isospin', [31, 32, 33]),

$$\vec{J} \equiv \vec{L} + \vec{T}, \qquad \vec{J}^2 \to j(j+1), \qquad j \in \{l-1, l, l+1\} , \tag{16}$$

where $\vec{L}$ denotes $\vec{M}$ or $\vec{N}$, respectively. Replacing angular momenta by their eigenvalues and exchanging $j \to n$, $m \to j$, equation (14) turns into (15). This amounts exactly to the action of the gauge transformations h from (5).

The symmetry considerations above suggest the following form of the ground state,

$$\phi(x) = Y_{(j,l)}(\hat{x})\varphi(r) , \tag{17}$$

where the Y's denote the spherical harmonics on S^3 (see Appendix A). Note that there are two competing angular momentum terms in (14) and (15), so that it is not obvious in which angular momentum sector the groundstate will be. By simply looking at the radial potentials in the different sectors, we can only state the following bound on the energy in an arbitrary sector,

$$E_{(j,l)} \geq \min\{E_{(0,1)},\, E_{(1/2,1/2)},\, E_{(1,0)}\} . \tag{18}$$

The quantum numbers of the ground state candidates on the r.h.s. correspond to the representations (0,1), (1/2, 1/2) and (1,0) of $su(2)_j \oplus su(2)_l$ and thus have degeneracies 3, 4 and 3, respectively. Note that the singlet $(0,0)$ is excluded by the selection rules for $t=1$, see (16). Accordingly, for any of the possible choices in (18), the groundstate will be degenerate. The spherical harmonics for the three cases are listed in Appendix A.

At this point two further remarks are in order: First, the radial part φ shows power law behaviour in r, both for small and large r, independent of R and ρ,

$$\varphi^{\mathrm{sg}}(r \to 0) \to r^{2j} , \qquad \varphi^{\mathrm{sg}}(r \to \infty) \to r^{-2m} , \tag{19}$$

$$\varphi^{\mathrm{reg}}(r \to 0) \to r^{2n} , \qquad \varphi^{\mathrm{reg}}(r \to \infty) \to r^{-2j} . \tag{20}$$

Second, upon substituting $\varphi \equiv (r^2 + R^2) \cdot \chi$ and $\lambda \equiv ER^2 + 2$, one can absorb the dilatation term,

$$\left[-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4\vec{M}^2}{r^2} + \frac{4\rho^2(\vec{J}^2 - \vec{M}^2)}{r^2(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} - \frac{4\lambda R^2}{(r^2+R^2)^2} \right] \chi^{\mathrm{sg}} = 0 , \tag{21}$$

$$\left[-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4\vec{N}^2}{r^2} + \frac{4(\vec{J}^2 - \vec{N}^2)}{(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} - \frac{4\lambda R^2}{(r^2+R^2)^2} \right] \chi^{\mathrm{reg}} = 0 . \tag{22}$$

Setting $R = \rho$, the differential equation (21) coincides with the one considered by 't Hooft in his analysis of the fluctuations around instantons [34]. The eigenvalues are $\lambda_k = (k + j + l + 1 - t)(k + j + l + t + 2)$. The lowest energy corresponds to $k = 0$ and $j + l = 1$, consistent with the three possible groundstates of (18). Together they form

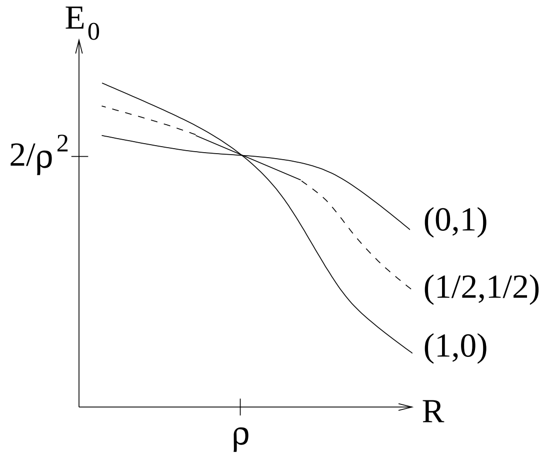

Figure 1: Energy of the lowest-lying states in the relevant angular momentum sectors as functions of the compactification radius R (singular gauge). At the point $R = \rho$ the two triplets and the quadruplet meet, while for $R \to \infty$ the triplet $(1, 0)$ has lowest energy. For symmetry reasons we expect the dashed line to stay inbetween the other two for $R \neq \rho$.

the 10-dimensional adjoint representation[3] of SO(5) [35]. The value of the ground state energy is $E = 2/R^2$.

