11 Supersymmetric $N = 1$ WZ-models in two Dimensions

The purpose of this section is to present again the ideas of supersymmetry and to elaborate upon them using the simpler examples of two dimensional SUSY-models. Also we shall investigate the Euclidean models and the relation between the Minkowskian and Euclidean formulation in some detail. On the way the Ward-identities are discussed and are checked explicitly for some simple models.
In two dimensions the Lorentz group is Abelian. It is the twofold covering of

$$SO(1,1) \quad \text{and} \quad SO(2)$$

in Minkowski spacetime and Euclidean spacetime, respectively. According to our general analysis in chapter 2 there are Majorana or Weyl spinors in two dimensions for any choice of signature. In addition, in two-dimensional Minkowki spacetime there are Majorana-Weyl fermions.

11.1 Wess-Zumino model in $2d$-Minkowski spacetime

When discussing supersymmetric models it is always useful to recall the relevant properties of spinors which are relevant for the model under consideration. Although we have discussed these properties for fermions in spaces of arbitrary dimensions and signatures, it is useful to recall and further extend these results for the probably simplest case, namely 2-dimensional Minkowski space. The general Fierz identity (3.64) simplifies to

$$2\psi\bar{\chi} = -(\bar{\chi}\psi) - \gamma_\mu(\bar{\chi}\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu}\psi) = -(\bar{\chi}\psi) - \gamma_\mu(\bar{\chi}\gamma^\mu\psi) - \gamma_*(\bar{\chi}\gamma_*\psi) \tag{11.1}$$

There is a charge conjugation matrix with

$$\eta = \epsilon = 1$$

and when we choose the Majorana representation

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \text{such that} \quad \gamma_* = -\gamma_0\gamma_1 = \sigma_3 \tag{11.2}$$

then the antisymmetric charge conjugation matrix reads $\mathcal{C} = -\gamma^0$. For Majorana spinors the symmetry properties (3.53) read

$$\bar\psi\chi=\bar\chi\psi,\quad \bar\psi\gamma^\mu\chi=-\bar\chi\gamma^\mu\psi\quad\text{and}\quad \bar\psi\gamma_*\chi=-\bar\chi\gamma_*\psi. \tag{11.3}$$

Some further Fierz identities are easily derived from these properties: For Majorana spinors we find

$$\begin{aligned}(\bar\alpha\psi)(\bar\alpha\psi) &= (\bar\psi\alpha)(\bar\alpha\psi)=\bar\psi\big(\alpha\bar\alpha\big)\psi=-\tfrac{1}{2}(\bar\alpha\alpha)(\bar\psi\psi)\\ (\bar\psi\gamma^\mu\alpha)\psi &= (\psi\bar\psi)\gamma^\mu\alpha=-\tfrac{1}{2}(\bar\psi\psi)\gamma^\mu\alpha.\end{aligned}$$

In addition

$$\bar\psi\chi,\ \bar\psi\gamma_*\chi\quad\text{are hermitean and}\qquad \bar\psi\gamma^\mu\chi\quad\text{is antihermitean.} \tag{11.4}$$

To construct the Wess-Zumino model in $1+1$ dimensions we need a *real superfield* $\phi(x,\alpha)$ which has the following expansion

$$\Phi(x,\alpha)=A(x)+\bar\alpha\psi(x)+\tfrac{1}{2}\bar\alpha\alpha F(x) \tag{11.5}$$

with real (pseudo)scalar fields A,F and Majorana spinor ψ. Its supersymmetry variation is generated by the supercharge,

$$\delta_\beta\Phi=i\bar\beta Q\Phi,\qquad Q=-i\frac{\partial}{\partial\bar\alpha}-(\gamma^\mu\alpha)\partial_\mu. \tag{11.6}$$

The supersymmetry variation of the component fields follows from

$$\begin{aligned}\delta_\beta\Phi &= \big(\bar\beta\frac{\partial}{\partial\bar\alpha}-i(\bar\beta\gamma^\mu\alpha)\partial_\mu\big)\Phi=\bar\beta\psi+\bar\beta\alpha F-i(\bar\beta\gamma^\mu\alpha)\partial_\mu A-i(\bar\beta\gamma^\mu\alpha)\bar\alpha\partial_\mu\psi\\ &= \delta A+\bar\alpha\delta\psi+\tfrac{1}{2}\bar\alpha\alpha\delta F.\end{aligned}$$

We read off the following transformations of the component fields

$$\delta A=\bar\beta\psi,\quad \delta\psi=(F+i\partial\!\!\!/ A)\beta\Rightarrow\delta\bar\psi=\bar\beta(F-i\partial\!\!\!/ A),\quad \delta F=i\bar\beta\partial\!\!\!/\psi. \tag{11.7}$$

Calculating the commutator of two susy transformations is considerably simpler as in four dimensions:

$$\begin{aligned}[][\delta_1,\delta_2]A &= 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu A\\ [\delta_1,\delta_2]\psi &= i(\beta_2\bar\beta_1-\beta_1\bar\beta_2)\partial\!\!\!/\psi+i\gamma^\mu(\beta_2\bar\beta_1-\beta_1\bar\beta_2)\partial_\mu\psi=2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu\psi\\ [\delta_1,\delta_2]F &= 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu F.\end{aligned}$$

The corresponding anticommutator of two supercharges are

$$\{Q_\alpha,\bar Q^\beta\}=2(\gamma^\mu)_\alpha{}^\beta P_\mu. \tag{11.8}$$

Multiplying with γ_0 and summing over the spinor indices yields

$$\sum_{\alpha=1,2}\{Q_\alpha,Q^\dagger_\alpha\}=4P_0\geq 0.$$

The supercovariant derivatives read

$$D=\frac{\partial}{\partial\bar{\alpha}}+i(\gamma^\mu\alpha)\partial_\mu \quad\text{and}\quad \bar{D}=-\frac{\partial}{\partial\alpha}-i(\bar{\alpha}\gamma^\mu)\partial_\mu, \tag{11.9}$$

and they anticommute with the supercharges.
To construct an invariant action we take the $\bar{\alpha}\alpha$ term of $\bar{D}\Phi D\Phi$. We need the covariant derivatives of the superfield, which are found to be

$$D\Phi=\psi+\alpha F+i\partial\!\!\!/ A\alpha-\tfrac{i}{2}(\bar{\alpha}\alpha)\partial\!\!\!/\psi \quad\text{and}\quad \bar{D}\Phi=\bar{\psi}+\bar{\alpha}F-i\bar{\alpha}\partial\!\!\!/ A+\tfrac{i}{2}(\bar{\alpha}\alpha)\partial_\mu\bar{\psi}\gamma^\mu.$$

Hence we obtain

$$\begin{aligned}\tfrac{1}{2}\bar{D}\Phi D\Phi &= \tfrac{1}{2}\bar{\psi}\psi+(\bar{\alpha}\psi)F-i\bar{\alpha}\gamma^\mu\psi\partial_\mu A+\bar{\alpha}\alpha\mathcal{L}_0\\ \mathcal{L}_0 &= \tfrac{1}{2}\partial_\mu A\partial^\mu A-\tfrac{i}{4}\bar{\psi}\partial\!\!\!/\psi+\tfrac{i}{4}\partial_\mu\bar{\psi}\gamma^\mu\psi+\tfrac{1}{2}F^2.\end{aligned}$$

In (11.7) we have calulculated the variations of the components for an arbitrary real superfield (11.5). We read off, that

$$\delta_\beta\mathcal{L}_0=\partial_\mu(\bar{\beta}V_0^\mu),\qquad V_0^\mu=\tfrac{1}{2}\big(i\gamma^\mu\psi F+\gamma^\mu\gamma^\nu\psi\partial_\nu A\big).$$

