12 Kaluza-Klein-Reduction from 4 to 2 dimensions

As an exercise in getting familiar with the Kaluza Klein reduction, I shall show how the 4 dimensionsional Wess-Zumino-model is reduced to an two-dimensional model with extended supersymmetry.

12.1 Reduction to a $N = 2$ Lorentzian model in $2d$

In 4 dimension a Majorana spinor has 4 real components which are two times the 2 real components of Majorana spinor in 2 dimensions (which exist for both signatures $(+,-)$ and $(+,+)$).
We start with the Dirac matrices γ^μ in 2-dimensional Minkowski space and give an explicit realization for the matrices $\Gamma^m, m = 0, \ldots, 3$ in 4 dimensions: We make the ansatz

$$\Gamma_\mu = \Delta \otimes \gamma_\mu, \quad \Gamma_{1+a} = \Delta_a \otimes \gamma_*, \quad \mu = 0, 1, \quad a = 1, 2, \qquad \gamma_* = -\gamma_0 \gamma_1$$

The condition on the Δ-factors such that

$$\{\Gamma_m, \Gamma_n\} = 2\eta_{mn}, \qquad \eta = \mathrm{diag}(1, -1, -1, -1) \tag{12.1}$$

is easily seen to read

$$\Delta^2 = \mathbb{1}_2, \quad [\Delta, \Delta_a] = 0, \qquad \{\Delta_a, \Delta_b\} = -2\delta_{ab}\mathbb{1}_2.$$

Since Γ^0 must be hermitean and the $\Gamma^{m>0}$ antihermitean and since $\Gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ is hermitean it also follows that

$$\Delta^\dagger = \Delta, \quad \Delta_a^\dagger = -\Delta_a.$$

Since Δ commutes with all matrices we may choose it to be the identity,

$$\Delta = \mathbb{1}_2.$$

Note that the hermitean $i\Delta_a$ generate the Euclidean Clifford algebra in 2 dimensions and that the $[\Delta_a, \Delta_b]$ generate the group $spin(2)$. Earlier we have shown that in 2 dimensions there is a Majorana representation. Hence we may choose Δ_a to be imaginary and hence symmetric. To construct this Majorana representation we take

$$\Delta_1 = i\tau_1 \quad \text{and} \quad \Delta_2 = i\tau_3,$$

With our earlier convention the hermitean Γ_5 takes the form

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = -i\Delta_1\Delta_2 \otimes \gamma_0\gamma_1 = -\tau_2 \otimes \gamma_*. \tag{12.2}$$

We choose a Majorana representation for the 2-dimensional γ_μ such that they are imaginary (see section 3.5) and $\gamma_* = -\gamma_0\gamma_1$ is real. Since the Δ_a are imaginary it follows that the Γ_m are purely imaginary as well and that $\mathcal{C}_4 = -\Gamma_0 = -\mathbb{1}_2 \otimes \gamma_0$. Hence we are lead to take the following relation between the conjugation matrices in 4 and 2 dimensions,

$$\mathcal{C}_4 = -\Gamma_0 = \mathbb{1}_2 \otimes \mathcal{C}_2, \tag{12.3}$$

where $\mathcal{C}_2 = -\gamma_0$ is the antisymmetric charge conjugation matrix in 2 dimensions. We have

$$\mathcal{C}_4^{-1}\Gamma_m\mathcal{C}_4 = -\Gamma_m^T, \qquad \mathcal{C}_2^{-1}\gamma_\mu\mathcal{C}_2 = -\gamma_\mu^T. \tag{12.4}$$

Now its easy to reduce a Majorana spinor, since the Majorana condition reads

$$\Psi = \xi \otimes \chi = \Psi_c = \Psi^* \Longleftrightarrow \xi \in \mathbb{R}^2,\ \chi = \chi_c = \chi^*, \tag{12.5}$$

and an arbitrary Majorana spinor in 4 space-time dimensions has the expansion

$$\Psi = \sum_{r=1}^{2} e_r \otimes \chi_r, \tag{12.6}$$

where the χ_r are Majorana spinors in 2 dimensions and the e_r form a (real) base in $\mathbb{R}^2$.

