Z. Phys. C – Particles and Fields 35, 351–354 (1987)

Zeitschrift für Physik C Particles and Fields

Finite temperature $\lambda\phi^4$ theory in two and three dimensions and symmetry restoration

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Received 5 November 1986

Abstract. $\lambda\phi^4$ Theory is studied in 2 and 3 dimensions to examine the validity of the finite temperature perturbation theory. We find that in some cases it is good even at high temperature in contrast to the case in 4 dimensions. We also discuss the problem of symmetry restoration and show an example of symmetry restoration within a safe perturbation at high temperature.

1 Introduction

In this letter we investigate the $\lambda\phi^4$ theory at finite temperature ($\equiv T \neq 0$) in 2 and 3 dimensions. Our primary interest is in the nature of $T \neq 0$ perturbation at high temperature i.e. $\beta m \ll 1$. $T \neq 0$ perturbation involves not only the coupling constant but also the temperature as expansion parameter. At very high temperature the perturbation breaks down even when the coupling constant is small. As will be shown, $T \neq 0$ perturbation for $\lambda\phi^4$ theories in $D(<4)$ dimensions differ in a non-trivial way from that in 4 dimensions. Unlike in 4 dimensions there are cases where the one-loop correction becomes dominant even at high temperature, $\beta m \ll 1$. Another difference is in the infrared behaviour. In $D(<4)$ dimensions the infrared behaviour is worse than in 4 dimensions. Even at the lowest order level one faces the infrared singularity which bars one from predicting the critical temperature. In the first half of the paper we shall clarify to what extent $T \neq 0$ perturbation is reliable. In the latter half of the paper we shall discuss, as a related subject, the problem of symmetry restoration which has been argued for and against by several people [1, 2] in the context of the $O(N) \times O(N)$ model in 4 dimensions. It will be shown that in 3 dimensions symmetry is restored at high temperature.

* Work supported in part by Schweizerischer Nationalfonds

** Work in part supported by the Deutsche Forschungsgemeinschaft, Address after 1.1.1987: Theoretical Division, Los Alamos National Laboratory, USA

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2 General character of $T \neq 0$ perturbation

We first discuss the features of ordinary $T \neq 0$ perturbation and thus the parameters λ and m are defined at zero temperature.

The temperature independent part is left out of consideration since we are interest only in temperature effects. Further we consider only the high temperature case ($\beta m \ll 1$). At low temperature ($\beta m \gg 1$) perturbation is always good so long as we deal with massive theories because each loop goes with an exponential suppression factor $e^{-\beta m}$. The general tendency of $T \neq 0$ perturbation can be read off by calculating a few typical diagrams (Figs. 1–6). The results are presented below.

1. 3-Dimensional $\lambda\phi^4$

$$\text{Fig. } 1 = \frac{-1}{2\pi}\frac{\lambda}{\beta}\ln(1-e^{-\beta m}) \cong \frac{-\lambda}{2\pi}\frac{1}{\beta}\ln\beta m, \tag{1.1}$$

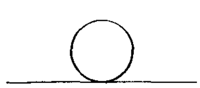

Fig. 1. One-loop self-energy diagram

$$\text{Fig. } 2 \cong C_1 \left(\frac{\lambda}{2\pi}\right)^2 \frac{1}{\beta^2 m^2} \ln \beta m, \tag{1.2}$$

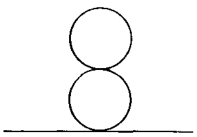

Fig. 2. A two-loop self-energy diagram

$$\text{Fig. } 3 \cong C_2 \left(\frac{\lambda}{2\pi}\right)^3 \frac{1}{\beta^3 m^3} \ln \beta m, \tag{1.3}$$

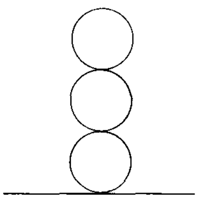

Fig. 3. A three-loop self-energy diagram

$$\text{Fig. } 4 \cong C_3 \left(\frac{\lambda}{2\pi}\right)^2 \frac{1}{\beta^2 m^2} \ln \beta m, \tag{1.4}$$

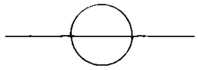

Fig. 4. A non-planar two-loop diagram

$$\text{Fig. } 5 \cong C_4 \left(\frac{\lambda}{2\pi}\right)^3 \frac{1}{m^3} \left(\frac{1}{\beta} \ln \beta m\right)^2, \tag{1.5}$$

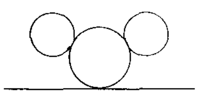

Fig. 5. A three-loop diagram

$$\text{Fig. } 6 = -\frac{3\lambda^2}{8\pi} \frac{1}{m} \frac{1}{e^{\beta m}-1} \cong -\frac{3\lambda^2}{8\pi} \frac{1}{\beta m^2}. \tag{1.6}$$