As expected, the Langrangian density $\mathcal{L}_0$ transforms into a spacetime derivative, and the action for the free and massless theory,

$$S_0=\int d^2x\mathcal{L}_0,$$

is invariant under the supersymmetry transformations (11.7). The equations of motion are simply

$$\Box A=0,\quad \partial\!\!\!/\psi=0 \quad\text{and}\quad F=0. \tag{11.10}$$

Since

$$\sum_\phi\frac{\partial\mathcal{L}_0}{\partial_\mu\phi}\delta_\beta\phi=\partial^\mu A\,\delta_\beta A+\tfrac{i}{2}\delta_\beta\bar{\psi}\gamma^\mu\psi=\bar{\beta}\big(\partial^\mu A\,\psi+\tfrac{i}{2}F\gamma^\mu\psi+\tfrac{1}{2}\gamma^\nu\gamma^\mu\partial_\nu A\psi\big)$$

we obtain the following *conserved Noether current* for the free massless model

$$J_0^\mu=\big(\eta^{\mu\nu}+\gamma_*\epsilon^{\mu\nu}\big)\partial_\nu A\,\psi,\qquad \epsilon_{01}=-\epsilon^{01}=1. \tag{11.11}$$

By using the Klein-Gordon equation for A and the free Dirac equation for ψ one sees at once that J^μ is a conserved spinorial supercurrent.
Now we can add an interaction of the form

$$S_1=\int d^2xd^2\alpha W(\Phi), \tag{11.12}$$

which has the component expansion

$$W(\Phi)=W(A)+\bar{\alpha}\psi\,W'(A)+\tfrac{1}{2}\bar{\alpha}\alpha\mathcal{L}_1,\qquad \mathcal{L}_1=FW'(A)-\tfrac{1}{2}W''(A)\bar{\psi}\psi.$$

The variation of $\mathcal{L}_1$ is found to be

$$\delta_\beta \mathcal{L}_1 = \partial_\mu(\bar{\beta} V_1^\mu), \qquad V_1^\mu = iW'(A)\gamma^\mu\psi$$

such that the Noether current of the interacting model with action

$$S = \int \mathcal{L}, \qquad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{11.13}$$

becomes

$$J^\mu = \big(\eta^{\mu\nu} + \gamma_*\epsilon^{\mu\nu}\big)\partial_\nu A\,\psi - iW'(A)\gamma^\mu\psi.$$

Again F is a nonpropagating auxiliary field, which satisfies the algebraic equation of motion $F = -W'(A)$. After its elimination the Lagrangian density aquires the by now familiar form

$$\mathcal{L} = \tfrac{1}{2}(\partial A)^2 - \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\,\psi - \tfrac{1}{2}W'^2(A) - \tfrac{1}{2}W''(A)\bar{\psi}\psi. \tag{11.14}$$

The Hamiltonian density is found to be

$$\mathcal{H} = \tfrac{1}{2}\pi_A^2 + \tfrac{1}{2}A'^2 + \tfrac{i}{2}\bar{\psi}\gamma^1\psi' + \tfrac{1}{2}W'(A) + \tfrac{1}{2}W''(A)\bar{\psi}\psi, \quad \pi_A = \dot{A}, \quad \pi_\psi = -\tfrac{i}{2}\bar{\psi}\gamma^0. \tag{11.15}$$

The (anti)commutation relations for the field read

$$[\pi_A(x), A(y)] = i\delta(x-y) \quad \text{and} \quad \{\psi_\alpha(x), \psi_\beta(y)\} = -(\gamma^0\mathcal{C})_{\alpha\beta}\delta(x-y).$$

One can check, that the equations of motion are correctly gotten from[1]

$$\dot{O} = i[O, H].$$

Consider the following superpotential

$$W(\Phi) = g\Phi + \tfrac{1}{2}m\Phi^2 + \tfrac{1}{3}\lambda\Phi^3$$

which leads to the Langrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \qquad \mathcal{L}_1 = -\tfrac{1}{2}(g + mA + \lambda A^2)^2 - \tfrac{m}{2}\bar{\psi}\psi - \lambda A\bar{\psi}\psi.$$

To simplify the following discussion let us assume that $m = 0$. Then the quartic potential has a minimum at $A = 0$ and at $A^2 = -g/\lambda$, provided that g/λ is negativ.

• For $g/\lambda > 0$ we have

$$\langle A\rangle = 0, \quad \langle F\rangle = -g, \quad V\big(\langle A\rangle\big) > 0 \quad \text{and} \quad m_A = \sqrt{2g\lambda}, \quad m_\psi = 0$$

such that supersymmetry is broken at the classical level. The vanishing $\psi-$mass signals a Nambu-Goldstone fermion sometimes also called *goldstino*, associated with the spontaneous supersymmetry breaking at the tree level.

• For $g/\lambda < 0$ we have

$$\langle A\rangle = \sqrt{-g/\lambda}, \quad \langle F\rangle = 0, \quad V\big(\langle A\rangle\big) = 0 \quad \text{and} \quad m_A = m_\psi = 2\sqrt{-g\lambda}.$$

The fields A and ψ have equal masses, supersymmetry is exact at the tree level but the reflection symmetry $A \to -A$ is broken.

[1] we had to change the conventions used in section (5), since the kinetic term for the Dirac fermions have changed signs and since $\{\psi^\dagger, \psi\}$ must be positive. The matrix $-\gamma_0\mathcal{C}$ leads to $\{\psi^\dagger_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y)$.

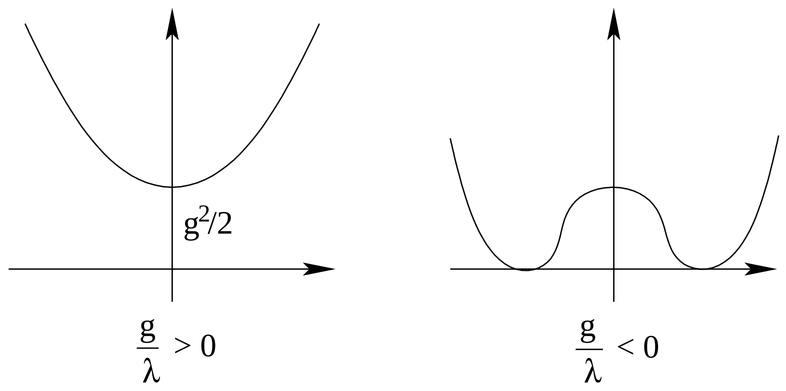

Figure 11.1: *Potentials for the Wess-Zumino model with and without susy breaking.*

11.1.1 2-point functions

Here I follow the book of Bogoliubov and Shirkov, suitable generalized. The positive frequency part of the Pauli-Jordan function in d space-dimensions is

$$\Delta_+(\xi) = \frac{1}{i(2\pi)^{d-1}} \int e^{-ip\xi}\theta(p_0)\delta(p^2-m^2)dp = \frac{1}{i(2\pi)^{d-1}} \int \frac{1}{2\omega} e^{-i(\omega\xi^0+p_i\xi^i)}\, d\vec{p}.$$