12.1.1 Reduction of the $4d$ action

In 4 and 2 dimensions a spinor field has the dimension

$$[\Psi] = L^{-3/2} \quad \text{and} \quad [\chi] = L^{-1/2},$$

respectively. We start with the expansion (12.6) for a Majorana spinor and its adjoint in 4 dimensions. We rescale the spinors such that the χ_r in

$$\Psi = \frac{1}{\sqrt{V_2}}\sum_{r=1}^{2} e_r \otimes \chi_r, \quad \bar\Psi = \frac{1}{\sqrt{V_2}}\sum_{r=1}^{2} e_r^T \otimes \bar\chi_r$$

have the dimension of a spinorfield in 2-dimensional Minkowski spacetime. The spinor should be independent of the internal coordinates which are the coordinates x^2 and x^3. We obtain for the fermionic bilinears

$$\begin{aligned}\int d^4x\, \bar\Psi\Gamma^m\partial_m\Psi &= \sum_r \int d^2x\, \bar\chi_r \not\partial \chi_r \\ \int d^4x\, \bar\Psi\Psi &= \sum_r \int d^2x\, \bar\chi_r\chi_r\end{aligned}$$

$$\int d^4x\,\bar\Psi\Gamma_5\Psi \quad = \quad -\int d^2x\,(\tau_2)_{rs}\,\bar\chi_r\gamma_*\chi_s$$

Now we are ready to reduce the Wess-Zumino model with Lagrangian

$$\begin{aligned}\mathcal{L}_4 \quad = \quad & \tfrac{1}{2}\partial_m A\partial^m A + \tfrac{1}{2}\partial_m B\partial^m B - \tfrac{1}{2}m^2(A^2+B^2) + \tfrac{i}{2}\bar\Psi\Gamma^m\partial_m\Psi - \tfrac{1}{2}m\bar\Psi\Psi \\ & -mgA(A^2+B^2) - \tfrac{1}{2}g^2(A^2+B^2)^2 - g\bar\Psi(A - i\Gamma_5 B)\Psi\end{aligned} \tag{12.7}$$

to two dimensions. For that we observe that a scalar field in $d=4$ has dimension L^{-1}, whereas in $d=2$ it has no dimension. Hence we make the replacements

$$A \longrightarrow \frac{1}{\sqrt{V_2}}A \quad \text{and} \quad B \longrightarrow \frac{1}{\sqrt{V_2}}B.$$

If we finally relate the coupling constants in 4 and 2 dimensions as

$$g_2 = \frac{g_4}{\sqrt{V_2}},$$

then the action of the 2 dimensional model, given by

$$S_4 = \int d^4x\mathcal{L}_4 = S_2 = \int d^2x\mathcal{L}_2$$

reads

$$\begin{aligned}\mathcal{L}_2 \quad = \quad & \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{1}{2}\partial_\mu B\partial^\mu B - \tfrac{1}{2}m^2(A^2+B^2) + \tfrac{i}{2}\bar\chi\not\partial\chi - \tfrac{1}{2}m\bar\chi\chi \\ & \qquad -mgA(A^2+B^2) - \tfrac{1}{2}g^2(A^2+B^2)^2 - g\bar\chi(A + i\tau_2\otimes\gamma_* B)\chi.\end{aligned} \tag{12.8}$$

Here χ is a Majorana spinor with the two components χ_1 and χ_2. These two flavours maybe combined to a Dirac spinor. The first factor in $\tau_2\otimes\gamma_*$ affects the flavour indices.
The Hamiltonian density is

$$\begin{aligned}\mathcal{H}_2 \quad = \quad & \tfrac{1}{2}\Big(\pi_A^2 + \pi_B^2 + (\nabla A)^2 + (\nabla B)^2 + m^2A^2 + m^2B^2\Big) - \tfrac{i}{2}\chi^\dagger\gamma^0\gamma^i\partial_i\chi + \tfrac{1}{2}m\bar\chi\chi \\ & \qquad +mgA(A^2+B^2) + \tfrac{1}{2}g^2(A^2+B^2)^2 + g\bar\chi(A + i\tau_2\otimes\gamma_* B)\chi.\end{aligned} \tag{12.9}$$