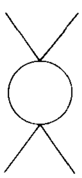

Fig. 6. One-loop vertex diagram

2. 2-Dimensional $\lambda\phi^4$

$$\text{Fig. } 1 = \frac{\lambda}{2\pi} \int_1^\infty dx \frac{1}{(x^2-1)^{\frac{1}{2}}} \frac{1}{e^{\beta m x}-1}$$

$$\cong \frac{\lambda}{\pi} \left\{ \frac{\pi}{2} \frac{1}{\beta m} - \frac{1}{4}(-2\gamma + 2\log 4\pi - \log \beta^2 m^2) + O(\beta m) \right\}$$

$$\cong \frac{\lambda}{2} \frac{1}{\beta m}, \tag{2.1}$$

$$\text{Fig. } 2 \cong C_5 \frac{\lambda^2}{\beta^2 m^4}, \tag{2.2}$$

$$\text{Fig. } 3 \cong C_6 \frac{\lambda^3}{\beta^3 m^7}, \tag{2.3}$$

$$\text{Fig. } 4 \cong C_7 \frac{\lambda^2}{\beta^2 m^4}, \tag{2.4}$$

$$\text{Fig. } 5 \cong C_8 \frac{\lambda^3}{\beta^2 m^6}, \tag{2.5}$$

$$\text{Fig. } 6 = \frac{3\lambda^2}{2} \frac{1}{2m} \frac{\partial}{\partial m} \int_1^\infty dx \frac{1}{(x^2-1)^{\frac{1}{2}}} \frac{1}{e^{\beta m x}-1}$$

$$\cong -\frac{3\lambda^2}{4} \frac{1}{\beta m^3}. \tag{2.6}$$

In the above C_i $(i=1\sim 8)$ are numerical constants $\sim O(1)$. The results shows that the self-energy correction is dominated by the one-loop diagram (Fig. 1) so long as $\frac{\lambda}{2\pi} \frac{1}{\beta m^2} \ll 1$ in 3 dimensions and $\frac{\lambda}{\beta m^3} \ll 1$ in 2 dimensions. In other words there is a region of parameters where the perturbation makes good sense even at high temperature. This character in 2 and 3 dimensions is missing in 4 dimensions. In 4 dimensions we have

$$\text{Fig. } 1 = \frac{\lambda m^2}{4\pi^2} \int_1^\infty dx \frac{(x^2-1)^{\frac{1}{2}}}{e^{\beta m x}-1}$$

$$\cong \frac{\lambda m^2}{4\pi^2} \left(\frac{\pi^2}{6} \frac{1}{\beta^2 m^2} - \frac{\pi}{2} \frac{1}{\beta m} + O(\log \beta m) \right), \tag{3.1}$$

$$\text{Fig. } 2 \cong C \frac{\lambda^2 m^2}{(\beta m)^3} \quad (C\text{: constant}). \tag{3.2}$$

Therefore at high temperature ($\beta m \ll 1$) the higher order diagrams such as Fig. 2 contributes more than the one-loop diagram (Fig. 1) and thus $T \neq 0$ perturbation breaks down.

Let us now leave the ordinary perturbation and turn to what we call the self-consistent method which involves the summation of self-energy diagrams to all orders. It was introduced by Dolan and Jackiw [3] to discuss the restoration of symmetry due to temperature in $O(N)$ models. In the single real scalar theory the self-consistent equations for the effective mass, m, is given diagrammatically as in Fig. 7. Unlike in $O(N)$ models in the large N limit non-planar diagrams become non-negligible. Still the self-consistent method is useful to discuss the symmetry restoration. To see that, we first truncate the self-consistent equa-

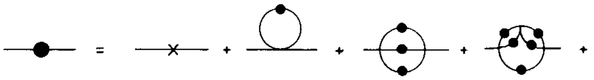

Fig. 7. Self-energy equation for the effective mass

tion for m at one-loop. Then the equation in 3 dimensions becomes

$$m^2=-\mu^2+\frac{-1}{4\pi}\frac{\lambda_0}{\beta}\log\beta m \tag{4}$$

where λ_0 and μ are zero temperature parameters. Equation (4) gives a solution with positive effective mass, which means symmetry restoration. What is more, for a sufficiently small βm βm^2 becomes large enough that one may safely neglect the higher order terms in the self-consistent equation. On the other hand (4) is not useful for predicting the critical temperature because at the critical temperature $m=0$ (4) does not make sense*. In 4 dimensions the corresponding equation is

$$m^2=-\mu^2+\frac{\lambda_0}{4\pi^2}\left(\frac{\pi^2}{6}\frac{1}{\beta^2}-\frac{\pi}{2}\frac{m}{\beta}+m^2\times O(\log m)\right). \tag{5}$$

