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Finite Size Effects from General Covariance and Weyl Anomaly

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Abstract

By exploiting the diffeomorphism invariance we relate the finite size effects of massless theories to their Weyl anomaly. We show that the universal contributions to the finite size effects are determined by certain coefficient functions in the heat kernel expansion of the related wave operators. For massless scalars confined in a 4-dimensional curved spacetime with boundary the relevant coefficients are given confirming the results of Moss and Dowker and also of Branson and Gilkey. We apply the general results to theories on bounded regions in two- and four-dimensional flat space-times and determine the change of the effective action under arbitrary conformal deformations of the regions.

Keywords: Schwinger model; finite temperature; Euclidean path integral; fermionic zero modes; Wilson loop; condensate; effective action; field theory, torus; two-point function

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1 Introduction

Finite size effects play an important role for critical systems having no intrinsic length scale except those dictated by the geometry. They are caused by the dependence of the vacuum or equilibrium state on the underlying space-time. For example, the energy-momentum of the vacuum depends on the boundary enclosing the systems and this leads to a measured Casimir force acting on the boundary. The geometry-dependence appears as anomalous contributions to the effective action which generates the correlation functions of the energy-momentum tensor. The anomalies are due to external gravitational fields and/or boundaries of space-time. Their consequences have been investigated in a wide range of theories like QED [1], QCD [2], Kaluza-Klein theories [3], 2-dimensional conformal field theories [4] and Stringtheories [5].

Actually the gravitational- and boundary anomalies are related by general covariance and this interrelation will be considerably exploited in our investigations of finite size effects. In what follows general covariance plays an essential role and thus we choose a manifestly covariant regularization, namely the zeta-function regularization which immediately connects to Schwingers proper time (heat kernel) expansion. This regularization scheme is convenient to extract the geometry dependence of expectation values and in particular the relation between the bulk- and surface terms. One only needs to separate the bulk- and surface contributions to the heat kernel expansion for the wave operators of interest. Recently much efforts have been made to work out the relevant expansion coefficients for various field theories and different boundary conditions [7, 8, 9, 10]. Our surface terms agree with the earlier results of Dowker and Moss [8] and Branson and Gilkey [9].

In this paper we study massless particles which are Weyl-invariantly coupled to gravity and which are confined in a finite space-time region. Although we are mainly concerned with finite size effects in flat Euklidean space-time it pays off to couple the particles to a gravitational field and only assume space-time to be euklidean at the end of the computations. This allows us to exploit the consequences of general covariance which relates the bulk- and surface terms. For example, the identification of the central charge of two-dimensional models (defined via the short distance behaviour of the energy-momentum correlators which is determined by the bulk term of the effective action) as Casimir effect (determined by the surface term of the effective action) follows immediately when one couples the system to gravity.

Our results apply to arbitrary massless particles interacting with the gravitational field and the boundary. Since for different particles the finite size effects are related if certain constants in the heat kernel expansion are adjusted accordingly we give the explicit results for scalar particles only, since they play a prominent role in the inflationary cosmological scenarios.

The paper is organized as follows. In section 2 the significance of the Weyl anomaly for finite size effects and the interplay between bulk and surface terms for conformal field theories is discussed. In particular we relate certain coefficients in the heat kernel expansion to the finite size effects. In the following section 3 these general results are applied to 2-dimensional systems and the response of the quantum system to arbitrary changes of the boundary is derived. In section 4 we outline the computation of the relevant heat kernel coefficients for 4-dimensional curved space-times with boundaries. We used Seeley's method [6] to determine these Seeley-deWitt coefficients. This project has been undertaken independently from [7, 8, 9], has not yet been published and was only privately communicated [10]. But the trilogy of papers [7, 8, 9] makes clear, that it is worth having several derivations of this important result obtained by different methods. In the last section 5 the general results are applied to 4-dimensional systems and the finite size effects for simple geometries are evaluated. In appendix A the notation and conventions used in the main body of the paper are explained and in appendix B all relevant Seeley-deWitt coefficients in 4 dimensions for scalar particles obeying Dirichlet boundary conditions are listed. The reader who is less interested in technical details may skip part of section 4 and take formula (83) as main result of this section.

2 Finite Size Effects From Weyl Anomaly

In this paper we shall investigate the change of field theoretical quantities under conformal transformations. A *conformal transformation* $f : \{\mathcal{M}, g\} \to \{\mathcal{N}, \tilde{g}\}$ is a map that preserves angles but not necessarily distances. The spacetimes $\mathcal{M}$ and $\mathcal{N}$ may possess boundaries $\partial\mathcal{M}$ and $\partial\mathcal{N}$ and for simplicity we shall assume that both are submanifolds of the same d-dimensional Lorentzian (Riemannian) spacetime X and their boundaries are hypersurfaces in X. Then g and $\tilde{g}$ are the metric of X restricted to $\mathcal{M}$ and $\mathcal{N}$, respectively, and in the following both are denoted by g. Since f leaves angles invariant the distance between neighbouring points can only change by a local scale factor. Choosing local coordinates on X, so that

$$f : \{\mathcal{M}, \partial\mathcal{M}\} \longrightarrow \{\mathcal{N}, \partial\mathcal{N}\}; \quad x^\mu \longrightarrow y^\mu = f^\mu(x) \tag{1}$$

this means that

$$g_{\mu\nu}(y)\,dy^\mu dy^\nu = e^{2\varphi(x)} g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{2}$$

where the local scale factor is determined by the metric and conformal transformation as

$$e^{2\varphi(x)} = \frac{1}{d}\, g_{\mu\nu}\big(y(x)\big) \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial y^\nu}{\partial x^\rho}\, g^{\sigma\rho}(x). \tag{3}$$

It is important to distinguish between conformal transformations and *diffeomorphism*. A map $f : \mathcal{M} \to \mathcal{N}$ is conformal with respect to prescribed geometries on $\mathcal{M}$ and $\mathcal{N}$ and the proper length may change by a (local) scale factor. Of course, we may also interpret such an f as diffeomorphism (or coordinate transformation), but then the metric is carried along, i.e.

$$\hat{g}_{\mu\nu} dy^\mu dy^\nu = g_{\mu\nu} dx^\mu dx^\nu \Longrightarrow \hat{g}_{\mu\nu}(y) = e^{-2\varphi} g_{\mu\nu}(y) \tag{4}$$

and differs from the prescribed metric on $\mathcal{N}$ by a Weyl factor. Thus, an arbitrary conformal transformation f is a composition of a diffeomorphism (defined by the same f) and a compensating Weyl transformation. Note in particular that the conformal group is not a subgroup of the diffeomorphism group since by diffeomophisms we mean maps (1) together with the associated transformations of the metric tensor, $g = f^*\hat{g}$, and matter fields. Also note that contrary to the diffeomorphism group which is always infinite dimensional, it may happen that there are no conformal maps from $\mathcal{M}$ to $\mathcal{N}$.

In Minkowski space-time the conformal transformations consist of translations, Lorentz transformations, dilatations

$$y^\mu = \lambda x^\mu \tag{5}$$

and special conformal transformations

$$y^\mu = \frac{x^\mu + x^2 b^\mu}{1 + 2b\cdot x + b^2 x^2} \tag{6}$$

and form a $SO(d,2)$ (in Euklidean space a $SO(d+1,1)$) group. The scale factor is one for the Poincare subgroup and it is

$$e^\varphi = \lambda, \qquad e^\varphi = \big(1 + 2b\cdot x + b^2 x^2\big)^{-1} \tag{7}$$

for the dilatations and special conformal transformations, respectively.

Let us now assume that a massless matter field $\Phi_{\mu\nu\dots}$ (which may be a spinor or tensor field) couples Weyl invariantly to the gravitational field, i.e. that the classical action is invariant under the Weyl transformations of the metric and matter fields

$$\big\{x^\mu, g_{\mu\nu}(x), \Phi_{\mu\nu\dots}(x)\big\} \longrightarrow \big\{x^\mu, e^{2\varphi(x)} g_{\mu\nu}(x), e^{\alpha\varphi(x)}\Phi_{\mu\nu\dots}(x)\big\}. \tag{8}$$

For example, a massless scalar has Weyl weight $\alpha = \frac{1}{2}(2-d)$, a spinor $\alpha = \frac{1}{2}(1-d)$ and a photon in $d = 4$ dimensions has $\alpha = 0$. For fermions one needs to introduce a

d-bein field. It is understood that the d-bein inherits its Weyl transformation from the metric field, i.e. $e^a_{\ \mu} \to e^{\varphi} e^a_{\ \mu}$.

Since a conformal transformation is a composition of a diffeomorphism and a compensating Weyl transformation a generally covariant and Weyl invariant classical field theory is automatically conformally invariant. We prefer to change the order and first Weyl transform and then act with the diffeomorphism f in order for the Weyl transformation to act on the original manifold $\mathcal{M}$ rather then on $\mathcal{N}$. Thus, given a conformal map (1) and the correponding Weyl factor (3), we first Weyl transform the metric and matter fields with this Weyl factor and then act with f, now interpreted as diffeomorphism. The net result is the conformal transformation

$$\big\{x^\mu, g_{\mu\nu}(x), \Phi_{\mu\nu\dots}(x)\big\} \longrightarrow \big\{y^\mu, g_{\mu\nu}(y), e^{\alpha\varphi}\Lambda_\mu^{\ \sigma}\Lambda_\nu^{\ \rho}\dots\Phi_{\sigma\rho\dots}(y)\big\}\,, \tag{9}$$

Since Weyl transformations and diffeomorphisms are both classical symmetries this shows that conformal maps (9) are indeed classical symmetries. For example, in Minkowski space-time (9) leaves $\eta_{\mu\nu}$ invariant and are the well-known conformal symmetry transformations of a Minkowskian field theory.