The radial eigenfunctions are rational,

$$\varphi^{\text{sg}}(r) = R\,\frac{(r/R)^{2j}}{r^2 + R^2}\;, \qquad \varphi^{\text{reg}}(r) = R\,\frac{(r/R)^{2-2j}}{r^2 + R^2}\;, \tag{23}$$

and obey the asymptotics (19) and (20), respectively. In accordance with the node theorem for the one-dimensional radial equation, the lowest-lying states have no zeros apart from $r = 0$ and $r = \infty$.

For the cases $R > \rho$ and $R < \rho$ we cannot solve the radial equation analytically. However, we are able to prove the following statements:

For the singular gauge and $R > \rho$, the triplet $(1, 0)$ has lower energy than the triplet $(0, 1)$. For $R < \rho$, the situation is vice versa with the triplet $(0, 1)$ having lower energy. Analogous statements hold for the regular gauge (see Fig. 1). These results are a straightforward consequence of the Feynman-Hellmann theorem (cf. Appendix B).

For the quadruplet (1/2, 1/2), the situation is somewhat more complicated. Using perturbation theory in $\delta = \rho^2 - R^2$ (see Appendix B), one finds that these states have energy inbetween the two disjoint triplet states. For symmetry reasons we do not expect the spectral flow $E_{(1/2,1/2)}(R)$ to intersect the others for some $R \neq \rho$ (see Fig. 1).

Finally, the node theorem again guarantees that ϕ vanishes only at $r = 0$ and $r = \infty$, in accordance with the asymptotics (19,20).

[3]Using the conventions of [35], this representation is labelled by the integers $\{n_1, n_2\} = \{0, 2\}$ which are the coefficients of the highest weight when expanded in terms of the fundamental weights.

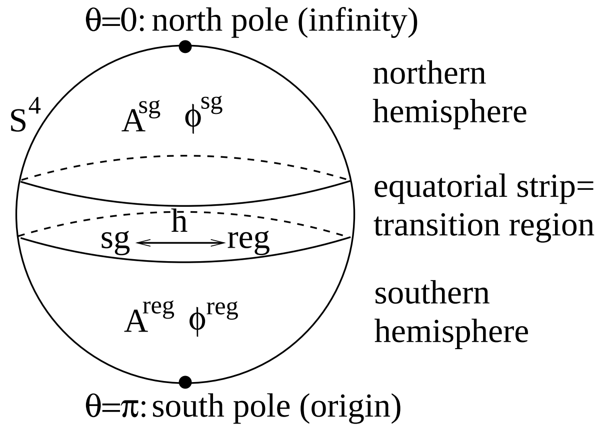

Figure 2: In the bundle picture on S^4 there are two gauge and Higgs fields, which are smoothly defined on their domains (hemispheres). In the transition region, they are related by the gauge transformation (transition function) h. Note that the four-dimensional radius can be expressed in terms of the azimuthal angle θ, $r = R\cot(\theta/2)$.

5. Properties of the solutions

Before characterising the zeros of the solutions ϕ, let us point out the following subtlety: Near the origin, the (0,1) wave functions (Higgs fields) in the singular gauge are bilinear in $\hat{x}_\mu$ and thus *discontinuous* there. They inherit this singularity from the instanton field[4]. Nevertheless, the wave functions are square integrable on S^4 due to the measure factor r^3. The same, of course, is true for the regular gauge states near infinity. In order to work with smooth Higgs fields, it is appropriate to use the *principal fibre bundle* picture. This can be viewed as a non-Abelian S^4-analogue of the Wu-Yang construction for the Dirac monopole on S^2 [36]. The A-field in the regular gauge represents the connection smoothly[5] on the southern hemisphere (the chart containing the origin), while the A-field in the singular gauge does the same on the northern hemisphere (the chart containing infinity). In the transition region formed by the equatorial strip displayed in Fig. 2, the gauge transformation h from (5) interpolates between the two. For simplicity we will retract the transition region to a single three-sphere S^3_r of fixed four-dimensional radius r (fixed azimuthal angle θ).