12.1.2 R-symmetry

In 4 dimensions a scalar- and spinorfield transform under Lorentz transformations as

$$A(x) \longrightarrow A(\Lambda_4^{-1}x) \quad \text{and} \quad \Psi(x) \longrightarrow S_4\Psi(\Lambda_4^{-1}x), \quad \bar\Psi(x) \longrightarrow \bar\Psi(\Lambda_4^{-1}x)S_4^{-1},$$

where

$$S_4 = \exp\left(\tfrac{1}{2}\omega_{mn}\Gamma^{mn}\right) \quad \text{and} \quad (\Lambda_4)^m_{\ n} = \left(e^\omega\right)^m_{\ n}, \qquad \Gamma_{mn} = \tfrac{1}{2}[\Gamma_m,\Gamma_n]$$

are the spinrotation and Lorentz transformation with parameter ω_{mn} in 4-dimensional spacetime. They are related via

$$S_4^{-1}\Gamma^m S_4 = (\Lambda_4)^m_{\ n}\Gamma^n.$$

The Lagrangian density is as scalar field. When we reduce the theory to $\mathbb{R}^2$ we must require that the fields do not depend on the internal coordinates. Clearly this condition is not compatible with the 4 dimensional Lorentz invariance. Only those Lorentz transformations survive which do not mix the coordinates on $\mathbb{R}^4$ with those in the internal space, hence

$$SO(1,3) \longrightarrow SO(1,1)\times SO(2) \quad \text{or} \quad \Lambda_4 \longrightarrow \begin{pmatrix} \Lambda & 0 \\ 0 & R \end{pmatrix},$$

where Λ is a 2-dimensional Lorentz transformation and $R \in O(2)$[1]. With our choice for the Γ_m the generators of the corresponding spin transformations in 4 dimensions read

$$\Gamma_{\mu\nu} = \mathbb{1}_2 \otimes \gamma_{\mu\nu}, \quad \Gamma_{1+a,1+b} = \Delta_{ab}\otimes \mathbb{1}_2 \quad \text{and} \quad \Gamma_{\mu,1+a} = 2\Delta_a \otimes \gamma_\mu\gamma_* \tag{12.10}$$

The Δ_{ab} generate the spin(2) subalgebra of *spin*$(1,3)$ and the $\gamma_{\mu\nu}$ the spin$(1,1)$ subalgebra. Since the $\Gamma_{\mu\nu}$ act trivially on the first factor in the decomposition $\Psi = \xi\otimes\chi$, and the $\Gamma_{1+a,1+b}$ trivially on the second one, the spin rotations compatible with the dimensional reduction decompose as follows:

$$S_4 = S_2\otimes S, \quad S\Psi = \sum_r S_2 e_r \otimes S\chi_r, \qquad S_2^{-1}\Delta^a S_2 = R^a_{\ b}\Delta^b, \quad S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu,$$

where S is a spin rotation acting on spinors in 2-dimensional Minkowski spacetime. To summarize, the remaining Lorentz transformations reduce to Lorentz transformations on the two dimensional spacetime,

$$S: \ \chi_a(x) \longrightarrow S\chi_a(\Lambda^{-1}x) \quad , \quad A(x)\to A(\Lambda^{-1}x), \quad B(x)\to B(\Lambda^{-1}x),$$

and to a symmetry which does not act on the coordinates of $\mathbb{R}^2$:

$$S_2 : e_r \longrightarrow S_2 e_r \quad , \quad A(x)\to A(x), \quad B(x)\to B(x)$$

We see that the spin rotations S_2 reduce to *internal symmetries* which rotate the Majorana fields without changing the coordinates (on $\mathbb{R}^2$). With our choice for the Δ_a we have $\Delta_{12} = i\tau_2$ such that

$$S_2 = e^{\omega\Delta_{12}} = e^{i\omega\tau_2} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\otimes \mathbb{1}_2 \equiv R,$$

where we have set $\omega^{23} = \omega$. Hence, spinors transform under the R transformations as

$$\Psi \longrightarrow \sum_r Re_r\otimes\chi_r = \sum_{rs} R_{sr}e_s\otimes\chi_r = \sum_s e_s\otimes (R\chi)_s.$$

Hence we may interpret the R-symmetry to act on the Majorana spinors. We conclude, that the reduced model has the following $U(1)$ invariance,

$$\chi \longrightarrow R\chi = e^{i\alpha\tau_2}\chi.$$

[1]actually, only $\det\Lambda\cdot\det R = 1$ is required.