Here dx, dp and denote integrals over time and space coordinates, whereas $d\vec{p}$ denotes the integral over the spatial components only. The corresponding Wightman function $i\Delta_+$ defines a positive inner product, as it must be,

$$\begin{aligned} \langle A(f)A(g)\rangle &= i\int dxdy\; f(x)\Delta_+(x,y)g(y) \\ &= \frac{1}{(2\pi)^{d-1}} \int d\vec{p}\,dx\,dy\; f(x)e^{-ipx}\theta(p^0)\delta(p^2-m^2)e^{ipy}g(y) \\ &= \int d\vec{p}\,\theta(p^0)\delta(p^2-m^2)\tilde{f}^*(p)g(p) = \int d\vec{p}\,\frac{1}{2\omega}\,\tilde{f}^*(\omega,\vec{p})g(\omega,\vec{p}), \end{aligned}$$

where we made use of the Fouriertransform

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int e^{ipx} f(x).$$

Note that only the on-shell values of the test function enters. Let us find the explicit form of the Wightman function in two spacetime dimensions. First we change variables,

$$\omega = \sqrt{p^2+m^2} = m\cosh\phi \quad \text{and} \quad p_1 = m\sinh\phi$$

which results in

$$\Delta_+(\xi) = \frac{1}{4\pi i}\int_{-\infty}^{\infty} e^{-im(\cosh\phi\,\xi^0+\sinh\phi\,\xi^1)}d\phi$$

Fortunately, this integral has been evaluated in the book cited. We take the result

$$\Delta_+(\xi) = \begin{cases} \frac{1}{4}\Big(iN_0(m\sqrt{\xi^2}) - \epsilon(\xi^0)J_0(m\sqrt{\xi^2})\Big) & \text{for } \xi^2 > 0 \\ \frac{1}{2\pi i}K_0\big(m\sqrt{-\xi^2}\big) & \text{for } \xi^2 < 0 \end{cases} \tag{11.16}$$

The Pauli-Jordan function becomes then

$$\Delta(\xi) = \Delta_+(\xi) - \Delta_+(-\xi) = -\tfrac{1}{2}\epsilon(x^0)\,\theta\big(\xi^2\big)\; J_0\big(m\sqrt{\xi^2}\big),$$

and gives rise to the commutator

$$[A(x), A(y)] = i\Delta(\xi) = -\tfrac{i}{2}\epsilon(\xi^0)\theta(\xi^2)\, J_0\big(m\sqrt{\xi^2}\big)$$

with support for timelike separated points, $\xi = x - y$. If we would quantize the free model on the cylinder

$$(x^0, x^1) \in \mathbb{R} \times S^1,$$

where the circumference of S^1 is L, then for spacelice x the function would has the form

$$\Delta_+(x) = \frac{1}{2\pi i}\sum_n K_0\Big(m\sqrt{(Ln + x^1)^2 - x_0^2}\Big). \tag{11.17}$$

Furthermore, the time derivative at $x^0 = 0$ is

$$(\partial_0\Delta_+)(0, x^1) = -\frac{1}{4\pi}\int e^{-ipx^1}\, dp = -\tfrac{1}{2}\delta(x^1) \Longrightarrow (\partial_0\Delta)(0, x^1) = -\delta(x^1).$$

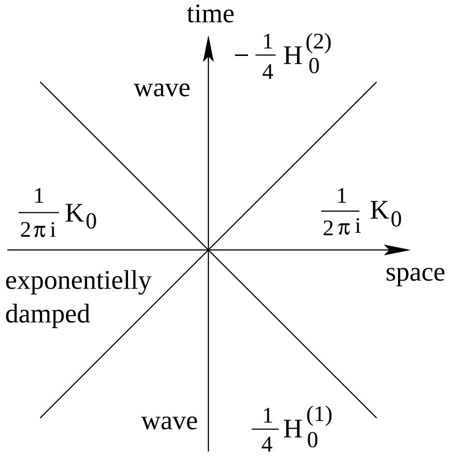

Figure 11.2: *The Wightmanfunction* Δ_+ *in two dimensions.*

To obtain the Wightmanfuntion for the fermions we calculate

$$S \;=\; (i\partial\!\!\!/ - m)(i\Delta_+) = \frac{1}{(2\pi)^{d-1}}\int (p\!\!\!/ - m)\theta(p^0)\delta(p^2 - m^2)e^{-ip\xi}$$

$$= \frac{1}{(2\pi)^{d-1}} \int \frac{1}{2\omega} (\not{p} - m) e^{-ip\xi},$$

where in the last formula $p_0 = \omega(\vec{p})$. Note that

$$S = (i\not{\partial} - m)(i\Delta_0) = \begin{pmatrix} -m & \partial_0 - \partial_1 \\ -\partial_0 - \partial_1 & -m \end{pmatrix} \begin{pmatrix} i\Delta_+ & 0 \\ 0 & i\Delta_+ \end{pmatrix}.$$

The inner product induced by the Wightman function is

$$\int dx\, dy\; g^*_\alpha (S\gamma^0)_{\alpha\beta}(x-y) f_\beta(y) = \frac{1}{(2\pi)^{d-1}} \int \frac{d\vec{p}}{2\omega} \tilde{g}^*(\omega,\vec{p}) \big(\omega - h_f\big) \tilde{f}(\omega,\vec{p}), \quad h_f = \alpha^i p_i + m\gamma^0,$$

where we introduced the hermitean matrices $\alpha^i = \gamma^0\gamma^i$. Since $h_f^2 = (\vec{p}^{\,2} + m^2)\mathbb{1} = \omega^2\mathbb{1}$, the eigenvalues of h_f are $\pm\omega$. Hence the inner product of f with itself vanishes if and only if f is on the mass shell.

To proceed we need

$$(i\partial_1\Delta_+)(0,x^1) = -\frac{m}{2\pi} \sum K_1(mLn + nx^1)$$

which finally yiels

$$\big\langle \psi(0,x^1)\bar{\psi}(0,y^1)\big\rangle = S(0,\xi^1) = \tfrac{1}{2}\delta(\xi^1)\sigma_2 + \frac{m}{2\pi}\big(K_1(\xi^1)\sigma_1 - K_0(\xi^1)\sigma_0\big).$$

The two point function of two fermion fields are

$$\langle \psi_\alpha(0,x^1)\psi_\beta(0,y^1)\rangle = \tfrac{1}{2}\delta(\xi^1)\delta_{\alpha\beta} + \frac{m}{2\pi}\big(iK_1(\xi^1)\sigma_3 - K_0(\xi^1)\sigma_2\big)_{\alpha\beta}.$$

After smearing with real testfunctions $f : \mathbb{R}^2 \to \mathbb{R}^2$ we obtain

$$\langle \psi(0,f)\psi(0,f)\rangle = \tfrac{1}{2} \int dx^1 f_\alpha(x^1) f_\alpha(x^1).$$

11.2 Wess-Zumino model in $2d$-Euclidean spacetime

Instead to analytically continue the Minkowskian model to the Euclidean one, we construct the Euclidean model starting from the known facts about the Clifford algebra on E_2 with metric diag$(1,1)$. The Dirac conjugation is just

$$\bar{\psi} = \psi^\dagger. \tag{11.18}$$

and there is a charge conjugation matrix $\mathcal{C}$ satifying

$$\gamma_\mu^T = \mathcal{C}^{-1}\gamma_\mu\mathcal{C} \quad \text{and} \quad \mathcal{C}^T = \mathcal{C}. \tag{11.19}$$

For example, for the real representation

$$\gamma_0 = \sigma_1, \quad \gamma_1 = \sigma_3 \quad \text{we have} \quad \mathcal{C} = \mathbb{1}, \qquad \gamma_* = i\gamma_0\gamma_1 = \sigma_2$$

a Majorana spinor is just real. For Majorana spinors the symmetry properties are

$$\bar\psi\chi=-\bar\chi\psi,\quad \bar\psi\gamma_\mu\chi=-\bar\chi\gamma_\mu\psi\quad\text{and}\quad \bar\psi\gamma_*\chi=\bar\chi\gamma_*\psi,\quad\text{where}\quad \bar\psi\equiv\psi^\dagger,$$

and the hermiticity poperties become

$$\bar\psi\chi,\,\bar\psi\gamma_\mu\chi\quad\text{are antihermitean and}\quad \bar\psi\gamma_*\chi\quad\text{is hermitean.}$$