The Higgs field is a section in an associated fibre bundle: on each of the two charts there is a Higgs field. In the transition region, the same transition function h relates the two (see Fig. 2). Our results obtained so far can immediately be carried over to the bundle picture, since, for every solution in the singular gauge, there is a corresponding gauge transformed 'mirror' solution in the regular gauge with the same energy (and vice versa). Moreover, the angular momenta are interchanged by h in such a way that

[4]which results in the asymptotics $\varphi(r) \sim r^0$, see (19).

[5]to be precise: The A-fields are the pullbacks of the connection under smooth sections of the bundle.

the *radial* wavefunctions (23) are smooothly defined on the whole of S^4. The complete eigenfunctions ϕ are continuous in their respective charts but 'jump' (in their isospin direction) due to the action of the transition function h in the transition region.

Along these lines, let us discuss the ground state in the $(1/2, 1/2)$ sector, which has zeros localised on *loops*. To simplify the discussion, we choose the fourth of the spherical harmonics in (37) or (38), for which

$$\phi^{\rm sg}(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \frac{1}{r^2+R^2} = \phi^{\rm reg}(x), \tag{24}$$

since h from (5) commutes with $\phi^{\rm sg}$. This Higgs field ϕ is of hedgehog type. Thus, its diagonalisation induces a Dirac monopole at $\vec{x}=0$ and a Dirac string along the negative 3-axis. The world-line of the monopole is the great circle in 4-direction (which degenerates to the 4-axis in the infinite-volume limit). For the other three states in the multiplet the same holds true upon permutation of the coordinates.

It is well-known [37] that the monopole charge is characterised by the winding number of the *normalised* Higgs field $n = \phi/|\phi|$, explicitly given by

$$n^{\rm sg}(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} /|\vec{x}| = n^{\rm reg}(x) \ . \tag{25}$$

The n-field is singular at $\vec{x}=0$, where ϕ vanishes. This singularity has the following topological characterisation. Consider the two-sphere, S^2: $\vec{x} = const.$, surrounding the singularity. There, the n-field provides a smooth mapping, $S^2 \to S^2 \cong SU(2)/U(1)$, labelled by an integer, the winding number, which in our case is just one.

The LAG Higgs field also serves as an illustration of the relation between instanton number and monopole charge recently proposed in [26]. We note that in the (1/2, 1/2) sector the two Higgs fields $n^{\rm sg}$ and $n^{\rm reg}$ coincide on the whole of S^4, their singularities being located at two points $p_1 = (\vec{0}, -r)$, $p_2 = (\vec{0}, r)$ in the retracted transition region S^3_r (see Fig. 3). Consider the submanifold $S^3_r \setminus \{p_1, p_2\} \cong S^2 \times I_{12}$. In terms of the polar angle $\vartheta \in (0,\pi)$ on the three-sphere, the intervall I_{12} is parametrised by $x_4 = r\cos\vartheta \in (-r, r)$, while the two-spheres are given by $|\vec{x}| = r\sin\vartheta$.

We already know that the magnetic charge measured on the two-sphere is $q=1$. If we express the transition function h in terms of n and ϑ,

$$h = \exp(i\vartheta\, n^a \sigma_a) \ , \tag{26}$$

the flux[6] $\Phi = \int_{I_{12}} {\rm d}(2\vartheta)$ is easily computed as

$$\Phi = 2\int_0^\pi {\rm d}\vartheta = 2\pi \ . \tag{27}$$

[6]notice the difference σ_a vs. $\sigma_a/2$ as compared to [26]