12.1.3 Chiral rotations

To investigate the behaviour under chiral rotations we rewrite the Lagrangian density in terms of the Dirac spinor

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2).$$

With the identities

$$\bar{\psi}\psi = \tfrac{1}{2}\bar{\chi}\chi, \quad \bar{\psi}\partial\!\!\!/\,\psi = \tfrac{1}{2}\bar{\chi}\partial\!\!\!/\,\chi + \tfrac{i}{2}\partial_\mu(\bar{\chi}_1\gamma^\mu\chi_2), \quad \bar{\psi}\gamma_*\psi = -\tfrac{1}{2}\bar{\chi}\tau_2 \otimes \gamma_*\chi,$$

where we made use of (11.3), the density for the massless model takes the following simple form,

$$\mathcal{L}_2 = \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{1}{2}\partial_\mu B\partial^\mu B + i\bar{\psi}\partial\!\!\!/\,\psi - \tfrac{1}{2}g^2(A^2+B^2)^2 - 2g\bar{\psi}(A - i\gamma_* B)\psi.$$

It is hermitean and possesses the following $U(1)$ chiral symmetry,

$$\psi \longrightarrow e^{i\alpha\gamma_*}\psi, \qquad \bar{\psi} \longrightarrow \bar{\psi}e^{i\alpha\gamma_*} \quad \text{and} \quad \begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

12.1.4 Supersymmetry transformations

To find the susy transformations we dimensionally reduce (5.21) by expanding

$$\Psi = \sum e_r \otimes \chi_r \quad \text{and} \quad \alpha = \sum e_r \otimes \beta_r$$

with two dimensional real supersymmetry parameters β_r. We easily obtain

$$\begin{aligned} \delta A &= \sum \bar{\beta}_r\chi_r, \qquad \delta B = -i\sum(\tau_2)_{rs}\bar{\beta}_r\gamma_*\chi_s \\ \delta\chi_r &= -\big(i\partial\!\!\!/\,A + g(A^2 - B^2)\big)\beta_r - (\tau_2)_{rs}\gamma^\mu\gamma_*\beta_s\partial_\mu B + 2igAB\,(\tau_2)_{rs}\gamma_*\beta_s. \end{aligned}$$

Again we introduce the Dirac field and $\alpha = (\beta_1 + i\beta_2)/\sqrt{2}$ and obtain

$$\begin{aligned} \delta A &= \bar{\alpha}\psi + \bar{\alpha}_c\psi_c \quad , \quad \delta B = i\big(\bar{\alpha}\gamma_*\psi - \bar{\alpha}_c\gamma_*\psi_c\big) \\ \delta\psi &= -\big\{i\partial\!\!\!/\,(A + i\gamma_* B) + g(A + i\gamma_* B)^2\big\}\alpha. \end{aligned}$$

To reduce the susy algebra of the supercharges, we insert

$$Q = \sum e_r \otimes q_r, \quad \alpha = \sum e_r \otimes \beta_r$$

into the 4-dimensional algebra

$$[\bar{Q}\alpha_1, \bar{\alpha}_2 Q] = -2i(\bar{\alpha}_2\Gamma^m\alpha_1)\partial_m \longrightarrow -2i(\bar{\alpha}_2\gamma^\mu\alpha_1)\partial_\mu, \qquad \mu = 0, 1. \tag{12.11}$$

Then the susy algebra reduces to the following $N = 2$ algebra for the Majorana supercharges q_1 and q_2:

$$\{q_{i\alpha}, \bar{q}_j^{\,\beta}\} \;=\; 2i\delta_{ij}(\gamma^\mu)_\alpha{}^\beta\partial_\mu$$

Rewritten in terms of the Dirac-charges

$$Q_\alpha = \frac{1}{\sqrt{2}}(q_1 + iq_2)$$

this reads

$$\{Q_\alpha, \bar{Q}^\beta\} = 2i(\gamma^\mu)_\alpha{}^\beta \partial_\mu.$$

12.2 Reduction of the $4d$ WZ-model to a $2d$ Euclidean model

We reduce the model on 4-dimensional Minkowki spacetime (Γ_m) to a model on 2-dimensional Euclidean space with Euclidean γ-matrices γ_μ and $\gamma_* = -i\gamma_0\gamma_1$. We make the ansatz

$$\Gamma_0 = \Delta_0 \otimes \mathbb{1}_2, \quad \Gamma_1 = i\Delta_1 \otimes \mathbb{1}_2, \quad \Gamma_{2+\mu} = i\Delta_* \otimes \gamma_\mu, \quad \mu = 0, 1, \quad \Delta_* = -i\Delta_0\Delta_1,$$

where the hermitean γ_μ and hermitean Δ_a generate the 2-dimensional Euclidean Clifford algebras,