It does no harm to set $m=0$ and one predicts the critical temperature, β_c,

$$\beta_c=\left(\frac{24\mu^2}{\lambda}\right)^{\frac{1}{2}}. \tag{6}$$

One may compare it with a naive but physically reasonable estimate of β_c. One would think that the symmetry is restored at the point where the thermal fluctuation becomes as large as the distance between the two minima of the tree potential, $V(\phi)=-\frac{\mu^2}{2}\phi^2+\frac{\lambda_0}{4!}\phi^4$ [3]. Then the critical temperature turns out to be identical to that in (6). As stated in the beginning the infrared ($m\to 0$) behaviour in 3 dimensions is worse and the self-consistent equation becomes useless in the infrared region.

3 Symmetry restoration

In Sect. 2 we have investigated the symmetry restoration in the single real scalar theory. The subject becomes more subtle and interesting when there are two or more fields. The problem was first posed by Weinberg [5] in a $O(N)\times O(N)$ model in 4 dimensions whose potential is

$$V(\phi_1,\phi_2)=-\frac{\mu_1^2}{2}\phi_1^2-\frac{\mu_2^2}{2}\phi_2^2+\frac{\lambda_1}{4!}(\phi_1^2)^2-\frac{\lambda_{12}}{12}\phi_1^2\phi_2^2+\frac{\lambda_2}{4!}(\phi_2^2)^2. \tag{7}$$

Rephrased in 3 dim. context the problem is as follows. Let us perform the ordinary perturbation and use high temperature expansion. Then the effective masses, $m_i^2\,(i=1,2)$, at one-loop are given by

$$m_1^2=-\mu^2+\{(N+2)\lambda_1-N\lambda_{12}\}\frac{-1}{2}\frac{1}{\beta}\log\beta\mu \tag{8.1}$$

$$m_2^2=-\mu^2+\{(N+2)\lambda_2-N\lambda_{12}\}\frac{-1}{2\pi}\frac{1}{\beta}\log\beta\mu \tag{8.2}$$

(for simplicity we have set $\mu_1=\mu_2\simeq\mu$ in (7)). If the parameters are such that

$$(N+2)\lambda_1>N\lambda_{12}>(N+2)\lambda_2 \tag{9}$$

then m_2^2 never becomes positive. Therefore the broken $O(N)\times O(N)$ symmetry is not restored even at high temperature. This conclusion was challenged by the authors of Ref. [2] where they found symmetry restoration by solving self-consistent equations for the effective masses. We investigate the problem below in the context of 3 dim. two-scalar theory and $O(N)\times O(N)$ model. (We shall not consider the 2 dimensional case since it is known that spontaneous breaking does not occur in 2 dimensions [6]). The Lagrangian for the two-scalar theory is

$$L(\phi_1,\phi_2)=\frac{1}{2}\phi_1\Box\phi_1+\frac{1}{2}\phi_2\Box\phi_2+\frac{\mu^2}{2}\phi_1^2+\frac{\mu^2}{2}\phi_2^2-\frac{\lambda_1}{4.}\phi_1^4+\frac{\lambda_{12}}{4}\phi_1^2\phi_2^2-\frac{\lambda_2}{4!}\phi_2^4. \tag{10}$$

The coupling constants are required to satisfy

$$\lambda_1>\lambda_{12}>\lambda_2>0,\qquad \lambda_1\lambda_2-\lambda_{12}^2>0. \tag{11}$$

Let us set up self-consistent equations at one-loop.

$$m_1^2=-\mu^2+\lambda_1\times F(\beta m_1)-\lambda_{12}\times F(\beta m_2)\cong-\mu^2-\frac{\lambda_1}{2\pi}\frac{1}{\beta}\log\beta m_1+\frac{\lambda_{12}}{2\pi}\frac{1}{\beta}\log\beta m_2\qquad(\beta m_1,\beta m_2\ll 1) \tag{12.1}$$

$$m_2^2=-\mu^2+\lambda_2\times F(\beta m_2)-\lambda_{12}\times F(\beta m_1)\cong-\mu^2-\frac{\lambda_2}{2\pi}\frac{1}{\beta}\log\beta m_2+\frac{\lambda_{12}}{2\pi}\frac{1}{\beta}\log\beta m_1 \tag{12.2}$$

* To supplement this point we shall perform a lattice calculation in a companion paper [7]

where

$$F(\beta m) \equiv \frac{-1}{2\pi} \frac{1}{\beta} \{\log(e^{\beta m} - 1) - \beta m\}. \quad (12.3)$$