Note that under an infinitesimal conformal transformation

$$y^\mu = x^\mu - \epsilon X^\mu(x), \qquad X_{\mu;\nu} + X_{\nu;\mu} = \frac{2}{d}\, g_{\mu\nu}\nabla\cdot X \tag{10}$$

(X is a conformal Killing field for conformal transformations) the matter field transforms as

$$\Phi_{\mu\nu\dots} \longrightarrow \big(L_X - \frac{\alpha}{d}\nabla\cdot X\big)\,\Phi_{\mu\nu\dots}. \tag{11}$$

Which of these classical symmetries survive in the quantum theory depends on the chosen regularization. We shall use the manifestly covariant zeta-function regularization such that the theory is diffeomorpism invariant. However, it is well-known that classical Weyl invariance ceases to be a symmetry of a covariantly quantized theory if space time is curved and/or has boundaries. This implies then that the conformal invariance is broken as well. In particular the change of the effective quantum action under conformal transformations (1) is equals to the change under the corresponding Weyl transformation (8)

$$\delta\Gamma \equiv \Gamma[\mathcal{N}, g] - \Gamma[\mathcal{M}, g] = \Gamma[\mathcal{M}, e^{2\varphi}g] - \Gamma[\mathcal{M}, g] \tag{12}$$

if φ is related to the conformal transformation (1) by (3).

The variation of Γ under Weyl transformations is determined by the Weyl anomaly (or trace anomaly of the energy-momentum tensor). This anomaly is local in the curvature of space-time and its covariant derivatives and in the extrinsic and intrinsic curvature of the boundary. Is is determined by the t-independent term in the expansion of the heat kernel $K(t, x)$ of the relevant wave operator. Thus we may compute the change of Γ under conformal changes of $\mathcal{M}$ from the heat kernel expansion alone.

To be more specific we consider bosonic and fermionic theories with classical actions

$$S_B = \int\limits_{\mathcal{M}} \sqrt{g}\ \Phi\, A(g)\, \Phi, \quad \text{and} \quad S_F = \int\limits_{\mathcal{M}} \sqrt{g}\ \Phi\, D(e)\, \Phi, \tag{13}$$

where $A(g)$ and $D(e)$ are second and first order differential operators (e.g. $A(g) = -\Delta_g + \xi\,\mathcal{R}$ for scalars and $D(e) = \not{D}$ for Dirac fermions). The Weyl invariance (8) then requires that the wave operators transform as follows under Weyl transformations of the metric and d-bein:

$$A(e^{2\varphi} g_{\mu\nu}) = e^{-(\alpha+d)\varphi} A(g_{\mu\nu}) e^{-\alpha\varphi} \quad \text{and} \quad D(e^{\varphi} e^{a}_{\ \mu}) = e^{-(\alpha+d)\varphi} D(e^{a}_{\ \mu}) e^{-\alpha\varphi}. \tag{14}$$

Also we assume the matter fields to obey some conformally invariant boundary conditions on $\partial\mathcal{M}$.

According to (12) the response of the effective action

$$\Gamma[g] = -\log \int \mathcal{D}\Phi\, e^{-S[\Phi,g]} = \pm\frac{1}{2} \log\det\ A \tag{15}$$

(the plus sign holds for bosons and the minus sign for fermions for which $A = D^2$) to a conformal deformation of $\mathcal{M}$ is equals to the difference $\Gamma[e^{2\varphi} g] - \Gamma[g]$. To determine this difference one introduces the one-parameter family of metrics

$$g^{\tau}_{\mu\nu} = e^{2\tau\varphi} g_{\mu\nu} \tag{16}$$

which interpolates between the two metrics, and determines the τ-variation of the zeta-function regularized determinants

$$\log\det A(g^{\tau}) = -\frac{d}{ds}\zeta(\tau, s)|_{s=0}, \qquad \text{where} \quad \zeta(\tau, s) = \sum \lambda_n^{-s}(\tau). \tag{17}$$

In what follows we shall assume that all eigenmodes Φ_n of $A(g^{\tau})$ have positive eigenvalues $\lambda_n(\tau)$ for $0 \le \tau \le 1$. Clearly, the difference of the effective actions (for bosons) is now given by

$$\Gamma[\mathcal{M}, e^{2\varphi} g] - \Gamma[\mathcal{M}, g] = -\frac{1}{2} \int\limits_0^1 d\tau\, \frac{d}{ds}\frac{d}{d\tau}\zeta(\tau, s)|_{s=0}, \tag{18}$$

and similarly for fermions. That (17) indeed regularizes the determinants, i.e. the zeta-function is smooth at the origin, and that the τ-variation of Γ can be computed from the heat kernel expansion can be seen as follows:

1. First one rewrites the sum defining the zeta-function in (17) as a Mellin transform of the heat kernel as

$$\zeta(\tau, s) = \frac{1}{\Gamma(s)} \int\limits_0^{\infty} dt\, t^{s-1} \mathrm{Tr}\, e^{-tA(g^{\tau})}. \tag{19}$$

The τ-derivative of (19) is obtained if the Hellman-Feynman theorem

$$\frac{d}{d\tau}\lambda_n = \Big(\Phi_n, \frac{d}{d\tau}A(g^\tau)\Phi_n\Big)$$

and (14) with φ replaced by $\tau\varphi$ are used, resulting in a factor $s(2\alpha+d)$ and an insertion of a Weyl angle φ into the trace of (19).

2. For sufficiently smooth manifolds the curvature scalar is bounded and the wave operators of interest are self-adjoint (e.g. $(-\Delta_\tau + \xi\mathcal{R}_\tau)$ is self-adjoint due to a theorem of Kato and Rellich [11]). Using ordinary spectral theory one shows that

$$e^{-tA(g^\tau)} = -\frac{1}{2\pi i}\int\limits_{i\infty}^{-i\infty} e^{-t\lambda}\big(A(g^\tau)-\lambda\big)^{-1}d\lambda\ , \tag{20}$$

where the contour encloses the spectrum at infinity. The representation (20) will be the starting point for our explicit calculations of the heat kernel expansion in section 4.

3. For the eigenvalues, there is an estimate [12]

$$\lambda_n > Cn^\delta\ , \quad C>0\ ,\ \delta>0\ , \tag{21}$$

valid, because the operators under consideration are not only selfadjoint but also elliptic, a property explained below. Hence $\int_1^\infty dt\, t^{s-1}\mathrm{Tr}\,\exp(-tA(g^\tau))$ is an entire function of s. This suggests to split the integration region in (19) into $[0,1]$ and $[1,\infty]$. In the limit $s\to 0$ the second integral and its τ-derivative vanish, and we are left with the first integral.

4. In order to evaluate the first integral we construct the heat kernel in the limit $t\to 0^+$ up to regular parts. For the system $(A,\mathcal{M},\partial\mathcal{M})$ it is known that

$$\mathrm{Tr}\ e^{-tA(g^\tau)}\varphi\ \sim \frac{1}{t^{\frac{d}{2}}}\sum_{n=0}^{\infty}t^{n/2}\Big[\int\limits_{\mathcal{M}}\sqrt{g^\tau}\,a_{\frac{n}{2}}(\varphi,g^\tau_{\mu\nu}) + \int\limits_{\partial\mathcal{M}}\sqrt{\tilde g^\tau}\,b_{\frac{n}{2}}(\varphi,g^\tau_{\mu\nu})\Big], \tag{22}$$

where $\tilde g^\tau$ is the determinant of the metric on $\partial\mathcal{M}$ induced by $g^\tau_{\mu\nu}$. For constant φ the Seeley-deWitt coefficients $a_{\frac{n}{2}}$ are local polynomials in the curvature and its covariant derivatives and the $b_{\frac{n}{2}}$ are local polynomials in the intrinsic and extrinsic curvatures of the boundaries. The a's vanish for odd n and have dimensions (length)$^{-n}$. The b's have dimensions (length)$^{1-n}$.

Using this expansion the τ-derivative of the ζ-function is defined for $2s>d$ and can be analytically continued, apart from poles at $s=d/2\ ,(d-1)/2\ ,...,$ to all s. In the limit $s \to 0$ the pole at $s = 0$ combines with the asymptotic behaviour of $s/\Gamma(s) \sim s^2$ such that

$$\delta\Gamma = -\Big(\alpha + \frac{d}{2}\Big) \int\limits_0^1 d\tau \,\Big(\int\limits_{\mathcal{M}} \sqrt{g^\tau}\, a_{\frac{d}{2}}[\varphi, g^\tau_{\mu\nu}] + \int\limits_{\partial\mathcal{M}} \sqrt{\tilde{g}^\tau}\, b_{\frac{d}{2}}[\varphi, g^\tau_{\mu\nu}]\Big), \tag{23}$$

and this formula for the change of the quantum action will be used in the following. Of course, in general Γ depends on the chosen renormalization conditions. The possible counterterms are just the lower Seeley-deWitt coefficient-functions and their coefficients are determined by these renormalization conditions. But contrary to the universal (scheme-independent) result (23) these additional ambigues terms are not universal.

The corresponding formula for fermions is obtained similarly and one obtains -2 times the expression on the right hand side of (23) where $a_{\frac{d}{2}}$ and $b_{\frac{d}{2}}$ are now the Seeley-deWitt coefficients of $D^2(e)$.

We see that the variation of the free energy under Weyl transformations, and thus under conformal transformations, is determined by the t-independent terms in the small t expansion (22) of the (weighted) heat kernel. Since the Seeley-deWitt coefficients are computed iteratively the calculation of the relevant coefficients becomes rather involved in four or more dimensions. So far the coefficients $a_n,\ n \le 4$, which are of interest in 8 or less dimensions, have been determined [12, 13]. The boundary dependent b-terms are especially difficult to compute and only the $b_n,\ n \le 2$ are known [9, 10].

Note that (23) automatically yields a separation of $\delta\Gamma$ into its bulk and surface contributions. As we shall see later the individual bulk and surface terms are not invariant under general coordinate transformations, only their sum is invariant. This implies that they are not independent, and that the bulk terms partly determine the surface terms. Before computing the relevant b_2-term in four dimensions, we first investigate the consequences of (12) and (23) in two dimensions.