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}_2 \quad \text{and} \quad \{\Delta_a, \Delta_b\} = 2\delta_{ab}\mathbb{1}_2.$$

With our earlier convention the hermitean Γ_5 reads

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = -\Delta_0\Delta_1 \otimes \gamma_0\gamma_1 = \Delta_* \otimes \gamma_*. \tag{12.12}$$

A 4-dimensional Majorana spinor has the expansion

$$\Psi = \frac{1}{\sqrt{2}}\sum_{r=1}^{2} e_r \otimes \chi_r, \tag{12.13}$$

where the χ_r are spinors in 2 Euclidean dimensions and the e_r form a (real) base in $\mathbb{R}^2$.

12.2.1 The Majorana conditions

We write the 4-dimensial charge conjugation matrix as

$$\mathcal{C}_4 = a \otimes \mathcal{C}_2, \qquad \mathcal{C}^{-1}\Gamma_m\mathcal{C} = -\Gamma_m^T.$$

Because of the particular form of Γ_0 and Γ_1 the last condition implies for $m = 0, 1$

$$a^{-1}\Delta_a a = -\Delta_a^T \Longrightarrow a^{-1}\Delta_* a = -\Delta_*^T.$$

With the last identity the factor $\mathcal{C}_2$ must then fulfil

$$\mathcal{C}_2^{-1}\gamma_\mu\mathcal{C}_2 = \gamma_\mu^T,$$

which just means, that it is a charge conjugation matrix in an Euclidean spacetime with signature $(+,+)$. In such a spacetime there are *real and symmetric* γ_μ giving rise to an imaginary and antisymmetric γ_*. In this representation we have $\mathcal{C}_2 = \mathbb{1}_2$. To obtain purely imaginary Γ_m in which case $\mathcal{C}_4 = -\Gamma_0$ and the 4-dimensional Majorana spinors become real, we could take

$$\Delta_0 = \tau_2 \quad \text{and} \quad \Delta_1 = \tau_3 \Longrightarrow \Delta_* = \tau_1. \tag{12.14}$$

To summarize we may choose a representation such that all Γ_m are imaginary and

$$\mathcal{C}_4 = -\Gamma_0 = -\tau_2 \otimes \mathbb{1}_2. \tag{12.15}$$

It follows, that Majorana spinors Ψ in $4d$ Minkowski spacetime and Majora spinors χ in $2d$ Euclidean space are both real. The last statement comes from

$$\chi = \chi_c = \mathcal{C}_2(\chi^\dagger)^T = \chi^*.$$

The choosen representation is very convenient, since in the expansion (12.13) all spinors happen to be real Majoran spinors. For the 2-dimensional Majorana spinors we have the identities listed above and below (11.4).

12.2.2 Reduction of $4d$ Wess-Zumino Lagrangian

Again we start with the expansion (12.13) for a Majorana spinor and its Dirac conjugate

$$\bar\Psi = \frac{1}{\sqrt{V_2}} \sum_{r=1}^{2} e_r^T \Delta_0 \otimes \chi_r^\dagger$$

The spinor should be independent of the internal coordinates which are the coordinates x^0 and x^1. We obtain for the fermionic bilinears

$$\begin{aligned}
\int d^4x\, \bar\Psi \Gamma^m \partial_m \Psi &= -\int d^2x\, (\Delta_1)_{rs} \chi_r^\dagger \not{\partial} \chi_s \\
\int d^4x\, \bar\Psi \Psi &= \int d^2x\, (\Delta_0)_{rs} \chi_r^\dagger \chi_s \\
\int d^4x\, \bar\Psi \Gamma_5 \Psi &= -i \int d^2x\, (\Delta_1)_{rs}\, \chi_r^\dagger \gamma_* \chi_s
\end{aligned}$$