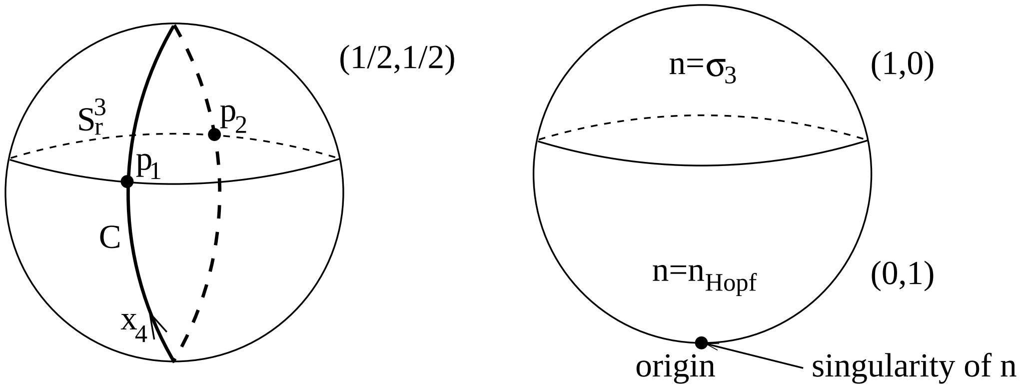

Figure 3: Submanifolds of S^4 on which the ground state wave function vanishes so that the normalised Higgs field becomes singular. The sector (1/2,1/2) gives rise to monopole loops C, while the generic sectors (0,1) and (1,0) lead to pointlike singularities with Hopf index as topological invariant.

We thus recover the instanton number,

$$\nu[A] = q\,\frac{\Phi}{2\pi} = 1\ . \tag{28}$$

Note that for linear combinations of states from different multiplets the monopole loops become *tilted*. As an example, take a combination of the sectors (1,0) and (1/2, 1/2),

$$\phi^{\rm reg} = \frac{1}{\sqrt{2}}\left(\phi^{\rm reg}_{(1,0)} - \phi^{\rm reg}_{(1/2,1/2)}\right) = \frac{1}{\sqrt{2}(r^2+R^2)}\left(\begin{array}{c} x_1 \\ x_2 \\ R - x_3 \end{array}\right)\ . \tag{29}$$

This Higgs field vanishes for $x_\mu = (0,0,R,x_4)$, a set of zeros which is still a great circle but does no longer include the poles.

As we have already argued, the quadruplet states (1/2, 1/2) will occur as ground states only for $R=\rho$. In the general case, $R \neq \rho$, the ground states will be the triplet states $(0,1)$ and $(1,0)$, which have *isolated, pointlike* zeros. Let us specialise to the physical region $R > \rho$ which includes the infinite-volume limit, $R \to \infty$. (For $R < \rho$ one has to perform the appropriate 'mirror' transformation.) We know that for the southern hemisphere (regular gauge) and for the northern hemisphere (singular gauge) we have to take $(j,l) = (0,1)$ and $(j,l) = (1,0)$, respectively, since these multiplets consist of the lowest-lying states. If we choose the third of the spherical harmonics in (36) and (40), the normalised Higgs field is given by

$$n(x) = \begin{cases} n^{\rm reg}(x) = \left(\begin{array}{c} 2(\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_2\hat{x}_3 + \hat{x}_1\hat{x}_4) \\ -\hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 \end{array}\right) & \text{southern hem.} \ni 0\ , \\[2ex] n^{\rm sg}(x) = \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) & \text{northern hem.} \ni \infty\ . \end{cases} \tag{30}$$

$n^{\rm reg}$ is singular at the origin $r = 0$ and closely resembles the standard Hopf map [38, 39]. For any finite radius $r \neq 0$, it provides a smooth mapping $S^3_{r=\text{fixed}} \to S^2 \cong SU(2)/U(1)$ with Hopf index one [40].

As a result, we have obtained the simplest realisation of the connection between instanton number and Hopf indices derived in [25]: The (signed) sum of all Hopf indices of n around its singularities equals the instanton number ν. This statement is analogous to results from residue calculus where the singularities of n (the zeros of ϕ) are replaced by the poles of a meromorphic function: The (signed) sum of all residues equals the residue at infinity. Like the magnetic monopoles in the PAG, n must possess singularities in any non-trivial instanton sector ($\nu \neq 0$). In addition, pairs of singularities may occur which do not contribute to the instanton number.

Coming back to the LAG, the remaining task is to diagonalise the ground-state Higgs field $\phi(x)$. On the northern hemisphere, where ϕ is already diagonal, there is nothing to be done. No gauge transformation is needed and the gauge field remains in the singular gauge. On the southern hemisphere, we basically have to diagonalise the standard Hopf map. This is achieved by the gauge transformation h, which transforms the gauge field A from regular to singular gauge. Independent of where we choose the transition region, *the LAG-fixed configuration on the orbit of the single 't Hooft instanton is in the singular gauge* (for $R > \rho$). Notice that the gauge fixed configuration inherits a singularity only at the point where n is singular; there are no further 'Dirac strings'.