Finally, the Fierz identity reads

$$\psi\bar\chi=-\tfrac{1}{2}\bar\chi\psi-\tfrac{1}{2}\gamma_\mu(\bar\chi\gamma_\mu\psi)-\tfrac{1}{2}\gamma_*(\bar\chi\gamma_*\psi). \tag{11.20}$$

Some further Fierz identities are easily derived from these properties: For Majorana spinors we have

$$\begin{aligned}(\bar\alpha\psi)(\bar\alpha\psi)&=-(\bar\psi\alpha)(\bar\alpha\psi)=-\bar\psi\big(\alpha\bar\alpha\big)\psi=\tfrac{1}{2}(\bar\alpha\gamma_*\alpha)(\bar\psi\gamma_*\psi)\\(\bar\psi\gamma^\mu\alpha)\psi&=(\psi\bar\psi)\gamma^\mu\alpha=-\tfrac{1}{2}(\bar\psi\gamma_*\psi)\gamma_*\gamma^\mu\alpha\end{aligned}$$

A *hermitean superfield* has the following component expansion,

$$\Phi(x,\alpha)=A(x)+\bar\alpha\gamma_*\psi(x)+\tfrac{1}{2}(\bar\alpha\gamma_*\alpha)F(x)$$

with real fields A,F and Majorana spinorfield ψ. As supercharges we may take

$$Q=-i\frac{\partial}{\partial\bar\alpha}-\gamma^\mu\alpha\partial_\mu=\quad\text{and}\quad \bar Q=-i\frac{\partial}{\partial\alpha}-\bar\alpha\gamma^\mu\partial_\mu=Q^T\sigma_1.$$

One finds the following nontrivial anticommutation relations

$$\{Q_\alpha,\bar Q_\beta\}=2i(\gamma^\mu\partial_\mu)_{\alpha\beta}. \tag{11.21}$$

Thus we have recovered the correct anticommutation relations for the supercharges. The superderivatives are

$$D=\frac{\partial}{\partial\bar\alpha}+i(\gamma^\mu\alpha)\partial_\mu\quad,\quad \bar D=-\frac{\partial}{\partial\alpha}-i(\bar\alpha\gamma^\mu)\partial_\mu=-D^T\sigma_1.$$

Up to a sign they satisfy the same anticommutation relations as the supercharges. The supersymmetry variation of the superfield is

$$\begin{aligned}\bar\beta\big(\frac{\partial}{\partial\bar\alpha}-i\gamma^\mu\alpha\partial_\mu\big)\Phi&=\bar\beta\big(\gamma_*\psi+\gamma_*\alpha F-i\partial\!\!\!/\,A\alpha+\tfrac{i}{2}(\bar\alpha\gamma_*\alpha)\partial\!\!\!/\,\psi\big)\\&=\delta A(x)+\bar\alpha\gamma_*\delta\psi(x)+\tfrac{1}{2}(\bar\alpha\gamma_*\alpha)\delta F(x),\end{aligned}$$

from which we read off the following transformation rules for the component fields

$$\delta A=\bar\beta\gamma_*\psi,\quad \delta\psi=(F+i\gamma_*\partial\!\!\!/\,A)\beta\Longrightarrow\delta\bar\psi=\bar\beta(F-i\partial\!\!\!/\,A\gamma_*),\quad \delta F=i\bar\beta\partial\!\!\!/\,\psi. \tag{11.22}$$

Now we are ready to calculate the action of the free model. First we need the covariant derivatives of the superfield

$$\begin{aligned}D\Phi&=\gamma_*\psi+\gamma_*\alpha F+i\partial\!\!\!/\,A\alpha-\tfrac{i}{2}(\bar\alpha\gamma_*\alpha)\partial\!\!\!/\,\psi\\\bar D\Phi&=\bar\psi\gamma_*+\bar\alpha\gamma_*F-i\bar\alpha\partial\!\!\!/\,A+\tfrac{i}{2}(\bar\alpha\gamma_*\alpha)\partial_\mu\bar\psi\gamma^\mu\end{aligned}$$

Since $\gamma_* D\Phi$ is a Majorana field we obtain

$$\tfrac{1}{2}\bar{D}\Phi D\Phi = -\tfrac{i}{4}(\bar{\alpha}\gamma_*\alpha)\partial_\mu\big(\bar{\psi}\gamma_*\gamma^\mu\psi\big) = 0,$$

and this can be checked by a direct calculation using the Fierz identities in Euclidean spacetime. The supersymmetric and Lorentz-invariant action is derived from the density

$$\begin{aligned}\tfrac{1}{2}\bar{D}\Phi\gamma_* D\Phi = &= \tfrac{1}{2}\bar{\psi}\gamma_*\psi + (\bar{\alpha}\gamma_*\psi)F - i\bar{\alpha}\gamma^\mu\psi\partial_\mu A - \bar{\alpha}\gamma_*\alpha\mathcal{L}_0\\ \mathcal{L}_0 &= \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{i}{4}\bar{\psi}\partial\!\!\!/\,\psi - \tfrac{i}{4}\partial_\mu\bar{\psi}\gamma^\mu\psi - \tfrac{1}{2}F^2.\end{aligned}$$

The variation of the Lagrangian density $\mathcal{L}_0$ with respect to supersymmetry transformations is

$$\delta\mathcal{L}_0 = \partial_\mu(\bar{\beta}V_0^\mu), \quad V_0^\mu = \tfrac{1}{2}\big(\gamma_*\gamma^\mu\gamma^\nu\,\partial_\nu A - i\gamma^\mu F\big)\psi.$$

Note that $\mathcal{L}_0$ gives rise to an action which is *unbounded below* and above. Again we switch on an interaction by adding a superpotential $W(\Phi)$ with component expansion,

$$W(\Phi) = W(A) + \bar{\alpha}\gamma_*\psi W'(A) + \tfrac{1}{2}\bar{\alpha}\gamma_*\alpha\big(W'(A)F - \tfrac{1}{2}W''\psi_\alpha\gamma_*\psi\big),$$

and which gives rise to the Lagrangian density

$$\mathcal{L}_1 = \tfrac{1}{2}W''(A)\psi^\dagger\gamma_*\psi - W'(A)F.$$

The total Lagrangian is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ and from the superfield formulation we read off that

$$\delta\mathcal{L} = \partial_\mu(\bar{\beta}V^\mu), \quad V^\mu = V_0^\mu - iW'(A)\gamma^\mu\psi. \tag{11.23}$$

This can be checked directly by using the explicit form for the tranformations of the component fields A, ψ and F. The auxiliary field fulfils the algebraic equation

$$F = -W'(A).$$

Eliminating F we obtain

$$\mathcal{L} = \tfrac{1}{2}(\partial A)^2 + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\,\psi + \tfrac{1}{2}W'^2(A) + \tfrac{1}{2}W''(A)\,\bar{\psi}\gamma_*\psi. \tag{11.24}$$

This density is invariant under the on-shell supersymmetry transformations

$$\delta A = \bar{\beta}\gamma_*\psi \quad\text{and}\quad \delta\psi = \big(i\gamma_*\partial\!\!\!/\,A - W'(A)\big)\beta, \qquad \delta\bar{\psi} = \bar{\beta}\big(i\gamma_*\partial\!\!\!/\,A - W'(A)\big). \tag{11.25}$$