Now we are ready to reduce the Wess-Zumino model with Lagrangian (12.7) to two Euclidean dimensions,

$$-\mathcal{L}_2 = \mathcal{L}_B + \mathcal{L}_F \tag{12.16}$$

with

$$\mathcal{L}_B = \tfrac{1}{2}\Big(\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + m^2(A^2+B^2) + 2mgA(A^2+B^2) + g^2(A^2+B^2)^2\Big). \tag{12.17}$$

and

$$\mathcal{L}_F = \tfrac{i}{2}(\Delta_1)_{rs}\chi_r^\dagger \not{\partial} \chi_s + \tfrac{1}{2}m(\Delta_0)_{rs}\chi_r^\dagger\chi_s + gA(\Delta_0)_{rs}\chi_r^\dagger\chi_s - gB(\Delta_1)_{rs}\chi_r^\dagger\gamma_*\chi_s, \tag{12.18}$$

where we choose the representation $\Delta_0 = \tau_2$ and $\Delta_1 = \tau_1$. With the hermiticity properties worked

out in section (11.2), above (11.20) one shows that this gives rise to a hermitean action. For that one uses that Δ_0 is imaginary and Δ_1 real.
To reduce the susy transformations (5.21) we expand the Majorana susy parameter in 4 dimensions similarly as the Majorana spinor field Ψ as follows,

$$\alpha = \sum e_r \otimes \beta_r \quad \text{and} \quad \bar{\alpha} = \sum e_r^T \Delta_0 \otimes \beta_r^\dagger$$

In terms of these Majorana parameters in 2 Euclidean dimensions we obtain

$$\begin{aligned} \delta A &= \beta^\dagger \tau_2 \chi \quad , \quad \delta B = \beta^\dagger \tau_3 \gamma_* \chi \\ \delta \chi &= \big\{ i\gamma_* \partial\!\!\!/ B - mA - g(A^2 - B^2) \big\} \beta + i \big\{ i\partial\!\!\!/ A - (mB + 2gAB)\gamma_* \big\} \tau_1 \beta. \end{aligned} \tag{12.19}$$

The summations convention used is evident. Later we shall combine the two flavours χ_1 and χ_2 to a Dirac spinor.

12.2.3 R-symmetry

We reduce the theory to $\mathbb{R}^2$ by requireing that the fields do not depend on the internal coordinates which we choose to be x^0 and x^1.

$$SO(1,3) \longrightarrow SO(1,1) \times SO(2) \quad \text{or} \quad \Lambda_4 \longrightarrow \begin{pmatrix} \Lambda & 0 \\ 0 & R \end{pmatrix},$$

where Λ is now considered as internal symmetry and and $R \in O(2)$ as Lorentz transformation in the Euclidean 2-space. With our choice for the Γ_m the generators of the corresponding spin transformations in 4 dimensions read

$$\Gamma_{01} = -\Delta_* \otimes \mathbb{1}_2, \quad \Gamma_{23} = -\mathbb{1}_2 \otimes \gamma_{01} = -i\mathbb{1}_2 \otimes \gamma_*. \tag{12.20}$$

They yield the noncompact internal transformations $R = \exp(\omega_{01}\Delta_*) \otimes \mathbb{1}_2$ and the compact spinor transformations $S_2 = \mathbb{1}_2 \otimes \exp(-i\omega_{23}\gamma_*)$. The internal symmetry is ($\omega = \omega_{01}$)

$$\chi \longrightarrow e^{\omega \Delta_*} \chi \quad , \quad \chi^\dagger \longrightarrow \chi^\dagger e^{\omega \Delta_*} \quad \text{where} \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{12.21}$$

The invariance of the action under this R-symmetry is evident, since both Δ_0 and Δ_1 anticommute with Δ_*. With the choice (12.14) the R-symmetry is given by

$$e^{\omega \Delta_*} = e^{\omega \tau_1} = \begin{pmatrix} \cosh\omega & \sinh\omega \\ \sinh\omega & \cosh\omega \end{pmatrix}.$$

12.2.4 Chiral symmetry

The model (12.18) seems not to be chirally invariant under a compact transformation. That is not true! It is actually invariant under the rotations

$$\chi \longrightarrow e^{i\alpha \Delta_* \otimes \gamma_*} \chi \quad \text{and} \quad \begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

To prove the invariance one uses

$$\begin{aligned} e^{-i\alpha\Delta_*\otimes\gamma_*}(\Delta_0\otimes \mathbb{1}_2)e^{i\alpha\Delta_*\otimes\gamma_*} &= \Delta_0\otimes\mathbb{1}_2\cos 2\alpha+\Delta_1\otimes\gamma_*\sin 2\alpha \\ e^{-i\alpha\Delta_*\otimes\gamma_*}(\Delta_1\otimes \gamma_*)e^{i\alpha\Delta_*\otimes\gamma_*} &= \Delta_1\otimes\gamma_*\cos 2\alpha-\Delta_0\otimes\mathbb{1}_2\sin 2\alpha. \end{aligned}$$