3 Finite Size Effects in 2 Dimensions

In recent years the postulate of conformal invariance for critical models made it possible to identify them quite successfully with 2-dimensional Euklidean conformal field theories [14]. Such theories are characterized by the central charge c which is determined by the singular part of the operator product expansion of the energy momentum tensor. In two dimensions (and for topologically trivial regions) one can always find coordinates for which the metric is conformally flat, $g_{\mu\nu} = e^{2\varphi}\delta_{\mu\nu}$ and thus (23) allows one to calculate the metric-dependence of the effective action for arbitrary 2-dimensional space-times. Thus in two dimension $\Gamma[g_{\mu\nu}] - \Gamma[\delta_{\mu\nu}]$ is completely determined by the Seeley-deWitt coefficients a_1 and b_1 (in other regularizations as the one chosen here $a_{\frac{1}{2}}$, $b_{\frac{1}{2}}$ and a_0 may be needed as counterterms leading to extra non-universal terms in Γ). Since vacuum expectation values of products of the energy momentum tensor can be computed from the effective action as

$$\langle T_{\mu\nu}(x_1)\cdots T_{\alpha\beta}(x_n)\rangle = \frac{2^n}{\sqrt{g(x_1)\cdots g(x_n)}}\frac{\delta^n\Gamma[g]}{\delta g^{\mu\nu}(x_1)\cdots\delta g^{\alpha\beta}(x_n)}, \tag{24}$$

they are determined by the volume part of the effective action alone, and thus by the coefficient a_1. In particular the central charge is determined by this coefficient.

Let us now apply the result (12) to a region in flat Euklidean space-time for which $g_{\mu\nu} = \delta_{\mu\nu}$ in (12). In two dimensions the (global) conformal group $SO(2,2)$ introduced in the previous section is only a small subgroup of all conformal transformations of the Euklidean plane since all analytic point transformations

$$w = w(z) \quad \text{and} \quad \bar w = \bar w(\bar z), \qquad z = x^0 + ix^1,\ w = y^0 + iy^1 \tag{25}$$

are conformal, and thus the result (12) applies to all of them with

$$e^{2\varphi(z,\bar z)} = \frac{dw}{dz}\frac{d\bar w}{d\bar z}\,. \tag{26}$$

According to the Riemann mapping theorem [15] any region with smooth boundary (and without hole) can be mapped into the unit disk by an analytic transformation. Thus the formula (12) determines the effective actions for arbitrary shaped regions relative to the effective action for the unit disk. For example, for $A(g) = -\Delta_g$ and Dirichlet boundary conditions the coefficients a_1 and b_1 are given in appendix B and the general formula (12) together with (25,26) yield

$$\begin{aligned}\Gamma[\mathcal{N},\delta_{\mu\nu}] - \Gamma[\mathcal{M},\delta_{\mu\nu}] = &-\frac{i}{48\pi}\int\limits_{\mathcal{M}}\Big(\frac{d}{dz}\log\frac{dw}{dz}\Big)\Big(\frac{d}{d\bar z}\log\frac{d\bar w}{d\bar z}\Big)dz d\bar z\\ &-\frac{i}{48\pi}\oint\limits_{\partial\mathcal{M}}\Big(\frac{d}{d\sigma}\log\frac{z'}{\bar z'}\Big)\log\Big(\frac{dw}{dz}\frac{d\bar w}{d\bar z}\Big)d\sigma\,,\end{aligned} \tag{27}$$

where $\mathcal{M}$ is mapped into $\mathcal{N}$ by the conformal transformation $w = w(z)$. We have used that for scalars $\alpha + d/2 = 1$. The line-integral along the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ contains the derivative $z' = \partial_\sigma z$ of the parametrised boundary curve $z(\sigma)$ with respect to the curve parameter σ. Note that on the Euklidean plane only the surface term in (23) contributes to the effective action since a_1 vanishes, and indeed the first term on the right hand side of (27) can be converted into a boundary term.

If one considers dilatations, the finite size effects are independent of the shape of the boundary, simply given by the Euler number χ^E of the manifold

$$\Gamma[\lambda\mathcal{M}] - \Gamma[\mathcal{M}] = -\frac{1}{6}\chi^E\log\lambda, \qquad \chi^E = \frac{1}{2\pi}\int\limits_{\partial\mathcal{M}}\mathcal{T}r\chi. \tag{28}$$

Here $\mathcal{T}r\chi$ is the trace of the extrinsic version of the second fundamental form (see appendix A) and the sign in the definition of χ^E is chosen to be positive for a sphere.

Let us now see how the bulk term $\int a_1$ (which determines the $T_{\mu\nu}$ correlators) and the surface term $\oint b_1$ (which determines the finites size effects) are related. For that one observes from the explicit expressions (107) and (110), that for a non-zero $\mathcal{R}$ both terms are not seperately invariant under the transformations (25) taken as diffeomorphism so that

$$\hat{\varphi}(w,\bar{w}) = \varphi(z(w),\bar{z}(\bar{w})) + \frac{1}{2}\log\Big(\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\Big).$$

Only the sum of the bulk and the surface term is invariant (up to φ-independent terms, reflecting the non-invariance of $\Gamma[\mathcal{M},\delta_{\mu\nu}]$ under the transformations (25) and leading to finite size effects in flat space-time) and this fixes the relative normalization of a_1 and b_1. Thus the correlators (24) and finite size effects (27) are very much related. This relation can only be seen when the scalar field is coupled to a non-trivial background metric. More generally, for an arbitrary conformal field theory the coefficient a_1 must be a local, dimension 2 object which is a scalar for $\varphi = 1$. The only such object is the Ricci scalar, so that a_1 must have the form (107), up to a constant factor c, and c is the central charge. From general covariance we conclude that b_1 must have the form (110) times the same constant c. It follows that the central charge define via the short distance expansion of $\langle T_{\mu\nu}(x_1)T_{\alpha\beta}(x_2)\rangle$ reappears in the formulae (27,28). In particular [16]

$$\Gamma[\lambda\mathcal{M}] - \Gamma[\mathcal{M}] = -\frac{c}{6}\,\chi^E\log\lambda, \tag{29}$$

for a conformal field theory with central charge c. Stricly speaking the volume terms do not determine the surface terms uniquely. But the ambiguous surface terms must be scalars under analytic coordinate transformations. The only ambiguous term in 2 dimensions is $\int_{\mathcal{M}} \Delta\varphi$, and such a term does not contribute on flat space-time.

4 Heat Kernel Expansion for Manifolds with Smooth Boundaries

In this section we outline the method used to calculate b_2. More extensive expositions may be found in [6] and [13, 17, 18, 19, 20]. In the previous sections we have seen that the singular structure of the heat kernel trace for small parameters t is required. Starting from (20), the techniques of pseudodifferential operators can be employed to investigate this singular behaviour. In this formalism two operators are identified if they possess the same singularity structure. In particular the inverse operator appearing in (20) is constructed up to smooth parts.

More precisely, an equivalence relation will be defined

$$A \sim B \quad \Leftrightarrow \quad A - B : D \to \mathcal{C}^\infty \ , \tag{30}$$

where the function space D will be specified later on. Operators like $A - B$ do not produce singularities and are viewed as being neglegible. The procedure reminds of Lebesgue theory, where all results are valid up to sets of measure zero.
In a first step we consider R^d instead of the general case $(\mathcal{M}, \partial\mathcal{M})$. Let $A(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$ be a differential operator with C^∞-coefficient functions a_α. We use the multi-index notations

$$\alpha = (\alpha_1, ..., \alpha_d)\,, \quad |\alpha| = \sum_{i=1}^{d} \alpha_i\,, \quad \alpha! = \alpha_1!.......\alpha_d!\,,$$
$$D_x^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial}{\partial_{x_1}^{\alpha_1}} ..... \frac{\partial}{\partial_{x_d}^{\alpha_d}} \ . \tag{31}$$

With A we associate a polynomial, called symbol, replacing the derivatives in x by the momentum $p \in R^d$

$$\sigma_A(x, p) = \sum_{|\alpha|\leq m} a_\alpha(x) p^\alpha \quad , \quad \sigma_A^m = \sum_{|\alpha|=m} a_\alpha(x) p^\alpha \ . \tag{32}$$

σ_A^m is called leading symbol and if it is different from zero for all non-zero p the operator is called elliptic. We recover the differential operator A from its symbol σ_A by Fouriertransformation

$$Au(x) = \int e^{i\langle x,p\rangle} \sigma_A(x, p) \hat{u}(p) dp, \qquad \hat{u}(p) = \frac{1}{(2\pi)^d} \int e^{-i\langle x,p\rangle} u(x) dx \ . \tag{33}$$

The calculations simplify considerably when these polynomials are used instead of the corresponding operators. The prize we pay is that we must introduce equivalence classes of operators to recover, for example, the inverse operator from the inverted symbol. The reason is that the inverted symbol usually has singularities which must be regularized by introducing a cut off function. In addition, to derive (22) from (20), we must scale λ as

$$\lambda \to \ -\frac{i\lambda}{t} \ , \tag{34}$$

so that t appears in the inverse $(A(g) + i\lambda/t)^{-1}$ and hence in the inverted symbol. Unfortunately the latter lacks a homogeneity property and t can not be removed from it. But it turns out that we can find a sequence of homogeneous symbols approaching the exact inverse in the sense of (30), if the function space D is the Schwartz class supplied with the Sobolev norm, denoted by $\mathcal{H}^s, s \in R$,

$$\|u\|_s = \left(\int \hat{u}(p)(1 + |p|^2)^s dp \right)^{\frac{1}{2}} \ .$$

These spaces have the following properties:

1. $D_x^\alpha : \mathcal{H}_s \to \mathcal{H}_{s-|\alpha|}$ is continuous.