Hence, on the classical level we have constructed an $N = 1$ supersymmetric and *stable* Euclidean Wess-Zumino model. Already for the simple Euclidean Wess-Zumino model we have the strange result that off-shell the action is unbounded from below. Only after eliminating the auxiliary field does the theory become stable. In theories with extended supersymmetry the models are unstable even on-shell.
The equations of motion read

$$\begin{aligned}\triangle A &= (W'W'')(A) + \tfrac{1}{2}W'''(A)\bar{\psi}\gamma_*\psi\\ \tfrac{i}{2}\partial\!\!\!/\,\psi &= -\tfrac{1}{2}W''(A)\gamma_*\psi.\end{aligned}$$

The solutions are integral curves to the hamiltonian vectorfield generated by

$$\mathcal{H} = \pi_A \dot{A} + \pi_\psi \dot{\psi} - \mathcal{L} = \tfrac{1}{2}\pi_A^2 - \tfrac{1}{2}(A')^2 - \tfrac{i}{2}\bar{\psi}\gamma^1\partial_1\psi - \tfrac{1}{2}W'^2(A) - \tfrac{1}{2}W''\bar{\psi}\gamma_*\psi,$$

where

$$\pi_A = \dot{A} \quad \text{and} \quad \pi_\psi = \tfrac{i}{2}\bar{\psi}\gamma^0 = \tfrac{i}{2}\psi^T(\mathcal{C}^{-1})^T\gamma^0$$

are the momentum field conjugate to A and ψ. The (anti)commutation relations read

$$[A(x), \pi_A(y)] = i\delta(x-y) \quad \text{and} \quad \{\psi_\alpha(x), \pi_\psi^\beta(y)\} \sim \delta_\alpha^{\,\beta}\delta(x-y).$$

Indeed one finds

$$\begin{aligned} \dot{A}(x) &= i[H, A(x)] = \pi_A(x) \\ \dot{\pi}_A(x) &= i[H, \pi_A(x)] = -(\partial_x A)^2 + (W'W'')(A) + \tfrac{1}{2}W'''(A)\bar{\psi}\gamma_*\psi \end{aligned}$$

which imply the correct equations of motion for the scalar field. To continue we choose the following Majorana representation,

$$\gamma_0 = \sigma_1, \quad \gamma_1 = \sigma_3 \Longrightarrow \gamma_* = \sigma_2.$$

The charge conjugation matrix is the identity and

$$\pi_\psi = \tfrac{i}{2}(\psi_2, \psi_1).$$

The anticommutation relations read[2]

$$\{\psi_1(x), \psi_2(y)\} = \delta(x-y), \qquad \{\psi_1, \psi_1\} = \{\psi_2, \psi_2\} = 0.$$

The fermionic part of the Hamiltonian density reads

$$\mathcal{H}_F = -\tfrac{i}{2}(\psi_1\psi_1' - \psi_2\psi_2') + \tfrac{i}{2}W''(A)(\psi_1\psi_2 - \psi_2\psi_1).$$

With $[AB, C] = A\{B, C\} - \{A, C\}B$ we obtain

$$\begin{aligned} \dot{\psi}_1 &= i[H, \psi_1] = +\psi_2' - W''\psi_1 \\ \dot{\psi}_2 &= i[H, \psi_2] = -\psi_1' + W''\psi_2, \end{aligned}$$

which yields the above Dirac equation in the given representation.

11.2.1 Noether currents

With

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi = \bar{\beta}\big(\partial^\mu A\gamma_* - \tfrac{i}{2}F\gamma^\mu - \tfrac{1}{2}\partial\!\!\!/ A\gamma_*\gamma^\mu\big)\psi,$$

the Noether current reads

$$\bar{\beta}J^\mu = \bar{\beta}\big(\partial^\mu\gamma_* + i\epsilon^{\mu\nu}\partial_\nu A + iW'(A)\gamma^\mu\big)\psi, \qquad \epsilon^{01} = 1,$$

[2] I choose the normalization such, that the I get the correct equations of state

and gives rise to the Noether charge

$$Q=\int dx\Big(\pi_A\gamma_*+iA'+iW'(A)\gamma^0\Big)\psi.$$

In our Majorana representation

$$\bar\beta Q=\int dx\bar\beta\Big(\pi_A\sigma_2+iA'+iW'(A)\sigma_1\Big)\psi.$$

It is easy to calculate

$$\delta A=i[\bar\beta Q,A]=\bar\beta\gamma_*\psi.$$

To compute the commutator of the supercharge with the fermionic field, we use

$$\int dy[\bar\beta_\alpha A_{\alpha\beta}(y)\psi_\beta(y),\psi_\sigma(x)]=\int dy\bar\beta_\alpha A_{\alpha\beta}(y)\{\psi_\beta(y),\psi_\sigma(x)\}=\bar\beta_\alpha(A\sigma_1)_{\alpha\sigma}(x).$$

With this formula one finds

$$\delta\psi=i\bar\beta\big(\pi_A\sigma_2+iA'+iW'\sigma_1\big)\sigma_1=\big(i\sigma_2\big(\sigma_1\partial_0+\sigma_3\partial_1\big)A-W'\big)\beta.$$

Hence we recover the transformation law for the scalar field A and the Majorana spinor ψ.
In components (and the Majorana representation) we find

$$\begin{aligned}Q_1 &= i\int dx^1\left(\left(W'-\pi_A\right)\psi_2+A'\psi_1\right)\\ Q_2 &= i\int dx^1\left(\left(W'+\pi_A\right)\psi_1+A'\psi_2\right).\end{aligned}$$

11.3 Euclidean path integral for the free massive model

Here we recall the quantisation of the free massive $N=1$ model. The main emphasis is put on the correct quantisation of Majorana fermions in the path integral formulation. Since all functional integrals are Gaussian, the generating functional and hence all correlations functions, can be calculated explicitely.
First, to get the free massive model we choose a linear superpotential

$$W=\tfrac{m}{2}\Phi^2=\tfrac{m}{2}A^2+m(\bar\alpha\gamma_*\psi)A+\tfrac{m}{2}\bar\alpha\gamma_*\alpha(AF-\tfrac{1}{2}\bar\psi\gamma_*\psi),$$

giving rise to a quadratic Langrangian density,

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu A\partial^\mu A+\tfrac{i}{4}\bar\psi\partial\!\!\!/\,\psi-\tfrac{i}{4}\partial_\mu\bar\psi\gamma^\mu\psi-\tfrac{1}{2}F^2-mAF+\tfrac{m}{2}\bar\psi\gamma_*\psi.$$

We have seen that this density is invariant under the following susy transformations

$$\delta A=\bar\beta\gamma_*\psi\quad\text{and}\quad\delta\psi=(F+i\gamma_*\partial\!\!\!/\,A)\psi,$$

and that the density is invariant up to a divergence,

$$\delta\mathcal{L}=\partial_\mu(\bar\beta V^\mu),\quad V^\mu=V_0^\mu-imA\gamma^\mu\psi.$$

After eliminating the auxiliary field via its equation of motion, $F = -mA$, we end up with the massive free theory for a real scalar field and a Majorana spinor,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu A \partial^\mu A + \tfrac{m}{2}A^2 + \tfrac{i}{2}\bar{\psi}\not{\partial}\psi + \tfrac{m}{2}\bar{\psi}\gamma_*\psi. \tag{11.26}$$

Next we wish to study the Gaussion functional integral for the generating functional of the Euclidean correlation functions

$$Z(j,\eta) = \int \mathcal{D}A\mathcal{D}\psi\, e^{-S+i\int(jA+\bar{\eta}\psi)} \equiv Z_B(j)Z_F(\eta) \tag{11.27}$$

in detail. Note that the bosonic source term is imaginary, whereas the fermionic source term is hermitean. Clearly the functional factorizes into bosonic and fermionic parts. We shall work on the two-dimensional torus such that the relevant differential operators possess discrete spectra. Hence we identify x^0 with $x^0+\beta$ and x^1 with x^1+L. The functional integral over the scalar field is simply

$$Z_B(j) = \frac{1}{\det^{1/2}(-\triangle + m^2)}\ \exp\Big\{ -\big(j, \frac{1}{-\triangle + m^2}\, j\big)\Big\}.$$