For the choice $\Delta_*=\tau_1$ the transformation reads

$$\begin{pmatrix}\chi_1\\ i\gamma_*\chi_2\end{pmatrix}\longrightarrow\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\chi_1\\ i\gamma_*\chi_2\end{pmatrix}$$

or equivalently

$$\begin{aligned} \chi_1+\chi_2 &\longrightarrow e^{i\alpha\gamma_*}(\chi_1+\chi_2)\\ \chi_1-\chi_2 &\longrightarrow e^{-i\alpha\gamma_*}(\chi_1-\chi_2). \end{aligned}$$

It follows that

$$\psi=\frac{1}{\sqrt{2}}(\chi_1+i\chi_2)\quad\text{and}\quad\psi_c=\frac{1}{\sqrt{2}}(\chi_1-i\chi_2) \tag{12.22}$$

transforms as follows under chiral rotations,

$$\begin{pmatrix}\psi\\ \psi_c\end{pmatrix}\longrightarrow\begin{pmatrix}\cos\alpha & -\gamma_*\sin\alpha\\ \sin\alpha\gamma_* & \cos\alpha\end{pmatrix}\begin{pmatrix}\psi\\ \psi_c\end{pmatrix}.$$

12.2.5 Rewriting the $2d$ model in a Dirac basis

The chiral symmetry and supersymmetry is more transparent if we use the Dirac spinors (12.22). With the help of the relations above (11.20) one shows, that the bilinears are related as follows:

$$\begin{aligned} \psi^\dagger\psi=-\psi_c^\dagger\psi_c=-\tfrac12\chi^\dagger\tau_2\chi \quad &,\quad \psi^\dagger\psi_c=-\psi_c^\dagger\psi=-\tfrac{i}{2}\chi^\dagger\tau_1\chi\\ \psi^\dagger\gamma_*\psi_c=\tfrac12\chi^\dagger\tau_3\gamma_*\chi-\tfrac{i}{2}\chi^\dagger\gamma_*\chi \quad &,\quad \psi_c^\dagger\gamma_*\psi=\tfrac12\chi^\dagger\tau_3\gamma_*\chi+\tfrac{i}{2}\chi^\dagger\gamma_*\chi\\ \psi^\dagger\not{\partial}\,\psi_c=\tfrac12\chi^\dagger\tau_3\not{\partial}\,\chi-\tfrac{i}{2}\chi^\dagger\tau_1\not{\partial}\,\chi \quad &,\quad \psi_c^\dagger\not{\partial}\,\psi=\tfrac12\chi^\dagger\tau_3\not{\partial}\,\chi+\tfrac{i}{2}\chi^\dagger\tau_1\not{\partial}\,\chi. \end{aligned} \tag{12.23}$$

Hence we may rewrite the fermionic Lagrangian as

$$\mathcal{L}_F=\tfrac{i}{2}\big(\psi^\dagger\not{\partial}\,\psi_c+\psi_c^\dagger\not{\partial}\,\psi\big)-m\psi^\dagger\psi-2gA\psi^\dagger\psi-gB\big(\psi^\dagger\gamma_*\psi_c+\psi_c^\dagger\gamma_*\psi\big). \tag{12.24}$$

To rewrite the susy transformations (12.19) in terms of Dirac spinors we introduce the complex supersymmetry parameter

$$\alpha=\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\quad\text{and}\quad\alpha_c=\frac{1}{\sqrt{2}}(\beta_1-i\beta_2),$$

and which should be distinguished from the 4-dimensional parameters, for which we used the same symbol. With the identities

$$\begin{aligned} 2\alpha^\dagger\psi=\beta^\dagger\chi-\beta^\dagger\tau_2\chi \quad &,\quad 2\alpha_c^\dagger\psi_c=\beta^\dagger\chi+\beta^\dagger\tau_2\chi\\ 2\alpha^\dagger\gamma_*\psi_c=\beta^\dagger\tau_3\gamma_*\chi-i\beta^\dagger\tau_1\gamma_*\chi \quad &,\quad 2\alpha_c^\dagger\psi=\beta^\dagger\tau_3\gamma_*\chi+i\beta^\dagger\tau_1\gamma_*\chi \end{aligned}$$