2. Let d be the dimension of space. For $s > d/2$

$$
\begin{aligned}
&u \in \mathcal{H}_{s+k} \Rightarrow u \in \mathcal{C}^k \quad \text{(Sobolev-Lemma)} \\
&u|_{\mathcal{M}} \to u|_{\partial\mathcal{M}} \quad \text{is continuous and surjectiv .}
\end{aligned}
$$

To treat the inverse $(A-\lambda)^{-1}$ let us introduce the notion of parameter dependent symbols. $\sigma \in \mathcal{S}^m$ if

$$
\begin{aligned}
i)&\ \ \sigma(x,p,\lambda) \text{ is holomorphic in } \lambda\,, \\
ii)&\ \ |D_x^\alpha D_p^\beta D_\lambda^\gamma \sigma| \le C_{\alpha\beta\gamma}(1+|p|+|\lambda|^{1/2})^{m-|\beta|-2\gamma}\ .
\end{aligned}
\tag{35}
$$

We then establish the following equivalence relation on symbols

$$
\begin{aligned}
\sigma \sim \sigma' \Leftrightarrow \sigma - \sigma' \in \mathcal{S}^{-\infty} \quad &, \quad \mathcal{S}^{-\infty} := \bigcap_{m\in\mathcal{R}} \mathcal{S}^m \\
\Leftrightarrow A - A' : \mathcal{H}_s \to C^\infty &\text{ for all } \ s\ .
\end{aligned}
\tag{36}
$$

The next step is to define a symbol product

$$
\sigma(AQ) = \sum_\alpha \frac{1}{\alpha!}\partial_p^\alpha \sigma_A\, D_x^\alpha \sigma_Q\ . \tag{37}
$$

Now we are ready to approximate, in the sense of (30), the inverse A^{-1} of A, ($\sigma_A \in \mathcal{S}^m$) by an operator Q. The symbol q of Q is obtained as follows:

1. Using truncations one proves the existence of a q

$$
\sigma_A q \sim q\sigma_A \sim 1 \ \text{ and } \ q \in \mathcal{S}^{-m}\ . \tag{38}
$$

2. For q one makes an ansatz

$$
q = q_{-m} + q_{-m-1} + \ldots\ldots \quad , \quad q_i \in \mathcal{S}^{-i} \tag{39}
$$

   where the sum on the right hand side uniquely determines a symbol.

3. Calculate the q_i iteratively from (38,39) by using (37).

So far we haven't made any particular choice for the wave operator A. In what follows we shall consider scalar particles for which this operator has the form

$$
A = -\Delta + \xi\mathcal{R}. \tag{40}
$$

Let us now apply the above algorithm to scalar particles for which $\sigma_{A+i\lambda/t}$ is the second degree polynomial

$$\sigma_{A+\frac{i\lambda}{t}} = a_2 + a_1 + a_0 \quad , \quad a_i \in \mathcal{S}^i$$
$$a_2 = \ g^{\alpha\beta} p_\alpha p_\beta + \frac{i\lambda}{t}, \qquad a_1 = \ i g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} p_\gamma, \qquad a_0 = \ \xi\, \mathcal{R}\ . \tag{41}$$

The a_i are homogeneous in the momentum p and $\sqrt{\lambda}$. It is of main importance to include the λ dependent part into a_2. The ansatz for the symbol of the approximating inverse reads

$$q \sim q_{-2} + q_{-3} + q_{-4} + \dots\,, \qquad q_{-i} \in \mathcal{S}^{-i}\ . \tag{42}$$

Using (38,39) one finds

$$q_{-2} = \frac{1}{a_2}, \tag{43}$$

which is nowhere singular and homogeneous in p and $\sqrt{\lambda}$. Having made this choice the product of (41) with (42) yields (1+ lower order terms). The lower order terms are grouped together, each group belonging to S^{-l} for some l, and then separately set equal to zero

$$a_2\, q_{-2-l} + \sum_{j<l} \frac{1}{\alpha!} \partial^\alpha_p a_k D^\alpha_x q_{-2-j} = 0 \qquad |\alpha| + 2 + j - k = l \quad \forall\, l \geq 0\ . \tag{44}$$

This algebraic system of equations must be solved for the q's, which are easily seen to have the homogeneity property

$$q_{-2-l}\,(x, p, i\lambda/t) = t^{1+l/2}\, q_{-2-l}(x, \sqrt{t}\, p, i\lambda)\ . \tag{45}$$

By this procedure it is guaranteed, that after infinitely many steps the remaining contribution is in $\mathcal{S}^{-\infty}$. Because of homogeneity the substitution

$$p \to \sqrt{t}\, p \tag{46}$$

allows us to factorize the t-dependence in all integrals to be performed.

As explained in section 2, we are need to determine the heat kernel trace with the insertions of a conformal angle φ. Denoting as usual by $\hat{\varphi}$ the Fourier transformed angle we arrive at

$$\begin{aligned} \exp[-tA(g)]\varphi(x) &= \int_U dz \langle x| e^{-tA(g)} |z\rangle \langle z|\varphi\rangle \\ &\sim \frac{1}{2\pi} \sum_l \int d\lambda \int dp\, e^{i\lambda} e^{i\langle x,p\rangle} q_{-2-l}\, \hat{\varphi}(p)\ . \end{aligned} \tag{47}$$

The integrals are absolutely convergent, and we may interchange the order of integration. Therefore

$$\langle x| e^{-tA(g)} \varphi |x\rangle \sim t^{-\frac{d}{2}} \sum_l a_{\frac{l}{2}}[\varphi, g_{\mu\nu}]\, t^{\frac{l}{2}} \tag{48}$$

with

$$a_{\frac{l}{2}}[\varphi, g_{\mu\nu}] = \frac{1}{(2\pi)^{d+1}} \int\limits_{\mathcal{R}^d} dp \int\limits_{-\infty}^{\infty} d\lambda\, e^{i\lambda} q_{-2-l}\, \varphi \,. \tag{49}$$

The notation $a_{\frac{l}{2}}[\varphi, g_{\mu\nu}]$ is used in accordance with the existing literature on the subject.

To generalize from R^d to $\mathcal{M}$ the following remarks are in order: Using a decomposition of one on $\mathcal{M}$, compatible with an atlas, one can derive the results (48,49) in each chart. It follows that the trace in (22) is a sum of the contributions from the different charts covering space-time. It can be shown that the result is independent of the chosen atlas.

We now turn to the general case with boundary. Near the boundary we introduce geodesic coordinates $x^\alpha = (r, x^a)$, $\alpha, \beta, .. = 0, 1, 2, \ldots, d-1$, $a, b, .. = 1, 2, \ldots, d-1$. x^a is the point on the boundary minimizing the geodesic distance to x^α and r is the geodesic distance. Also we use Riemann normal coordinates on the boundary. This way we may identify a neighbourhood of a point on the boundary with a region in (R, R^{d-1}), the boundary being given by $r = 0$. Points in $\mathcal{M}$ have then positive r and those in $X \setminus M$ have negative r. Again using a decomposition of one we may even assume this region to be (R, R^{d-1}). In these coordinates the metric has the form

$$g_{rr} = g^{rr} = 1, \qquad g_{ra} = g^{ra} = 0, \qquad ds^2 = dr^2 + g_{ab}(r, x)\, dx^a dx^b \tag{50}$$

and then the intrinsic version of the second fundamental form (see appendix A) simplifies to

$$\mathcal{K}_{ab} = \Gamma^r_{ab} = -\frac{1}{2} g_{ab,r}.$$

At the origin we have

$$g_{ab}|_0 = \delta_{ab}, \qquad \Gamma^c_{ab}|_0 = 0, \qquad \mathcal{T}r\mathcal{K}|_0 = \mathcal{K}_{aa}|_0 \,. \tag{51}$$

In what follows we denote the restriction of g_{ab}, $g_{ab,r}$ etc. to $\partial\mathcal{M}$ as

$$g_{ab}|_{r=0} =: \tilde{g}_{ab}, \qquad g_{ab,r}|_{r=0} =: \tilde{g}_{ab,r} \qquad \text{etc.}$$

Now we would like to generalize (48,49) when boundaries are present. We supplement (40) with the conformally invariant Dirichlet boundary conditions

$$\Phi|_{\partial\mathcal{M}} = 0 \tag{52}$$

for Φ in some Sobolev space. It follows that

$$(A + i\lambda)^{-1}\Phi|_{\partial\mathcal{M}} = 0 \,. \tag{53}$$

Now we define restriction and extension operators $\hat{r}$,$\hat{e}$ as follows: For Φ defined on $\mathcal{M}$

$$\hat{e}\Phi := \begin{cases} \Phi & \text{if } \ r \geq 0 \,, \\ 0 & \text{if } \ r < 0 \,, \end{cases} \tag{54}$$

and $\hat{r}$ restricts functions on X to $\mathcal{M}$. Assume that we have already constructed an approximative inverse Q of $A + i\lambda$ on X (as explained above). Its restriction $\hat{r}Q\hat{e}$ does not obey to the boundary condition and hence

$$G_\lambda \ := \left(\hat{r}Q\hat{e} - \frac{1}{A+i\lambda} \right) . \tag{55}$$

does not vanish in the sense of (30). However we can use the previous results for the boundary-less case to calculate this correction. It is determined by

$$(A+i\lambda)G_\lambda = \ 0 \qquad \text{and} \qquad G_\lambda|_{\partial\mathcal{M}} = \ \hat{r}Q\hat{e}\ |_{\partial\mathcal{M}} . \tag{56}$$

It can be shown, that G_λ is only relevant near the boundary [14], because if a, b are truncations having arbitrarily small support around $\partial\mathcal{M}$ then the last two terms in

$$G_\lambda = aG_\lambda b + (1-a)G_\lambda + aG_\lambda(1-b) . \tag{57}$$

have C^∞ kernels and are neglegible. Thus the system (56) need only be solved on (R^{d-1}, R^+) near $r = 0$. In addition to (56) one must demand that

$$G_\lambda|_{r\to\infty} = 0 \tag{58}$$

which is one of the reasons why we can ignore the finite sizes of charts [6].