If we choose an arbitrary linear combination of the triplet spherical harmonics, the diagonalising gauge transformation includes an additional *global* SU(2) rotation. Together with $A^{\rm sg}$, all its global rotations are located on the gauge fixing hypersurface defined by the LAG. We thus find a whole S^3 of gauge-equivalent configurations (Gribov copies).

6. The Laplacian gauge

The importance of pointlike defects (as compared to loops) is corroborated by their occurence in a closely related gauge, the Laplacian gauge (LG)[7]. The Laplacian gauge [41, 42, 43, 44] is defined via a Higgs field q in the *fundamental* representation, being the ground state of the covariant Laplacian,

$$-\mathrm{D}_\mu^2[A]q = E\,q\,, \qquad \mathrm{D}_\mu = \partial_\mu - iA_\mu\,. \tag{31}$$

It is a *complete* gauge fixing (up to defects) if the two-component complex vector q is rotated into a fixed isospin direction and made real,

$$^{\Omega}q \equiv \Omega^{-1}q = \begin{pmatrix} |q| \\ 0 \end{pmatrix}, \qquad A_{\rm LG} \equiv {}^{\Omega}A\,. \tag{32}$$

Our formalism is easily adapted to this gauge by choosing the isospin $t = 1/2$ representation in terms of the Pauli matrices $T_a = \sigma_a/2$. For $R = \rho$ one again has to minimise $j+l$, whence $(j,l) = (0,1/2)$ or $(j,l) = (1/2,0)$. As before, these states form an irreducible

[7]The authors thank P. de Forcrand for drawing their attention to this issue.

representation[8] of SO(5). For $R > \rho$ and the singular gauge, the state $(1/2, 0)$ has lowest energy (by the same Feynman-Hellmann argument) so that the singular gauge instanton again satisfies the gauge condition. The relevant spherical harmonics are

$$Y^{\rm reg}_{(1/2,0)} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{33}$$

$$Y^{\rm reg}_{(0,1/2)} = \left\{ h \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{x}_4 + i\hat{x}_3 \\ -\hat{x}_2 + i\hat{x}_1 \end{pmatrix}, h \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 + i\hat{x}_1 \\ \hat{x}_4 - i\hat{x}_3 \end{pmatrix} \right\}, \tag{34}$$

which are nonzero throughout S^4. In analogy with (20), we have the following behaviour near the origin, $q(r) \sim r^{2n} = r$ for $(j, n) = (0, 1/2)$. Thus, the modulus of the Higgs field is proportional to the four-dimensional distance r from the origin (where the topological charge of the instanton is concentrated). This perfectly agrees with latest results from lattice simulations [45].

Again, a topological description is possible. On a three-sphere surrounding the origin, one can define $n \equiv q/|q| : S^3 \to S^3$ with integer winding number. In the case above, the n-field simply reduces to the identity map,

$$n \equiv Y = \begin{pmatrix} \hat{x}_4 + i\hat{x}_3 \\ -\hat{x}_2 + i\hat{x}_1 \end{pmatrix}, \tag{35}$$

the winding number k of which coincides with the instanton number, $k = \nu = 1$.

7. Conclusions

We have investigated the Laplacian Abelian gauge on the sphere S^4 in the background of a single 't Hooft instanton. This amounts to solving the eigenvalue problem for the covariant Laplacian in the adjoint representation. For any sphere radius R we have determined the angular dependence and isospin structure of the ground state wave functions (Higgs fields). Diagonalisation of the latter shows that the instanton in the singular gauge is in the LAG if R is larger than the instanton size ρ; for the regular gauge the same is true for $R < \rho$. The gauge fixing procedure thus selects one of the two instanton configurations, although, in a bundle picture, they represent the same connection.

It is interesting to note that the situation for the MAG on the sphere is similar: Singular and regular gauge instantons both satisfy the differential MAG condition, but the MAG functional $F_{\rm MAG}$ picks out one of them in the very same way as $F_{\rm LAG}$: for $R > \rho$ ($R < \rho$) the singular (regular) gauge instanton minimizes $F_{\rm MAG}$ (see Appendix C). It is, however, a highly nontrivial task to check whether a given configuration, say the 't Hooft instanton, really corresponds to the absolute minimum along its orbit. In general, one can never be sure that there is no other gauge equivalent configuration that lowers the functional even further.