The integral over the Majorana fermions is more subtle and hence we shall be a bit more explicit. The eigenvalue equation for the Euclidean Dirac equation,

$$(i\not{\partial} + m\gamma_*)\psi = \lambda\psi,$$

reads in the Majorana representation

$$i\begin{pmatrix} \partial_1 & \partial_0 - m \\ \partial_0 + m & -\partial_1 \end{pmatrix}\psi = \lambda\psi.$$

On the space of real Majorana spinors this euclidean Dirac equation has no solution. But we can solve

$$(i\not{\partial} + m\gamma_*)\chi_\lambda = i\lambda\phi_\lambda \quad\text{and}\quad (i\not{\partial} + m\gamma_*)\phi_\lambda = -i\lambda\chi_\lambda \Longrightarrow (-\triangle + m^2)\chi_\lambda = \lambda^2\chi_\lambda.$$

on real modes. The possible eigenvalues are

$$\lambda_{rs} = \pm\sqrt{p^2+m^2}, \qquad p_0 = \frac{2\pi r}{\beta}, \quad p_1 = \frac{2\pi s}{L}.$$

To construct these solutions one starts with a (necessarily) complex and normalized eigenfunction ψ_λ of the Dirac operator with eigenvalue λ. We assume that λ is positive. Then ψ^*_λ is an eigenfunction as well with eigenvalue $-\lambda$. Now we may take

$$\chi_\lambda = \frac{1}{\sqrt{2}}(\psi_\lambda + \psi^*_\lambda) \quad\text{and}\quad \phi_\lambda = \frac{1}{i\sqrt{2}}(\psi_\lambda - \psi^*_\lambda), \qquad \lambda > 0.$$

Since the ψ_λ form an orthonormal set of (complex) eigenfunctions, we have

$$(\chi_\lambda, \chi_\lambda) = (\phi_\lambda, \phi_\lambda) = 1 \quad\text{and}\quad (\chi_\lambda, \phi_\lambda) = 0.$$

Now we expand the Majorana field and source in the functional integral as

$$\psi = \sum_\lambda \big(b_\lambda \chi_\lambda + c_\lambda \phi_\lambda\big), \qquad \bar\eta = \sum_\lambda \big(u_\lambda \chi_\lambda^T + v_\lambda \phi_\lambda^T\big),$$

where the sum extends over all *positive* eigenvalues λ of the Dirac operator. It follows that

$$\tfrac{1}{2}\big(\bar\psi, (i\partial\!\!\!/ + m\gamma_*)\psi\big) = \tfrac{i}{2}\sum \lambda(b_\lambda c_\lambda - c_\lambda b_\lambda) \quad \text{and} \quad (\bar\eta, \psi) = \sum (u_\lambda b_\lambda + v_\lambda c_\lambda).$$

Next we take linear combinations of the real Grassmannian parameters and introduce the complex objects

$$a = \frac{1}{\sqrt{2}}(b + ic), \quad , \quad s = \frac{1}{\sqrt{2}}(u + iv)$$

such that

$$a^\dagger a = \tfrac{i}{2}(bc - cb), \qquad s^\dagger a + s a^\dagger = ub + vc.$$

Now we may rewrite the path integral in terms of complex Grassmann variables as follows,

$$Z_F(\eta) = \int da_\lambda da_\lambda^\dagger \, \exp\Big(\sum \big\{ -\lambda a_\lambda^\dagger a_\lambda + i s_\lambda^\dagger a_\lambda + i s_\lambda a_\lambda^\dagger \big\}\Big).$$

The remaining manipulations are wellknown. First we shift variables

$$-\lambda a^\dagger a + i(s^\dagger a + s a^\dagger) = \lambda \tilde a^\dagger \tilde a + \frac{1}{\lambda} s^\dagger s, \quad \tilde a = a + \frac{i}{\lambda} s, \quad \tilde a^\dagger = a^\dagger - \frac{1}{\lambda} s^\dagger,$$

such that the path integral simplifies to

$$Z_F(\eta) = \exp\big(-\sum \frac{1}{\lambda} s_\lambda^\dagger s_\lambda\big) \int d\tilde a_\lambda d\tilde a_\lambda^\dagger \, \exp\big(-\sum \lambda \tilde a_\lambda^\dagger \tilde a_\lambda\big) = \big(\prod \lambda\big) \, \exp\big(-\frac{1}{\lambda} s_\lambda^\dagger s_\lambda\big),$$

where the product and sum extends over the positive eigenvalues of the Dirac operator. Notice the difference as compared to the path integral for Dirac fermions. The eigenvalues of $-\partial\!\!\!/^{\,2}$ are at least double degenerate (the γ_*-doubling). When we quantize Dirac fermions, then the two eigenvalues are both included in the above formula. But for Majorana fermions we combined two eigenfunction in the subspace with fixed λ^2 to one complex eigenfunction. Hence we only must include one of the eigenvalues of the pair of eigenvalues $\pm\lambda$ of $i\partial\!\!\!/$. Hence the product is

$$\prod \lambda = \det{}^{1/2}(i\partial\!\!\!/ + \gamma_* m) = \det{}^{1/2}(-\triangle + m^2).$$

In the last step we used $\partial\!\!\!/^{\,2} = \triangle \mathbb{1}_2$. This way we arrive at the intermediate result

$$Z_F(\eta) = \det{}^{1/2}(-\triangle + m^2) \, \exp\big(-i \sum \frac{u_\lambda v_\lambda}{\lambda}\big).$$

To rewrite the last term we use the spectral decomposition of the Greensfunction of the Dirac operator,

$$S(x, y) \equiv \langle x | \frac{1}{i\partial\!\!\!/ + m\gamma_*} | y \rangle = i \sum_{\lambda > 0} \frac{1}{\lambda}\big(\phi_\lambda(x)\chi_\lambda^T(y) - \chi_\lambda(x)\phi_\lambda^T(x)\big) = \sum_{\text{all}\lambda} \frac{1}{\lambda} \psi_\lambda \psi_\lambda^\dagger.$$

This Greensfunction is purely imaginary and antisymmetric

$$S_{\alpha\beta}(x,y) = -S_{\beta\alpha}(y,x).$$

Using the eigenmode expansion of the source and the orthogonality of the eigenmodes one proves that

$$(\bar\eta, \frac{1}{i\partial\!\!\!/ + m\gamma_*}\eta) = -2i\sum \frac{u_\lambda v_\lambda}{\lambda}.$$

This finally proves the following nice formula for the path integral

$$Z(j,\eta) = \exp\Big(-\tfrac{1}{2}\int j(x)G(x,y)j(y) + \tfrac{1}{2}\int \bar\eta(x)S(x,y)\eta(y)\Big). \tag{11.28}$$

Notice that the (UV-singular) functional determinants have exactly canceled in the final result. This (partial) cancellation of divergences is one of the most attractive features of supersymmetric theories.

11.4 Do we get a sensible susy field theory?

After having computed the generating functional it is a easy exercise to derive the two-point function and hence all n-point function for the free $N=1$ model. The bosonic 2-point function is

$$\langle A(x)A(y)\rangle = \frac{1}{i^2}\frac{\delta^2}{\delta j(x)j(y)}\log Z(j,\eta)|_{j=0} = G(x,y). \tag{11.29}$$

It is positive in the following sense,

$$(f,Gf) \geq 0 \quad \text{and} \quad = 0 \quad \text{only for} \quad f=0,$$

implying that the in the bosonic sector of the corresponding Minkowskian theory the Hilbertspace has positive norm.