To handle the general case we introduce boundary symbols $\tilde{\sigma}$. They are polynomials in p^a but remain differential operators in r. For scalar particles

$$A + \frac{i\lambda}{t} = -\partial_r^2 - \ \frac{1}{2} g^{ab} g_{ab,r}\partial_r - g^{ab}\partial_a\partial_b + g^{ab}\Gamma^c_{ab}\partial_c + \alpha\mathcal{R} + \frac{i\lambda}{t} \tag{59}$$

has the full symbol

$$\begin{aligned} &\sigma_{A+\frac{i\lambda}{t}} = a_2 + a_1 + a_0 \\ &a_2 = \tau^2 + \varrho^2, \qquad a_1 = -\ \frac{i}{2} g^{ab} g_{ab,r}\tau + i g^{ab}\Gamma^c_{ab} p_c \qquad a_0 = \xi\,\mathcal{R} . \end{aligned} \tag{60}$$

where we have introduced

$$\varrho^2 := g^{ab} p_a p_b + \frac{i\lambda}{t} \tag{61}$$

and (τ, p^a) are the momenta conjugate to (r, x^a). Because of (57) it is natural to expand the boundary symbol around $r = 0$.

$$\begin{aligned} \tilde{\sigma}(x, r, p, D_r, \lambda) &= \sum_{k,l} r^k \partial_r^k a_l(x, 0, p, D_r) = \sum_l \tilde{a}^{(l)} \\ \tilde{a}^{(l)} &= \sum_{j-k=l} r^k \partial_r^k a_j / k! . \end{aligned} \tag{62}$$

In (62) terms of equal homogeneity are grouped together. $\tilde{a}^{(l)}$ has homogeneity $-(m+l)$ in $(1/r, p, D_r, \lambda^{1/m})$, if $\tilde{\sigma}$ is a symbol of order $(-m)$. For scalar particles the boundary symbols read explicitly:

$$\begin{aligned}
\tilde{a}^{(2)} &= -\partial_r^2 + \tilde{g}^{ab} p_a p_b + i\lambda \\
\tilde{a}^{(1)} &= r\tilde{g}^{ab}_{,r} p_a p_b - \frac{1}{2}\tilde{g}^{ab}\tilde{g}_{ab,r}\partial_r + i\tilde{g}^{ab}\tilde{\Gamma}^c_{\ ab} p_c \\
\tilde{a}^{(0)} &= \frac{1}{2}r^2\tilde{g}^{ab}_{,rr} p_a p_b - \frac{1}{2}r\tilde{g}^{ab}_{,r}\tilde{g}_{ab,r}\partial_r + \tilde{g}^{ab}\tilde{g}_{ab,rr}\partial_r \\
&\quad + ir(\tilde{g}^{ab}_{,r}\tilde{\Gamma}^c_{\ ab} + \tilde{g}^{ab}\tilde{\Gamma}^c_{\ ab,r})p_c + \sigma\xi\tilde{\mathcal{R}} \\
\tilde{a}^{(-1)} &= \frac{1}{6}r^3\tilde{g}_{ab,rrr} p_a p_b - \frac{1}{4}r^2\tilde{g}^{ab}_{,rr}\tilde{g}_{ab,r}\partial_r - \frac{1}{4}r^2\tilde{g}^{ab}\tilde{g}_{ab,rrr}\partial_r \\
&\quad - \frac{1}{2}r^2\tilde{g}^{ab}_{,r}\tilde{g}_{ab,rr}\partial_r + i\frac{1}{2}r^2\tilde{g}^{ab}_{,rr}\tilde{\Gamma}^c_{\ ab} p_c + ir^2\tilde{g}^{ab}_{,r}\tilde{\Gamma}^c_{\ ab,r} p_c \\
&\quad + \frac{i}{2}\,\tilde{g}^{ab}\tilde{\Gamma}^c_{\ ab,rr} p_c + r\sigma\xi\tilde{\mathcal{R}}_{,r} \, .
\end{aligned} \tag{63}$$

For the symbol of G_λ the ansatz reads

$$\sigma_{G_\lambda} \sim \sum d_{-2-l}$$

leading with (56) to

$$\tilde{\sigma}_{A+\frac{i\lambda}{t}} \sum d_{-2-l} = 0 \tag{64}$$

and to the boundary conditions

$$d_{-2-l}|_{r=0} = q_{-2-l}|_{r=0} \qquad \text{and} \qquad d_{-2-l} \to 0 \quad \text{for} \quad r \to \infty \, . \tag{65}$$

Using (62) the condition (64) can be rewritten as

$$\tilde{a}^{(2)} d_{-2-l} + \sum_{j<l} \frac{1}{\alpha!}\partial_p^\alpha \tilde{a}^{(k)} D_x^\alpha d_{-2-j} = 0, \qquad |\alpha| + 2 + j - k = l \qquad \forall\, l \geq 0 \, , \tag{66}$$

which is a system of ordinary differential equations for the d's. The solutions, subject to the boundary conditions (65), are now inserted into

$$\frac{1}{2\pi}\sum_l \int_{-\infty}^{\infty} d\lambda \int_{R^{d-1}} dp \int_{-\infty}^{\infty} d\tau\, e^{i\lambda} e^{i\langle x,p\rangle} d_{-2-l}\, \hat{\varphi}(p) \, . \tag{67}$$

There is no exponential factor related to the τ-integration because the symbols are taken at $r = 0$. Inserting

$$\hat{\varphi} = (2\pi)^{-d} \int dz ds\, e^{-i\langle s,\tau\rangle} e^{-i\langle z,p\rangle} \varphi$$

an exponential factor $e^{-i\langle r,\tau\rangle}$ reappears in the limit $x = z$, $r = s$. Scaling the variables according to

$$p' = \sqrt{t}\, p, \qquad \tau' = \sqrt{t}\, \tau \qquad r' = \frac{1}{\sqrt{t}} r \tag{68}$$

and expanding the conformal angle into a Taylor series

$$\varphi(x, t^{\frac{1}{2}} r) = \sum_{0}^{\infty} \frac{r^n}{n!} \partial_r^n \varphi \Big|_{r=0} t^{\frac{n}{2}}$$

the t-dependence factorizes again in the occuring integrals. The result is then

$$G_\lambda \sim t^{-\frac{d}{2}} \sum_{l} b_{\frac{l}{2}}[\varphi, g_{\mu\nu}]\, t^{\frac{l}{2}} \tag{69}$$

and

$$b_{\frac{l}{2}}[\varphi, g_{\mu\nu}] = -\frac{1}{(2\pi)^{d+1}} \sum_{n+k=l-1} \int_{R^{d-1}} dp \int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} dr \int_{-\infty}^{\infty} d\lambda \cdot e^{i\lambda} e^{-i\langle r,\tau\rangle} d_{-2-k}(\partial_r^n \varphi) \Big|_{r=0} . \tag{70}$$

The results of section 2 combined with (48,49), (69,70) show that the relevant Seeley-de Witt coefficients in d dimensions are those with $l = d$, that is

$$a_{\frac{l}{2}}[\varphi, g_{\mu\nu}] \qquad \text{and} \qquad b_{\frac{l}{2}}[\varphi, g_{\mu\nu}] . \tag{71}$$

Taking into account that

$$a_{\frac{1}{2}(2l+1)}[\varphi, g_{\mu\nu}] = 0 \qquad \text{and} \qquad b_{\frac{l}{2}}[\varphi, g_{\mu\nu}] = 0 \ \text{ for } \partial\mathcal{M} = 0 ,$$

the coefficients relevant for the zeta-function regularisation are listed in Table 1.

dimension	coefficients	
	$a_{\frac{l}{2}}[\varphi, g_{\mu\nu}]$	$b_{\frac{l}{2}}[\varphi, g_{\mu\nu}]$
1		$b_{\frac{1}{2}}[\varphi, g_{\mu\nu}]$
2	$a_1[\varphi, g_{\mu\nu}]$	$b_1[\varphi, g_{\mu\nu}]$
3		$b_{\frac{3}{2}}[\varphi, g_{\mu\nu}]$
4	$a_2[\varphi, g_{\mu\nu}]$	$b_2[\varphi, g_{\mu\nu}]$

Table 1: Coefficients relevant for the zeta-function regularisation.

A full list of these coefficients is given in appendix B. To calculate b_2, we need d_{-5} and solve (66) successively up to third order. To satisfy (65), we also need q_{-5}, which is the solution of the algebraic system

$$\begin{aligned} a_2 q_{-2-l} + &\sum_{j<l} (\partial_r^\alpha a_k D_r^\alpha q_{-2-j} + \partial_p^\alpha a_k D_x^\alpha q_{-2-j})/\alpha! = 0 \\ &k - |\alpha| - 2 - j = -l \quad \forall l \geq 0 . \end{aligned} \tag{72}$$

According to (70) we must integrate

$$d_{-5}\varphi \ , \ d_{-4}\,\partial_r\varphi \ , \ d_{-3}\,\partial_r^2\varphi \ , \ d_{-2}\,\partial_r^3\varphi \ . \tag{73}$$

to determine $b_2[\varphi, g_{\mu\nu}]$. Because of (66) the first one is determined by (recall $\tilde{\rho} = \rho(r=0)$)

$$\begin{aligned}
(\partial_r - \tilde{\varrho})(\partial_r + \tilde{\varrho})d_{-5} = {} & \frac{1}{i}\partial_a^p\tilde{a}^{(2)}\partial_a^x d_{-4} + \tilde{a}^{(1)}d_{-4} - \frac{1}{2}\partial_{ab}^p\tilde{a}^{(2)}\partial_{ab}^x d_{-3} \\
& + \frac{1}{i}\partial_a^p\tilde{a}^{(1)}\partial_a^x d_{-3} + \tilde{a}^{(0)}d_{-3} - \frac{1}{6i}\partial_{abc}^p\tilde{a}^{(2)}\partial_{abc}^x d_{-2} \\
& - \frac{1}{2}\partial_{ab}^p\tilde{a}^{(1)}\partial_{ab}^x d_{-2} + \frac{1}{i}\partial_a^p\tilde{a}^{(0)}\partial_a^x d_{-2} + \tilde{a}^{(-1)}d_{-2} \\
=: {} & (A + Br + Cr^2 + Dr^3 + Er^4 + Fr^5)e^{-\tilde{\rho}r} \ ,
\end{aligned} \tag{74}$$

and has the solution

$$\begin{aligned}
d_{-5} &= \tilde{q}_{-5}e^{-\tilde{\rho}r} + (ar + br^2 + cr^3 + dr^4 + er^5 + fr^6)e^{-\tilde{\rho}r} \\
a &= -\frac{15}{8\tilde{\varrho}^6}F - \frac{3}{4\tilde{\varrho}^5}E - \frac{3}{8\tilde{\varrho}^4}D - \frac{1}{4\tilde{\varrho}^3}C - \frac{1}{4\tilde{\varrho}^2}B - \frac{1}{2\tilde{\varrho}}A \\
b &= -\frac{15}{8\tilde{\varrho}^5}F - \frac{3}{4\tilde{\varrho}^4}E - \frac{3}{8\tilde{\varrho}^3}D - \frac{1}{4\tilde{\varrho}^2}C - \frac{1}{4\tilde{\varrho}}B \\
c &= -\frac{5}{4\tilde{\varrho}^4}F - \frac{1}{2\tilde{\varrho}^3}E - \frac{1}{4\tilde{\varrho}^2}D - \frac{1}{6\tilde{\varrho}}C \\
d &= -\frac{5}{8\tilde{\varrho}^3}F - \frac{1}{4\tilde{\varrho}^2}E - \frac{1}{8\tilde{\varrho}}D \\
e &= -\frac{1}{4\tilde{\varrho}^2}F - \frac{1}{10\tilde{\varrho}}E \\
f &= -\frac{1}{12\tilde{\varrho}}F \ .
\end{aligned} \tag{75}$$