the susy transformations become

$$\begin{aligned}\delta A &= \alpha_c^\dagger\psi_c - \alpha^\dagger\psi \quad , \quad \delta B = \alpha^\dagger\gamma_*\psi_c + \alpha_c^\dagger\gamma_*\psi \\ \delta\psi &= \{i\gamma_*\not\partial B - mA - g(A^2-B^2)\}\alpha - \{i\not\partial A - (mB+2gAB)\gamma_*\}\alpha_c \\ \delta\psi_c &= \{i\gamma_*\not\partial B - mA - g(A^2-B^2)\}\alpha_c + \{i\not\partial A - (mB+2gAB)\gamma_*\}\alpha\end{aligned}$$

To construct the susy algebra, we again decompose the 4-supercharges as

$$Q = \sum e_r \otimes q_r, \quad \bar{Q} = \sum e_r^T \Delta_0 \otimes q_r^\dagger$$

with Majorana supercharges q_1, q_2 in 2 Euclidean dimensions. We correspondingly expand the 4-dimensional supersymmetry parametes as

$$\alpha = \sum e_r \otimes \beta_r \quad \text{and} \quad \tilde{\alpha} = \sum e_r \otimes \tilde{\beta}_r$$

such that

$$\bar{Q}\tilde{\alpha} = q^\dagger \tau_2 \tilde{\beta} \quad \text{and} \quad \bar{\alpha}Q = \beta^\dagger \tau_2 q.$$

Now we introduce the two dimensional Dirac spinors

$$\epsilon = \frac{1}{\sqrt{2}}(\beta_1 + i\beta_2), \quad \tilde{\epsilon} = \frac{1}{\sqrt{2}}(\tilde{\beta}_1 + i\tilde{\beta}_2) \quad \text{and} \quad q = \frac{1}{\sqrt{2}}(q_1 + iq_2)$$

such that

$$\bar{Q}\tilde{\alpha} = q_c^\dagger\tilde{\epsilon}_c - q^\dagger\tilde{\epsilon} \quad \text{and} \quad \bar{\alpha}Q = \epsilon_c^\dagger q_c - \epsilon^\dagger q.$$

Inserting this into

$$[\bar{Q}\tilde{\alpha}, \bar{\alpha}Q] = -2i(\bar{\alpha}\Gamma^m\tilde{\alpha})\partial_m \longrightarrow -2(\bar{\alpha}\Delta_* \otimes \gamma^\mu\alpha)\partial_\mu = 2i\beta^\dagger\tau_3\gamma^\mu\tilde{\beta}\partial_\mu.$$

With

$$\epsilon^\dagger\gamma^\mu\tilde{\epsilon}_c = \tfrac{1}{2}\beta^\dagger(\tau_3 - i\tau_1)\gamma^\mu\tilde{\epsilon} \quad \text{and} \quad \epsilon_c^\dagger\gamma^\mu\tilde{\epsilon} = \tfrac{1}{2}\beta^\dagger(\tau_3 + i\tau_1)\gamma^\mu\tilde{\epsilon}$$

we obtain

$$[q_c^\dagger\tilde{\epsilon}_c - q^\dagger\tilde{\epsilon}, \epsilon_c^\dagger q_c - \epsilon^\dagger q] = 2i\big(\epsilon^\dagger\gamma^\mu\tilde{\epsilon}_c + \epsilon_c^\dagger\gamma^\mu\tilde{\epsilon}\big)\partial_\mu.$$

$$\{q_{i\alpha}, q_j^{\dagger\beta}\} = -2i(\tau_3)_{ij}(\gamma^\mu)_\alpha{}^\beta\partial_\mu.$$

For the Dirac supercharges

$$Q = \frac{1}{\sqrt{2}}(q_1 + iq_2) \quad \text{and} \quad Q^\dagger = \frac{1}{\sqrt{2}}(q_1 - iq_2)$$

we obtain

$$\{Q_\alpha, Q^{\dagger\beta}\} = 2(\gamma^\mu)_\alpha{}^\beta\partial_\mu \quad \text{and} \quad \{Q_\alpha, Q_\beta\} = -2(\gamma^\mu)_\alpha{}^\beta\partial_\mu.$$

(There must be an error here)!!!!!