[8] the four-dimensional spinor representation labelled by $\{n_1, n_2\} = \{0, 1\}$ in [35].

The LAG, on the other hand, has the big advantage that the ground state (and thus the absolute minimum of F_{LAG}) can be found explicitly. We have done so for $R = \rho$ and have given qualitative arguments concerning the angular and radial dependence for the case $R \neq \rho$. Apart from the degeneracies and the zeros (which we have under control), there are no further ambiguities.

We have found a whole S^3 of gauge equivalent configurations (obtained by *global* SU(2) rotations of A^{sg} similar to what has been observed in [46]) located on the gauge fixing hypersurface. These are Gribov copies of each other, generated by both finite and infinitesimal gauge transformations. The latter give rise to three flat directions in the configuration space along which the gauge fixing functional does not change. Only one of these directions is covered by the residual U(1) freedom. The other two are related to zero modes of the (coset part of the) Faddeev-Popov operator. We do see no reason why these Gribov ambiguities should not be present on the lattice. In contrast to the MAG (and related gauges), however, where gauge fixing is a "numerical problem of non-polynomial complexity" [42], there are no additional lattice Gribov copies beyond the denumerable ones we have encountered in the continuum. This clearly makes the LAG a superior gauge.

Once a ground states is chosen for diagonalisation, additional obstructions occur in terms of gauge fixing defects caused by the nodes of the possible ground states. These are the well-known source for magnetic monopoles in Abelian gauges. We have shown that these defects *must* be present whenever the LAG background is in a non-trivial instanton sector. Monopoles, however, only arise for a particular sphere radius $R = \rho$ and for a particular choice of ground states. *The generic defects are localised in space-time* (with codimension 4). Their topological invariant is the Hopf index $S^3 \to S^2$. Contrary to monopoles they have finite action even in the infinite volume limit. One may speculate that *these* defects condense in the low temperature phase of QCD, possibly giving rise to a new confinement mechanism. In view of the results presented in [26], they may as well be related to the solitonic excitations observed in recent effective theories for confinement [47, 48, 49, 50].

As we have calculated the LAG Higgs field only for a highly symmetric background, the question arises which features are generic also for other backgrounds. The degeneracy of the ground state is mainly due to the matrix structure of the 'Hamiltonian'. For a single instanton background, this was induced by nonvanishing angular momentum (like in quantum mechanical problems with spin). This should be contrasted with the case of a trivial background. For the vacuum, $A = 0$, the ground state obviously has a threefold degeneracy given by the canonical dreibein $\hat{e}^a$ in isospace. The associated constant wave functions do not have any zeros. We therefore conjecture that Singer's obstruction [18] against complete gauge fixing is reflected in the nodes rather than in the degeneracy of the ground state. To completely settle this issue, a full topological classification of Higgs field zeros would clearly be helpful.

A natural next step will be to analyse higher instanton sectors and instanton-anti-instanton pairs. The existence of fermionic zero modes in the background of Hopf defects is currently being investigated (for related work see [51] and references therein). Such zero modes may in the end lead to a relation between confinement and chiral symmetry breaking. Finally, the dynamical role of Hopf defects in QCD has to be analysed.

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A. Spherical harmonics

In the following we list the eigenfunctions of $\vec{J}^2$ and $\vec{L}^2$ for the three cases of interest (suppressing the two magnetic quantum numbers labelling the vectors in each multiplet).
(i) For $(j,l) = (1,0)$ the spherical harmonics are given by the canonical dreibein $\hat{e}^a$ of constant unit vectors,

$$Y^{\mathrm{sg}}_{(1,0)} = Y^{\mathrm{reg}}_{(1,0)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{36}$$