After that we perform subsequently the integrations

$$\int_{-\infty}^{\infty} d\tau \quad , \quad \int_0^{\infty} dr \quad , \quad \int_{-\infty}^{\infty} e^{i\lambda}d\lambda \quad , \quad \int_{-\infty}^{\infty} d^{(3)}\,p \tag{76}$$

and use the formula [21]

$$\int_{-\infty}^{\infty} d\tau\, e^{-ir\tau}\frac{\tau^m}{(\tau^2 + \tilde{\varrho}^2)^{n+1}} = \frac{(-i)^m(-1)^n}{n!}\pi\partial_{\tilde{\varrho}^2}^n(\tilde{\varrho}^{m-1}e^{-\tilde{\varrho}r}). \tag{77}$$

It is easily seen that terms of odd order in p vanish after integration. This explains why the coefficients a_n, n odd, do not appear in the table. After the step (66) has been performed the computation can be done at the origin of the Riemann normal coordinte system. The corresponding contribution to $b_2[\varphi = 1, g]$ is

$$\begin{aligned}
\pi^3\Big(&- \tfrac{11}{90}\mathcal{K}_{bb} g^{aa}_{,bb} + \tfrac{1}{9}\mathcal{K}_{aa} g^{aa}_{,bb} - \tfrac{1}{45}\mathcal{K}_{aa} g^{ab}_{,ab} + \tfrac{1}{18} g^{aa}_{,bb}\mathcal{T}r\mathcal{K} \\
&- \tfrac{2}{9} g^{ab}_{,ab}\mathcal{T}r\mathcal{K} + \tfrac{1}{3}\Gamma^{b}_{\ aa,rb} + \tfrac{2}{3}\mathcal{K}_{aa}\Gamma^{b}_{\ aa,b} - \tfrac{1}{3}\Gamma^{b}_{\ aa,b}\mathcal{T}r\mathcal{K} \\
&- \tfrac{1}{20} g^{aa}_{,rbb} + \tfrac{7}{30} g^{ab}_{,rab} - \tfrac{1}{6} g_{aa,rrr} - \tfrac{1}{15} g^{aa}_{,rrr} \\
&- \tfrac{2}{3}\mathcal{K}_{aa} g_{aa,rr} + \tfrac{9}{15}\mathcal{K}_{aa} g^{aa}_{,rr} + \tfrac{1}{3} g_{aa,rr}\mathcal{T}r\mathcal{K} + \tfrac{2}{15} g^{aa}_{,rr}\mathcal{T}r\mathcal{K} \\
&+ \tfrac{19}{189}(\mathcal{T}r\mathcal{K})^3 - \tfrac{109}{63}\mathcal{T}r\mathcal{K}\mathcal{T}r\mathcal{K}^2 + \tfrac{100}{189}\mathcal{T}r\mathcal{K}^3 - \tfrac{2}{3}\alpha\mathcal{R}_{,r} \\
&+ \tfrac{2}{3}\alpha\mathcal{R}\mathcal{T}r\mathcal{K}\Big)\,,
\end{aligned} \tag{78}$$

where all functions are to be evaluated at the origin $x^\alpha = 0$ and equal indices are summed over. This expression can be written covariantly by using the extrinsic version $\chi_{\mu\nu}$ of the second fundamental form (see appendix A) as follows

$$\begin{aligned}
(78) = \frac{1}{(4\pi)^2}\frac{1}{1890}\Big(&80\mathcal{T}r\chi^3 - 66\mathcal{T}r\chi\mathcal{T}r\chi^2 + 10(\mathcal{T}r\chi)^3 \\
&105\,\mathcal{R}\mathcal{T}r\chi - 21\mathcal{R}_{\mu\nu}\chi^{\mu\nu} - \tfrac{189}{2} n^\mu\nabla_\mu\mathcal{R} \\
&- 21\,\mathcal{R}_{\mu\nu}n^\mu n^\nu\mathcal{T}r\chi + 84\,\mathcal{R}_{\sigma\mu\rho\nu}n^\sigma n^\rho\chi^{\mu\nu} \\
&- \tfrac{63}{2}\Delta^{(3)}\mathcal{T}r\chi + 630\,\xi n^\mu\nabla_\mu\mathcal{R} \\
&- 630\,\xi\mathcal{T}r\chi\mathcal{R}\Big)\,.
\end{aligned} \tag{79}$$

Here n^μ is the inward pointing normal. For the remaining terms in (73) we find

$$\begin{aligned}
&-2\pi^3\Big(\tfrac{1}{2}\xi\mathcal{R} - \tfrac{1}{14}(\mathcal{T}r\chi)^2 + \tfrac{3}{28}\mathcal{T}r\chi^2 + \tfrac{1}{12}\mathcal{R}_{\mu\nu}n^\mu n^\nu - \tfrac{1}{12}\mathcal{R}\Big)\partial_r\varphi \\
&-2\pi^3\Big(\tfrac{3}{20}\mathcal{T}r\chi\partial_r^2\varphi - \tfrac{1}{12}\partial_r^3\varphi\Big)
\end{aligned} \tag{80}$$

which can be covariantly written as

$$\begin{aligned}
\frac{1}{(4\pi)^2}\frac{1}{1890}\Big(&45\mathcal{T}r\chi^2 n^\mu\nabla_\mu\varphi + 126\,\mathcal{T}r\chi\Delta\varphi - 9(\mathcal{T}r\chi)^2 n^\mu\nabla_\mu\varphi \\
&- \tfrac{315}{2}\mathcal{R}n^\mu\nabla_\mu\varphi - \tfrac{315}{2}n^\mu\nabla_\mu\Delta\varphi \\
&- 126\,\mathcal{T}r\chi\Delta^{(3)}\varphi + 945\,\xi\mathcal{R}n^\mu\nabla_\mu\varphi\Big)\,.
\end{aligned} \tag{81}$$

Using

$$\begin{aligned}
&\int_{\partial\mathcal{M}} \Delta^{(3)} n^\mu\nabla_\mu\varphi = 0 \\
&\int_{\partial\mathcal{M}} \varphi\Delta^{(3)}\mathcal{T}r\chi = \int_{\partial\mathcal{M}} \mathcal{T}r\chi\Delta^{(3)}\varphi \\
&\int_{\partial\mathcal{M}} \nabla^{(3)}_\mu\varphi\nabla^{(3)}_\nu\chi^{\mu\nu} = -\int_{\partial\mathcal{M}} \chi^{\mu\nu}\nabla^{(3)}_\mu\nabla^{(3)}_\nu\varphi
\end{aligned} \tag{82}$$

with an upper index ${}^{(3)}$ indicating derivatives inside the boundary, the final form that coincides with [8, 9] is

$$
\begin{aligned}
b_2[\varphi, g_{\mu\nu}] = \frac{1}{(4\pi)^2}\frac{1}{1890}\Bigg(\Big[&80\mathcal{T}r\chi^3 - 66\mathcal{T}r\chi\mathcal{T}r\chi^2 + 10(\mathcal{T}r\chi)^3 \\
&+ 105\mathcal{R}\mathcal{T}r\chi - 21\mathcal{R}_{\mu\nu}\chi^{\mu\nu} - \tfrac{189}{2}n^\mu\nabla_\mu\mathcal{R} \\
&- 21\mathcal{R}_{\mu\nu}n^\mu n^\nu\mathcal{T}r\chi + 84\mathcal{R}_{\sigma\mu\rho\nu}n^\sigma n^\rho\chi^{\mu\nu} \\
&+ 630\xi n^\mu\nabla_\mu\mathcal{R} - 630\xi\mathcal{T}r\chi\mathcal{R}\Big]\varphi \\
&+ 45\mathcal{T}r\chi^2 n^\mu\nabla_\mu\varphi + 126\mathcal{T}r\chi\Delta\varphi - 9(\mathcal{T}r\chi)^2 n^\mu\nabla_\mu\varphi \\
&- \tfrac{315}{2}\mathcal{R}n^\mu\nabla_\mu\varphi - \tfrac{315}{2}n^\mu\nabla_\mu\Delta\varphi + 945\xi\,\mathcal{R}n^\mu\nabla_\mu\varphi\Bigg)\,. \qquad (83)
\end{aligned}
$$

As always, explicit formulae are with respect to a chosen convention. To allow for a easy comparison of our results with those in the literature we specified our conventions in appendix A. Finally note that after using

$$f_{;nn} = \Delta^{(4)}f - \Delta^{(3)}f + \mathcal{T}r\chi\, f_{,n},$$

$R_{nnnn} = 0$ and

$$
\begin{aligned}
R_{abcb}L_{ac} &= R_{\mu\nu}\chi^{\mu\nu} - R_{anbn}L_{ab} \\
&= R_{\mu\nu}\chi^{\mu\nu} - R_{\sigma\mu\rho\nu}n^\sigma n^\rho\chi^{\mu\nu},
\end{aligned}
$$

the coefficient b_2 becomes identical to the one of Branson and Gilkey [9].

5 Applications to Simple Geometries

After having derived the explicit form of the heat kernel coefficient b_2 we can now apply the general formula (23) to determine the change of Γ under conformal transformation of an arbitrary region $\{\mathcal{M}, \partial\mathcal{M}\}$ in 4-dimensional (flat) Euklidean space-time. As we shall see the influence of a wall $\partial\mathcal{M}$ on the vacuum fluctuations is more subtle than in 2 dimensions.