One may rewrite the equations as follows

$$4\pi x = -\mu^2 \beta + (\lambda_1 - \lambda_{12}) \log \beta^{-1} - \lambda_1 \log x + \lambda_{12} \log y \quad (12.1)$$

$$4\pi y = -\mu^2 \beta + (\lambda_2 - \lambda_{12}) \log \beta^{-1} - \lambda_2 \log y + \lambda_{12} \log x \quad (12.2)$$

where $x \equiv \beta m_1^2$, $y = \beta m_2^2$. Although we omit the proof, the above equations always yield a solution with positive x and y for sufficiently high temperature, $\beta \ll 1$. A numerical result in the case $\lambda_1 = 0.5$ $\lambda_2 = 0.0475$ $\lambda_{12} = 0.05$ $\mu = 0.865$ is given in Table 1. As temperature increases the effective masses m_1, m_2 become larger, as expected. So our result indicates symmetry restoration. However there is a problem in the one-loop approximation we made. As discussed in the previous section the criteria for a safe $T \neq 0$ perturbation in 3 dims. are, $\beta m^2 \gg 1$ and $\beta m \ll 1$. As Table 1 shows $\beta m^2 \gg 1$ is not satisfied. In fact one can show that with the parameters we have chosen βm_2^2 is always less than 1. It means higher order terms cannot be neglected. Therefore one cannot safely conclude that symmetry is restored within this approximation.

The situation is different in $O(N) \times O(N)$ models, with potential

$$V(\phi_1, \phi_2) = -\frac{\mu^2}{2} (\phi_1^2 + \phi_2^2) + \frac{\lambda_1}{4!} (\phi_1^2)^2 - \frac{\lambda_{12}}{12} \phi_1^2 \phi_2^2 + \frac{\lambda_2}{4!} (\phi_2^2)^2. \quad (13)$$

If N is sufficiently large the non-planar diagrams may be neglected relative to the planar ones. Then the self-consistent equations become the same as (12.1), (12.2) apart from N and one finds

$$4\pi x = -\frac{\mu^2 \beta}{N} + \left(\frac{N+2}{N} \lambda_1 - \lambda_{12}\right) \log (\beta N)^{-1} - \frac{N+2}{N} \lambda_1 \log x + \lambda_{12} \log y \quad (14.1)$$

$$4\pi y = -\frac{\mu^2 \beta}{N} + \left(\frac{N+2}{N} \lambda_1 - \lambda_{12}\right) \log (\beta N)^{-1} - \frac{N+2}{N} \lambda_2 \log y + \lambda_{12} \log x \quad (14.2)$$

where

$$x \equiv \frac{\beta m_1^2}{N} \quad y \equiv \frac{\beta m_2^2}{N}.$$

If one sets $\frac{\mu}{N} = 0.865$, $\frac{N+2}{N} \lambda_1 = 0.5$, $\lambda_{12} = 0.05$, $\frac{N+2}{N} \lambda_2 = 0.0475$ the results of Table 1 is reproduced with the replacements of $m \to \frac{m}{N}$, $\beta \to \beta N$. If $N = 10^3$, $\beta = 10^{-5}$ then from the first row in Table 1 one reads off $m_1 = 4.6 \times 10^3$, $m_2 = 1 \times 10^3$ and thus $\beta m_1 = 4.6 \times 10^{-2}$, $\beta m_2 = 1 \times 10^{-2}$, $\beta m_1^2 = 2.1 \times 10^2$, $\beta m_2^2 = 10$. Therefore the truncation of self-consistent equations at one-loop is safe. The higher order contribution is suppressed by both $\frac{1}{N}$ and $\frac{1}{\beta m^2}$. In this case one may safely claim that symmetry is restored at high temperature. All the preceding analysis is concerned with the effective mass at the origin and thus there is left a possibility that the absolute minimum lies at some point away from the origin and the symmetry remains broken. We do not believe that is the case.

Table 1. Solution of self-consistent equations (12.1), (12.2)

β	0.01	0.001	0.0001	0.00001
m_1	4.6	16.5	60	200
m_2	1.0	3.0	10	50

4 Summary

We have investigated $\lambda \phi^4$ theories in 2 and 3 dimensions and pointed out that is a parameter region where $T \neq 0$ perturbation is safe and reliable even at high temperature, in contrast to the 4 dim. case. We have also shown within the safe region of parameters that the broken $O(N) \times O(N)$ symmetry is restored at high temperature using the self-consistent method.

Acknowledgement. One of us (Y.F.) wishes to thank Profs. J.T. Lewis and L. O'Raifeartaigh for hospitality at DIAS.

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