(ii) For $(j,l) = (1/2,1/2)$ there are four eigenfunctions, all linear in $\hat{x}_\mu$,

$$Y^{\mathrm{sg}}_{(1/2,1/2)} = \left\{ \begin{pmatrix} \hat{x}_4 \\ \hat{x}_3 \\ \hat{x}_2 \end{pmatrix}, \begin{pmatrix} -\hat{x}_3 \\ \hat{x}_4 \\ -\hat{x}_1 \end{pmatrix}, \begin{pmatrix} -\hat{x}_2 \\ \hat{x}_1 \\ -\hat{x}_4 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \right\}, \tag{37}$$

$$Y^{\mathrm{reg}}_{(1/2,1/2)} = \left\{ \begin{pmatrix} -\hat{x}_4 \\ \hat{x}_3 \\ \hat{x}_2 \end{pmatrix}, \begin{pmatrix} -\hat{x}_3 \\ -\hat{x}_4 \\ -\hat{x}_1 \end{pmatrix}, \begin{pmatrix} -\hat{x}_2 \\ \hat{x}_1 \\ \hat{x}_4 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \right\}. \tag{38}$$

The following remarks are in order. Obviously, Y^{reg} is obtained from Y^{sg} upon exchanging $\hat{x}_4 \to -\hat{x}_4$. This is achieved via conjugation with h, $Y^{\mathrm{sg}}_{(j=1/2,\,m=1/2)} \sim h^\dagger \, Y^{\mathrm{reg}}_{(j=1/2,\,n=1/2)} \, h$. Note that the 'intertwining' gauge transformation h is only defined up to rotations around the direction of the Higgs field ϕ in isospace. It is convenient to combine the members of each (1/2, 1/2) quadruplet into a 'four-vector' Y_μ. Introducing the basis matrices $\sigma_\mu \equiv (i\sigma^a, \mathbb{1})$, one finds the relation $Y^{\mathrm{reg}}_\mu = \sigma_\mu Y^{\mathrm{sg}}_\mu \, \sigma^\dagger_\mu$ for any $\mu = 1,\ldots,4$. Any component $Y_\mu(\hat{x})$ vanishes, if $\hat{x}_\mu = \pm\hat{e}_\mu$, the $\hat{e}_\mu$ denoting the canonical basis of $\mathbb{R}^4$. This means that the zeros of the quadruplet eigenfunctions are given by two points located on a three-sphere with fixed radius r (see Fig. 3).
(iii) For the case $(j,l) = (0,1)$ one has three basic eigenfunctions, now bilinear in $\hat{x}_\mu$,

$$Y^{\mathrm{sg}}_{(0,1)} = \left\{ \begin{pmatrix} \hat{x}_1^2 - \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_1\hat{x}_2 + \hat{x}_3\hat{x}_4) \\ 2(\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_2 - \hat{x}_3\hat{x}_4) \\ -\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_2\hat{x}_3 + \hat{x}_1\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{x}_4) \\ -\hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 \end{pmatrix} \right\} \tag{39}$$

$$Y^{\rm reg}_{(0,1)} = \left\{ \begin{pmatrix} \hat{x}_1^2 - \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_1\hat{x}_2 - \hat{x}_3\hat{x}_4) \\ 2(\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_2 + \hat{x}_3\hat{x}_4) \\ -\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_2\hat{x}_3 + \hat{x}_1\hat{x}_4) \\ -\hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 \end{pmatrix} \right\} \quad (40)$$

Again, the two sets of eigenfunctions are related via $\hat{x}_4 \to -\hat{x}_4$ and can most easily be obtained from case (i) by conjugation with h,

$$Y^{\rm sg}_{(j=0,\,m=1)} = h^\dagger \, Y^{\rm reg}_{(j=1,\,n=0)} \, h, \qquad Y^{\rm reg}_{(j=0,\,n=1)} = h \, Y^{\rm sg}_{(j=1,\,m=0)} \, h^\dagger, \tag{41}$$

which, in particular, implies that they *never vanish*.

B. Feynman-Hellmann theorem and perturbation theory

In order to obtain information when $R \neq \rho$, we keep R fixed and vary ρ. We restrict ourselves to the singular gauge. The ρ-dependent part of (14) contains two terms,

$$V^{\rm sg}_{\rho\,(j,m)}(r) \equiv 4e^{-\alpha_R(r)} \left[\frac{\rho^2(\vec{J}^2 - \vec{M}^2)}{r^2(r^2+\rho^2)} - \frac{\vec{T}^2\rho^2}{(r^2+\rho^2)^2} \right]. \tag{42}$$