Note that for a Weyl angle belonging to a diffeomorphism the volume integral in (23) vanishes and the conformal anomaly is solely a surface effect. This is of course true in arbitrary dimensions as long as the imbedding space-time X is flat. To evaluate the surface term in (23) we still must express the curvature terms in (83) as functions of $\tau\varphi$ (recall that $g^\tau_{\mu\nu} = e^{2\tau\varphi}\delta_{\mu\nu}$) and perform the τ-integration. The final result for massless scalars is

$$\delta\Gamma = -\frac{1}{(4\pi)^2}\frac{1}{1890}\int_{\partial\mathcal{M}}\sqrt{\tilde{g}}\,\Big\{\Big[80\mathcal{T}r\chi^3 - 66\mathcal{T}r\chi\mathcal{T}r\chi^2 + 10(\mathcal{T}r\chi)^3$$
$$+ 21(\mathcal{T}r\chi)^2\partial_n\varphi - 21\mathcal{T}r\chi^2\partial_n\varphi$$
$$- 14\mathcal{T}r\chi\partial_n\varphi^2 - 21\chi^{\mu\nu}\varphi_{,\mu\nu} + 14\chi^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$$
$$+ 21\mathcal{T}r\chi\varDelta\varphi - 28\varDelta\varphi\partial_n\varphi + 28n^\mu n^\nu\varphi_{,\mu\nu}\partial_n\varphi$$
$$- \tfrac{21}{2}\varphi_{,\nu}\varphi^{,\nu}\partial_n\varphi - 21\mathcal{T}r\chi\varphi_{,\mu\nu}n^\mu n^\nu\Big]\varphi$$
$$+ 45\mathcal{T}r\chi^2\partial_n\varphi - 18\mathcal{T}r\chi(\partial_n\varphi)^2 + 18(\partial_n\varphi)^3$$
$$- 9(\mathcal{T}r\chi)^2\partial_n\varphi + 126\mathcal{T}r\chi\varDelta\varphi + 126\mathcal{T}r\chi\varphi_{,\mu}\varphi^{,\mu}$$
$$+ 126\varDelta\varphi\partial_n\varphi + 126\,\varphi_{,\mu}\varphi^{,\mu}\partial_n\varphi - \tfrac{315}{2}\partial_n\varDelta\varphi$$
$$- \tfrac{315}{2}\partial_n(\varphi_{,\nu}\varphi^{,\nu})\Big\}\,, \tag{84}$$

where $\partial_n = n^\alpha\partial_\alpha$ is the (inward) normal derivative and all contractions in traces and derivatives are understood with respect to the original undeformed metric $g^0_{\mu\nu} = \delta_{\mu\nu}$. Of course $\tilde{g}$ is the determinant of the metric on $\partial\mathcal{M}$ induced by $g^0_{\mu\nu}$.

Dilatations

Now we generalize the two-dimensional result (28) for the change of Γ under dilatations (5) to four dimensions. Since then the Weyl angle is constant, $\varphi = \log\lambda$, all but the first three terms in (84) vanish so that

$$\Gamma[\lambda\mathcal{M}] - \Gamma[\mathcal{M}] = -\frac{1}{(4\pi)^2}\frac{\log\lambda}{1890}\int\limits_{\partial\mathcal{M}}\sqrt{\tilde{g}}\,\Big[80\,\mathcal{T}r\chi^3 - 66\,\mathcal{T}r\chi\mathcal{T}r\chi^2 + 10\,(\mathcal{T}r\chi)^3\Big]\,. \tag{85}$$

Contrary to the two-dimensional case the right hand side is not a topological invariant. To see that more clearly let us introduce the Euler number χ^E. Applying the index theorem to the De Rham-complex [22] the Euler number of a bounded flat manifold is

$$\chi^E = -\frac{1}{12\pi^2}\int\limits_{\partial\mathcal{M}}\sqrt{\tilde{g}}\,\Big[2\mathcal{T}r\chi^3 - 3\mathcal{T}r\chi\mathcal{T}r\chi^2 + (\mathcal{T}r\chi)^3\Big]. \tag{86}$$

It is just the winding number of the normal map $n:\partial\mathcal{M}\to S^3$ and the sign convention is such, that it is one for a sphere. Thus (85) can be written as

$$\delta\Gamma = -\frac{\log\lambda}{180}\,\chi^E - \frac{\log\lambda}{280\pi^2}\int\limits_{\partial\mathcal{M}}\sqrt{\tilde{g}}\,f(\chi) \tag{87}$$

where we have introduced the third order polynomial

$$f(\chi) = \mathcal{T}r\chi^3 - \mathcal{T}r\chi\,\mathcal{T}r\chi^2 + \frac{2}{9}(\mathcal{T}r\chi)^3\,. \tag{88}$$

Contrary to the Euler number the last term in (87) depends on the shape of the surface and thus is not topological. For simple geometries we obtain:
Spherical bubbles: For a spherical bubble, $\mathcal{M} = \mathcal{B}$, the boundary surface is a 3-sphere for which $f(\chi) = 0$. It follows that

$$\Gamma[\lambda\mathcal{B}] - \Gamma[\mathcal{B}] = -\frac{1}{180}\log\lambda \tag{89}$$

leading to a repulsive Casimir force as in 2 dimensions (see 28).
Squashed and stretched bubbles: We parametrize the surface of the ellipsoid as

$$\{x^0, x^1, x^2\} = A\sin\alpha\;\{\sin\beta\cos\gamma, \sin\beta\sin\gamma, \cos\beta\}, \quad x^3 = B\cos\alpha\;, \tag{90}$$

where

$$0 \le \gamma \le 2\pi \quad \text{and} \quad 0 \le \alpha, \beta \le \pi.$$

Inserting the corresponding second fundamental form into $f(\chi)$ in ()87,88) yields

$$\delta\Gamma = \log\lambda\; D(u), \qquad D(u) = -\Big[\frac{(u-1)^3(5u^3+20u^2+29u+16)}{5040u(1+u)} + \frac{1}{180}\Big], \tag{91}$$

where we have introduced the parameter $u = A/B$ which measures the deviation from a spherical bubble. The graph of $D(u)$ is displayed in Figure 1.

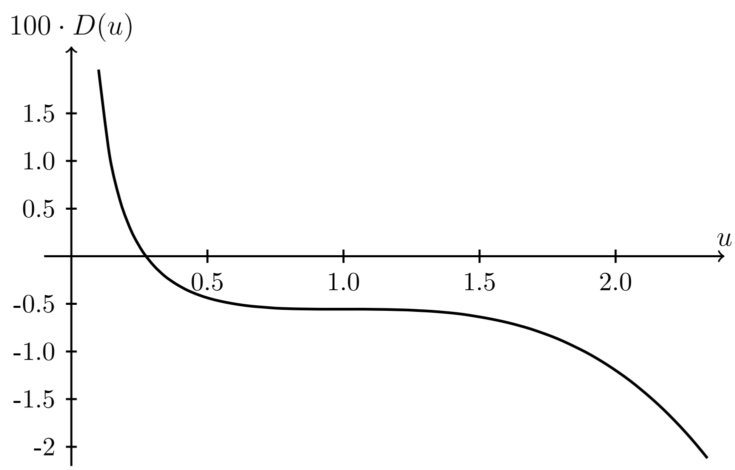

Figure 1: Change of the quantum actions of squashed or stretched bubbles under dilatations as a function of the bubble excentricity u. A stretched bubble has $u < 1$ and a squashed one $u > 1$.

One sees that the quantum action increases or decreases with the volume depending on whether $u < 0.274$ or $u > 0.274$. Thus, the vaccum fluctuations try to shrink bubbles stretched in the x^3-direction and expand squashed bubbles.

Special conformal transformations

A special conformal transformations (6) is a composition of an inversion, translation and again an inversion. We require that bounded regions are transformed into bounded ones which means that the transformed regions should contain the origin of the coordinate system if and only if the original body contained it. For an ellipsoid (90) and a special conformal transformation (6) with $b^\mu = (0,0,0,b)$ this is fulfilled for $Ab, Bb < 1$. Thus we may expand (84) (for special conformal transformations φ is x-dependent) in Ab and Bb, and the first non-vanishing terms are of second order. The explicit result up to second order reads

$$\delta\Gamma = (Ab)(Bb)\; S(u) \tag{92}$$

where

$$S(u) = -\frac{5u^9 + 10u^8 + 6u^7 + 2u^6 - 32u^5 - 66u^4 + 182u^3 + 38u^2 - 9u - 8}{5040\, u^2(1+u)^2}. \tag{93}$$

The function $S(u)$ is displayed in Figure 2. Again one finds that stretched bubbles

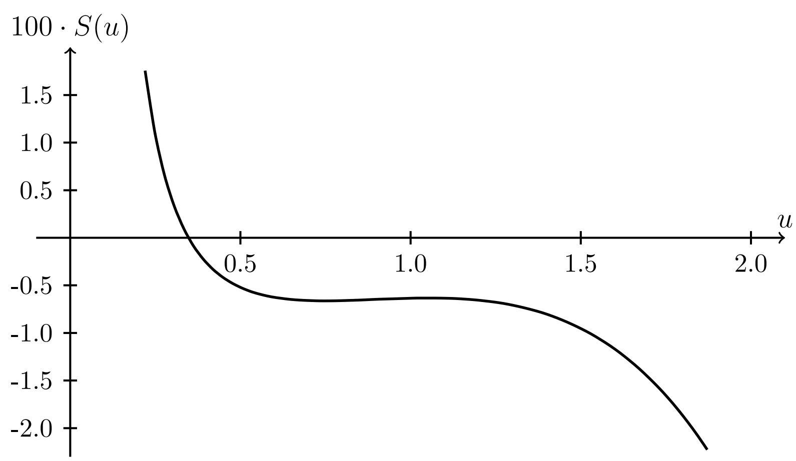

Figure 2: Change of the quantum actions of squashed or stretched bubbles under special conformal transformations. Depending on the excentricity the bubble favours or resists a conformal deformation.

resist a deformation and squashed ones are unstable against the special conformal deformations (6). Bubbles with $u = u_c = 0.348$ are marginal in the sense that their deformation does not change the quantum action induced by the vacuum fluctuations. The figures 3 shows two typical deformation of ellipsoids (the figures show the intersection of the ellipsoid with the (x^1, x^2)-plane). In Figure 3 a stretched bubble with $(A, B) = (0.5, 2)$, drawn with broken line, is deformed with $b = 0.3$ into the body drawn with the unbroken line. This deformation increases the quantum action. In Figure 4 a squashed bubble with $(A, B) = (2, 0.5)$ is deformed with the same b. The bubble is unstable against this deformation.