11.4.1 The requirements on the Euclidean Theory

Let us recall the relations between Minkowskian and Euklidean 2-point functions: In Minkowski spacetime one defines the two-point *Wightman funtion*

$$\begin{aligned}\langle\Omega|\hat A(x)\hat A(y)|\Omega\rangle &= \langle\Omega|e^{ix^0H}\hat A(0,\vec x)e^{-ix^0H}e^{iy^0H}\hat A(0,\vec y)e^{-iy^0H}|\Omega\rangle\\ &= \langle\Omega|\hat A(0,\vec x)e^{-i(x^0-y^0)H}\hat A(0,\vec y)|\Omega\rangle = W(x,y) = W(\xi),\end{aligned}$$

where $\xi = x-y$. This Poincaré invariant distribution must be positive, that is for non-vanishing test functions f we have $(f,Wf)>0$. From $H\geq 0$ it may be inferred that $W(\xi)$ admits an analytic continuation with respect to x^0. Writing $S(z,\vec\xi)$ for the analytic function with $z=\xi^0_E+i\xi^0$ restricted to the halfplane $\xi^0_E>0$,

$$S(z,\vec\xi\,) = \langle\Omega|\hat A(0,\vec x)e^{-zH}\hat A(0,\vec y)|\Omega\rangle, \qquad z = \xi^0_E + i\xi,$$

we have

$$W(\xi^0,\vec{\xi}) = \lim_{\xi^0_E\downarrow 0} S(\xi^0_E + i\xi^0,\vec{x}).$$

As a rule the boundary values only exist in the sense of distributions. Else there are no singularities for $\xi^0_E > 0$, and this is the domain where the Euclidean two-point function is evaluated:

$$S(\xi) = S(\xi^0_E,\vec{\xi}).$$

Now let us recall the basic axioms of Euclidean field theory such that it gives rise to a interpretable field theory in Minkowskian space.
The starting point is the normalized generating functional

$$Z(f) = \int d\mu(A)\, e^{iA(f)}, \qquad d\mu(A) \sim \mathcal{D}Ae^{-S[A]}, \quad A(f) = \int f(x)A(x), \quad f \in \mathcal{S}. \tag{11.30}$$

We have changed the notation. Here and below f (and not j) denotes the external source. The space $\mathcal{S}$ is a suitable real test function space, usually one takes the Schwartz space[3]. Obvious properties are

$$\begin{aligned} &|Z(f)| \leq Z[0] = 1 \\ &Z[\,f_1 + tf_2] \quad \text{is continuous function of} \quad t\in\mathbb{R} \quad \text{for all} \quad f_i \in \mathcal{S} \\ &a_{ij} = Z\big[f_i - f_j\big] \quad \text{is positive definite.} \end{aligned} \tag{11.31}$$

Next we introduce the space $\mathcal{S}^c$ of all complex $f = f_1 + if_2$, where $f_i \in \mathcal{S}$. Then we assume
Analyticity: *The Schwinger functional $Z[f]$ admits an extension to complex $f \in \mathcal{S}^c$ with the extension being analytic.*
This means that for $f = \sum_1^n z_i f_i$ with $z_i \in \mathbb{C}$ and $f_i \in \mathcal{S}^c$ the mapping $(z_1,\dots,z_n) \to Z[f]$ is entirely analytic in $\mathbb{C}^n$. Then Z admits a power series expansion which represents the functional. The next property we require is

Regularity: *There exists constants c_1 and c_2 such that for all $f \in \mathcal{S}^c$*

$$|Z[f]| \leq \exp\int dx\Big(c_1|f(x)| + c_2|f(x)|^2. \tag{11.32}$$

Moreover, the two point function $\langle A(x)A(y)\rangle$ is assumed to be locally integrable.
This postulate may physically not be as well founded as the postulate of
Invariance: *The functional $Z(f)$ is invariant under all symmetries of the Euclidean space.*
We shall use the Hilbert space $\mathcal{E} = L^2\big(\mathcal{S}', d\mu(A)\big)$ of square integrable functions $F : \mathcal{S}' \to \mathbb{C}$ with respect to the functional measure μ. The scalar product is given by

$$(F,G) = \int d\mu(A)\,\overline{F(A)}\,G(A). \tag{11.33}$$

[3]The Schwartz space consists of all $\mathbb{C}^\infty$-functions $f : \mathbb{R}^d \to \mathbb{R}$, such that f and all its derivatives decrease faster than any inverse power of $|x|$ as $|x| \to \infty$. The dual $\mathcal{S}'$ of $\mathcal{S}$ consists of all $\mathbb{R}$-linear functionals $\phi : \mathcal{S} \to \mathbb{R}$.

Let us explain the rather important lat axiom in more detail: The action of the Euclidean group on the test functions is described by

$$(a,R)f(x) = f\big(R^{-1}(x-a)\big), \qquad a \in \mathbb{R}^d, \quad R \in O(d). \tag{11.34}$$

A unitary representation $U(a,R) : \mathcal{E} \to \mathcal{E}$ is then given by

$$U(a,R)F(A) = \sum_k c_k \exp\big\{iA\big((a,R)f_k\big)\big\}, \qquad F(A) = \sum_{k=1}^n c_k \exp\big(iA(f_k)\big). \tag{11.35}$$

Particular elements of the Euclidean group are

• *Time reversal:* This is the element $\theta = (0,R)$, where R is a time reflection. The corresponding unitary operator Θ is an involution, $\Theta^2 = \mathbb{1}$.

• *Time shifts:* These are the elements $(a, \mathbb{1})$, where $a = (t, \vec{0})$ shifts the time coordinate. We obtain a one-parameter unitary group $U(t) = \exp(itB)$. But B is not to be confused with the Hamiltonian. Since

$$\Theta U(t) = U(-t)\Theta \tag{11.36}$$

the spectrum of B is symmetric about zero.

The last property we require is

Reflection positivity: *The matrix with elements $a_{ij} = Z(f_i - \theta f_j)$, $i,j = 1,\dots,n$ with arbitrary n, is poitive for all test functions $f_i \in \mathcal{S}$ that vanish in the halfplane $x^0 < 0$, i.e. $f(x) = 0$ for negative x^0.*

On may rephrase this crucial property in terms of the Schwinger functions, that is the moments of the Schwinger functional, as follows:

For each finite sequence $(f_0,\dots,f_n)$ of test functions $f_0 \in \mathbb{C}$ and $f_k \in \mathcal{S}(\mathbb{R}^{kd})$ with

$$f_k(x_1,\dots,x_k) = 0 \quad \text{except for} \quad 0 \geq x_1^0 \geq \dots \geq x_k^0$$

on has

$$\sum_{n,m} \mathcal{S}^{n+m}(\overline{\theta f_n} \otimes f_m) \geq 0. \tag{11.37}$$

Test functions with this property form a subspace $\mathcal{S}_+$ of $\mathcal{S}$. There is a corresponding subspace $\mathcal{E}_+ \subset \mathcal{E}$, which we define as the closed complex-linear hull of all vectors of the form $F(A) = \exp\big(iA(f)\big)$ with $f \in \mathcal{S}_+$. Reflection positivity implies that $(\Theta F, G) \geq 0$ for all $F, G \in \mathcal{E}_+$. Thus we introduce a second scalar product

$$\langle F, G\rangle \equiv (\Theta F, G) \tag{11.38}$$

under the provision that we factor out the null space

$$\mathcal{E}_0 = \{F \in \mathcal{E}_+ | \langle F, F\rangle = 0\}. \tag{11.39}$$