The ρ^2-dependence of the ground state energy is determined by the Feynman-Hellmann theorem,

$$\frac{\partial}{\partial\rho^2} E = \frac{\partial}{\partial\rho^2} \langle\phi|H|\phi\rangle = \langle\phi|\frac{\partial H}{\partial\rho^2}|\phi\rangle \equiv \langle\phi|\frac{\partial V_\rho}{\partial\rho^2}|\phi\rangle. \tag{43}$$

For the three angular momentum sectors of interest ($t = 1$) we have,

$$\begin{aligned} \frac{\partial V^{\rm sg}_{\rho\,(0,1)}(r)}{\partial\rho^2} &= \frac{(r^2+R^2)^2}{R^4} \frac{-4r^2}{(r^2+\rho^2)^3} < 0 \,, \\ \frac{\partial V^{\rm sg}_{\rho\,(1/2,1/2)}(r)}{\partial\rho^2} &= \frac{(r^2+R^2)^2}{R^4} \frac{2(\rho^2-r^2)}{(r^2+\rho^2)^3} \,, \\ \frac{\partial V^{\rm sg}_{\rho\,(1,0)}(r)}{\partial\rho^2} &= \frac{(r^2+R^2)^2}{R^4} \frac{4\rho^2}{(r^2+\rho^2)^3} > 0. \end{aligned} \tag{44}$$

According to (43), these functions have to be integrated with the positive factor $|\phi|^2\sqrt{g}$. Therefore, the ground state energies in the first and the third sector are monotonic in ρ^2, their slopes satisfying

$$\frac{\partial}{\partial\rho^2} E^{\rm sg}_{(0,1)} < 0 \,, \qquad \frac{\partial}{\partial\rho^2} E^{\rm sg}_{(1,0)} > 0 \,. \tag{45}$$

As the energies meet at $R = \rho$ ('level crossing') we conclude,

$$E^{\rm sg}_{(0,1)} < E^{\rm sg}_{(1,0)} \quad \text{for } R < \rho \,, \qquad E^{\rm sg}_{(0,1)} > E^{\rm sg}_{(1,0)} \quad \text{for } R > \rho \,. \tag{46}$$

This explains the behaviour of the full lines in Fig. 1.

For the sector $(1/2, 1/2)$ there is no such simple argument. Still, we can compute the slope of $E(\rho^2)$ at the point $\rho = R$ by simply inserting the known function ϕ. This amounts to ordinary perturbation theory in $\delta \equiv \rho^2 - R^2$,

$$H(\rho^2) = H(\delta = 0) + \delta \left.\frac{\partial H}{\partial \rho^2}\right|_{\delta=0} + O(\delta^2) = H_0 + H_{\mathrm{pert}} \ . \tag{47}$$

In this way we find a vanishing slope for the sector $(1/2, 1/2)$,

$$\left.\frac{\partial}{\partial \rho^2} E^{\mathrm{sg}}_{(1/2,1/2)}\right|_{\rho^2=R^2} \sim \int_0^\infty \frac{(1-r^2)}{(r^2+1)^7}\, r^5 \mathrm{d}r = 0 \ . \tag{48}$$

The lowest-lying state of this sector is thus pinched between the other two, at least for $R \approx \rho$ (cf. Fig. 1).

C. The MAG on the sphere

In [16] it has been shown that, due to their particular Lorentz and isospin structure, both A^{sg} and A^{reg} are in the MAG when defined on $\mathbb{R}^4$. This still holds true on S^4, where the gauge fixing functional has the values,

$$\begin{aligned} F_{\mathrm{MAG}}[A] &= \int \sum_{\bar{a}=1}^{2} A^{\bar{a}}_\mu A^{\bar{a}}_\nu \, g^{\mu\nu} \sqrt{g} \ \mathrm{d}^4 x \\ &= \frac{16\pi^2 R^4 [R^4 - 2R^2\rho^2 \ln(R^2/\rho^2) - \rho^4]}{\rho^2 (R^2 - \rho^2)^3} \times \begin{cases} 1 & \text{for } A^{\mathrm{sg}} \\ R^2/\rho^2 & \text{for } A^{\mathrm{reg}} \end{cases} \ . \end{aligned} \tag{49}$$

Obviously, $F_{\mathrm{MAG}}[A^{\mathrm{reg}}] = (R^2/\rho^2)\, F_{\mathrm{MAG}}[A^{\mathrm{sg}}]$, so that for $R > \rho$ ($R < \rho$) the singular (regular) gauge is singled out.

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