We thank H. Dorn for critical remarks.

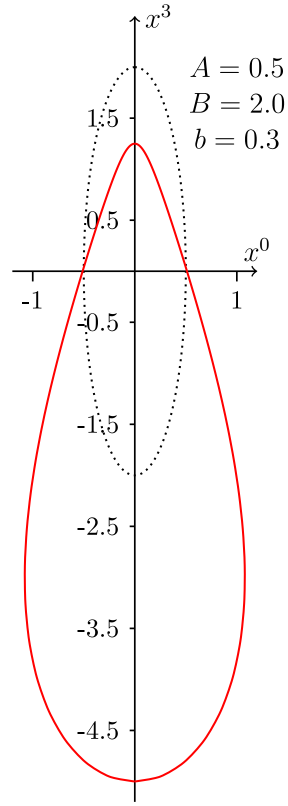

Figure 3: The stretched bubble enclosed by the broken line is mapped into the body enclosed by the full line by a special conformal transformations. The quantum system resists this deformation.

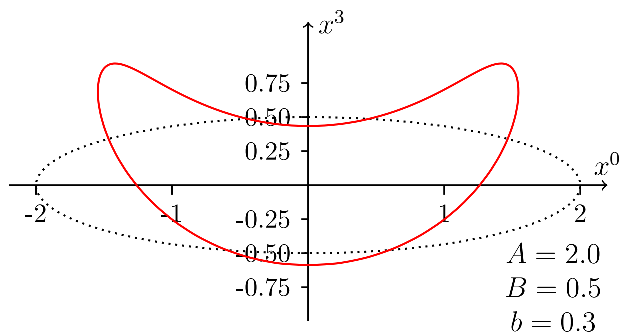

Figure 4: The squashed bubble enclosed by the broken line is mapped into the body enclosed by the full line. This deformation decreases the quantum action.

6 Appendix A: Conventions

Let the d-dimensional manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ be embedded a in space-time X. The metric on $\mathcal{M}$ is $g_{\alpha\beta}$. The definition of the curvature terms on $\mathcal{M}$ is

$$\mathcal{R}^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\delta\beta,\gamma} - \Gamma^{\alpha}_{\gamma\beta,\delta} + \Gamma^{\sigma}_{\delta\beta}\Gamma^{\alpha}_{\gamma\sigma} - \Gamma^{\sigma}_{\gamma\beta}\Gamma^{\alpha}_{\delta\sigma} \tag{94}$$

and

$$\mathcal{R}_{\beta\delta} = \mathcal{R}^{\alpha}_{\beta\alpha\delta} = \Gamma^{\alpha}_{\delta\beta,\alpha} - \Gamma^{\alpha}_{\alpha\beta,\delta} + \Gamma^{\sigma}_{\delta\beta}\Gamma^{\alpha}_{\alpha\sigma} - \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\alpha}_{\delta\sigma}. \tag{95}$$

The geometric properties of the boundary are usually described in terms of induced curvatures. Choosing local coordinates x^{α}, $\alpha = 0, \ldots, d-1$, in X the boundary can (locally) be parametrized through functions

$$x^\alpha = f^\alpha(u^i), \quad i = 1, \ldots, d-1, \tag{96}$$

and then the induced metric is defined by

$$\tilde{g}_{ij} := g_{\alpha\beta}\; x^\alpha_{|i} x^\beta_{|j} \; . \tag{97}$$

Here $x^\alpha_{|i}$ is the covariant derivative defined by the induced metric, but since the x^α are invariants under coordinate transformations of the u^i, it is just the ordinary derivative with respect to u^i. $x^\alpha_{|i}$ is a tangent vector and hence

$$g_{\alpha\beta} x^\alpha_{|i} n^\beta = 0 \; . \tag{98}$$

n^β is the unit inner pointing normal of $\partial\mathcal{M}$. The second fundamental form of the surface is the symmetric tensor

$$K_{ij} \; := g_{\alpha\beta} n^\beta x^\alpha_{|ij} \; , \tag{99}$$

which is equivalent to

$$K_{ij} = -x^\alpha_{|i} x^\beta_{|j} n_{\alpha;\beta} \; . \tag{100}$$

Here it is understood that the normal field n is extended to a neighbourhood of $\partial\mathcal{M}$ and that the covariant derivative is then computed with the connection on X. The value of $n_{\alpha;\beta}$ on the boundary does not depend on the extension. Let us introduce the extrinsic version $\chi_{\mu\nu}$ of the second fundamental form

$$\chi^{\alpha\beta} \; := x^\alpha_{|i} x^\beta_{|j} K^{ij} \; . \tag{101}$$

Introducing the projector

$$h_{\alpha\beta} = \; g_{\alpha\beta} - n_\alpha n_\beta = g_{\alpha\gamma} g_{\beta\delta}\; x^\gamma_{,i} x^\delta_{,j}\; \tilde{g}^{ij} \tag{102}$$

it can be cast into the form

$$\chi_{\mu\nu} = -h^\sigma_\mu h^\varrho_\nu n_{\sigma;\varrho} \; . \tag{103}$$

and this form is convenient since it only involves the metric on $\mathcal{M}$, the normal field n and its covariant derivative. It is easily shown that

$$\mathrm{Tr}\chi = g^{\mu\nu}\chi_{\mu\nu} = \tilde{g}^{ij} K_{ij} = \mathrm{Tr}K \; . \tag{104}$$

Using the relation

$$\nabla^{(d-1)}_\mu = h^\nu_\mu \nabla^{(d)}_\nu \tag{105}$$

between the extrinsic covariant derivative and the covariant derivative on $\mathcal{M}$ (the instrinsic one on $\partial\mathcal{M}$ is just the projection of ∇^{d-1} on $\partial\mathcal{M}$) one can prove the Gauss equation

$$\mathcal{R}^{(d-1)}_{\alpha\beta\gamma\delta} = \; h^{\alpha\sigma} h^{\beta\rho} h^{\gamma\mu} h^{\delta\nu} \mathcal{R}^{(d)}_{\sigma\rho\mu\nu} + \chi_{\alpha\gamma}\chi_{\beta\delta} - \chi_{\alpha\delta}\chi_{\beta\gamma} \; . \tag{106}$$

7 Appendix B: Seeley-de Witt coefficients

These are the volume Seeley-deWitt coefficients for massless scalars which are relevant in $d \leq 4$-spacetime dimensions:

$$a_1[\varphi, g_{\mu\nu}] = \frac{1}{(4\pi)^{d/2}} \left(\tfrac{1}{6} - \xi\right) \mathcal{R}^{(d)} \varphi \tag{107}$$

$$a_2[\varphi, g_{\mu\nu}] = \frac{1}{(4\pi)^{d/2}} \frac{1}{180} \Big(\mathcal{R}^{(d)}_{\sigma\mu\rho\nu} \mathcal{R}^{(d)\sigma\mu\rho\nu} - \mathcal{R}^{(d)}_{\mu\nu} \mathcal{R}^{(d)\mu\nu} + (6 - 30\xi) \Delta^{(d)} \mathcal{R}^{(d)} + 90 (\tfrac{1}{6} - \xi)^2 \, \mathcal{R}^{(d)^2} \Big) \varphi \; . \tag{108}$$

For Dirichlet boundary conditions

$$\Phi|_{\partial\mathcal{M}} = 0 \; ,$$

the relevant surface coefficients read

$$b_{\frac{1}{2}}[\varphi, g_{\mu\nu}] = \frac{1}{(4\pi)^{d/2}} \Big(-\frac{\sqrt{\pi}}{2} \Big) \varphi \tag{109}$$

$$b_1[\varphi, g_{\mu\nu}] = \frac{1}{(4\pi)^{d/2}} \Big(\tfrac{1}{3} \mathcal{T}r\chi\varphi - \tfrac{1}{2} n^\mu \nabla_\mu \varphi \Big) \tag{110}$$

$$\begin{aligned} b_{\frac{3}{2}}[\varphi, g_{\mu\nu}] = \frac{1}{(4\pi)^{d/2}} \frac{\sqrt{\pi}}{192} \Big(\big[-3(\mathcal{T}r\chi)^2 + 6\mathcal{T}r\chi^2 - 4\mathcal{R}^{(d-1)} + 12(8\xi - 1)\mathcal{R}^{(d)} \big] \varphi \\ + 30\mathcal{T}r\chi n^\mu \nabla_\mu \varphi - 24 n^\mu n^\nu \nabla_\nu \nabla_\mu \varphi \Big) \end{aligned} \tag{111}$$

$$\begin{aligned} b_2[\varphi, g_{\mu\nu}] = \frac{1}{(4\pi)^{d/2}} \frac{1}{1890} \Big\{ & \big[80\mathcal{T}r\chi^3 - 66\mathcal{T}r\chi\mathcal{T}r\chi^2 + 10(\mathcal{T}r\chi)^3 \\ & + 105\mathcal{R}^{(d)}\mathcal{T}r\chi - 21\mathcal{R}^{(d)}_{\mu\nu}\chi^{\mu\nu} - \tfrac{189}{2} n^\mu \nabla_\mu \mathcal{R}^{(d)} \\ & - 21\mathcal{R}^{(d)}_{\mu\nu} n^\mu n^\nu \mathcal{T}r\chi + 84\mathcal{R}^{(d)}_{\sigma\mu\rho\nu} n^\sigma n^\rho \chi^{\mu\nu} \\ & + 630\xi n^\mu \nabla_\mu \mathcal{R}^{(d)} - 630\xi\mathcal{T}r\chi\mathcal{R}^{(d)} \big] \varphi \\ & + 45\mathcal{T}r\chi^2 n^\mu \nabla_\mu \varphi + 126\mathcal{T}r\chi\Delta\varphi - 9(\mathcal{T}r\chi)^2 n^\mu \nabla_\mu \varphi \\ & - \tfrac{315}{2}\mathcal{R}^{(d)} n^\mu \nabla_\mu \varphi - \tfrac{315}{2} n^\mu \nabla_\mu \Delta\varphi \\ & + 945\xi\mathcal{R}^{(d)} n^\mu \nabla_\mu \varphi \Big\} \; . \end{aligned} \tag{112}$$

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