This then gives rise to the (physical) Hilbert space

$$\mathcal{H} = \overline{\mathcal{E}_+/\mathcal{E}_0}. \tag{11.40}$$

Two vectors F and G in $\mathcal{E}_+$ give rise to the same vector in $\mathcal{H}$ if they differ by some vector in $\mathcal{E}_0$. Thus one considers the equivalence class $F_\bullet$ associated with each $F \in \mathcal{E}_+$ and treat $F_\bullet$ as some vector in $\mathcal{H}$ so that the scalar product becomes

$$(F_\bullet, G_\bullet) = \langle F, G\rangle.$$

The Hamiltonian: It is remarkable that one may construct the Hamiltonian of the model within the Euclidian formulation of the model. First we observe that if a test function vanishes for negative euclidean times $f(x^0 < 0, \vec{x}) = 0$, so does the function

$$f_t(x^0, \vec{x}) = f(x^0 - t, \vec{x})$$

for all $t > 0$. This time shifts with positive t define a semigroup which maps $\mathcal{S}_+$ into itself. Hence

$$U(t) : \mathcal{E}_+ \longrightarrow \mathcal{E}_+.$$

One can show that $\mathcal{E}_+$ is left invariant under $U(t)$ such that the $U(t)$ maybe viewed as linear bounded operator $U_\bullet$ on the quotient space $\mathcal{E}_+/\mathcal{E}_0$ which is then extended to $\mathcal{H}$,

$$U(t)_\bullet F_\bullet = \big(U(t)F\big)_\bullet.$$

One can prove

$$0 \le U_\bullet^\dagger(t) = U_\bullet(t) \le 1 \Longrightarrow U_\bullet(t) = e^{-tH}, \qquad t \ge 0,\ H \ge 0. \tag{11.41}$$

The operator H is then just the selfadjoint and positive Hamiltonian of the theory under consideration.

11.4.2 The fermionic subspace

When we differentiate (11.27) with respect to the Grassman-valued fermionic sources we get

$$\frac{\delta^2}{\delta\bar{\eta}_\beta(y)\delta\eta_\alpha(x)} Z_F(\eta)\big|_{\eta=0} = Z_F \cdot \big\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\big\rangle,$$

where in the Majorana representation there is no difference between $\bar{\psi}_\alpha$ and ψ_α. When we differentiate (11.28) we obtain

$$\frac{\delta^2}{\delta\bar{\eta}_\beta(y)\delta\eta_\alpha(x)} Z_F(\eta)\big|_{\eta=0} = Z_F S_{\alpha\beta}(x,y).$$

Not unexpectedly this proves that

$$\big\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\big\rangle = \frac{\delta^2}{\delta\bar{\eta}_\beta(y)\delta\eta_\alpha(x)} \log Z_F(\eta)\big|_{\eta=0} = S_{\alpha\beta}(x,y). \tag{11.42}$$

This does not define a positive form on the set of functions $\mathbb{R}^2 \to \mathbb{R}^2$,

$$\int f_\alpha(x) S_{\alpha\beta}(x,y) f_\alpha(y) = 0.$$

However, this is not a serious problem. Only the positivity of the state space of the Minkowskian theory is really required.

11.4.3 Supersymmetric Ward identities:

In (11.26) we change variables $(A,\psi) \to (A',\psi')$, where the mapping is a supersymmetry transformation. We shall assume that the measure respects the supersymmetry. Then

$$\begin{aligned} Z(j,\eta) &= \int \mathcal{D}A\mathcal{D}\psi \;\exp(-S)\exp\left(i\int jA' + \bar\eta\psi'\right) \\ &= \int \mathcal{D}A\mathcal{D}\psi \;\exp(-S)\exp\left(i\int j'A + \bar\eta'\psi\right) = Z(j',\eta'). \end{aligned}$$

When calculate the transformations of the sources on needs some partial integrations and Fierz identities. The partial integrations are justified on the torus. It follows in particular, that

$$0 = \int \mathcal{D}A\mathcal{D}\psi \exp(-S)\exp\left(i\int jA + \bar\eta\psi\right)\cdot\left(\int j\delta A + \bar\eta\delta\psi\right). \tag{11.43}$$

where we have set $A' = A + \delta A$ and $\psi' = \psi + \delta\psi$. The Ward identities are gotten by differentiating (11.43) several times with respect to the sources j and η and setting them to zero after the differentiation has been performed. We give some examples:

Operator	Ward identity
$\frac{\delta}{\delta j(x)}$	$\langle\delta A(x)\rangle = 0$
$\frac{\delta^2}{\delta j(y)\delta j(x)}$	$\langle A(x)\delta A(y) - A(y)\delta A(x)\rangle = 0$
$\frac{\delta}{\delta\bar\eta_\alpha(x)}$	$\langle\delta\psi_\alpha(x)\rangle = 0$
$\frac{\delta^2}{\delta\bar\eta_\beta(y)\delta\bar\eta_\alpha(x)}$	$\langle\psi_\alpha(x)\delta\psi_\beta(y) - \psi_\beta(y)\delta\psi_\alpha(x)\rangle = 0$
$\frac{\delta^2}{\delta\bar\eta_\beta(y)\delta j(x)}$	$\langle A(x)\delta\psi_\beta(y) + \psi_\beta(y)\delta A(x)\rangle = 0$

For the free model only the last Ward identity yields a nontrivial relations between the two-point functions,

$$(i\partial\!\!\!/ + \gamma_* m)_x\langle A(x)A(y)\rangle = \langle\psi(x)\bar\psi(y)\rangle$$

which is indeed fulfilled, since

$$(i\partial\!\!\!/ + \gamma_* m)_x G(x,y) = S(x,y)$$

holds true. Next we investigate the Ward identities when we perform two successive supersymmetry transformations with parameters α and β:

$$\begin{aligned} A &\longrightarrow A + \delta_\alpha A + \delta_\beta A + \delta_\alpha\delta_\beta A + O(\alpha^2,\beta^2) = A + \delta_{\alpha\beta}A + O(\alpha^2,\beta^2). \\ \psi &\longrightarrow \psi + \delta_\alpha\psi + \delta_\beta\psi + \delta_\alpha\delta_\beta\psi + O(\alpha^2,\beta^2) = \psi + \delta_{\alpha\beta}\psi + O(\alpha^2,\beta^2). \end{aligned}$$

Clearly, we get the identity

$$0 = \int \exp\Big(-S + i\int (jA + \bar\eta\psi)\Big)\ \Big(i\int (j\,\delta_{\alpha\beta}A + \bar\eta\delta_{\alpha\beta}\psi) \\ -\tfrac{1}{2}\Big\{\int j(\delta_\alpha+\delta_\beta)A + \bar\eta(\delta_\alpha+\delta_\beta)\psi)\Big\}^2\Big) + O(\alpha^2,\beta^2).$$

Now exchange α and β and subtract the corresponding equation which results in

$$\begin{aligned} 0 &= \int \exp\Big(-S + i\int (jA + \bar\eta\psi)\Big) \int \Big(j\,[\delta_\alpha,\delta_\beta]A + \bar\eta[\delta_\alpha,\delta_\beta]\psi\Big) \\ &= 2i\bar\beta\gamma^\mu\alpha \int \exp\Big(-S + i\int (jA + \bar\eta\psi)\Big) \int \Big(j\partial_\mu A + \bar\eta\partial_\mu\psi\Big). \end{aligned} \tag{11.44}$$

But this is just the Ward identity for the translations in Euclidean spacetime. For example, it follows that the two-point functions only depend on the difference $x - y$, and not on x and y separately. We see, that on the level of Ward identities the supersymmetry transformations indeed close on spacetime translations.

11.4.4 Changing the representation:

If we would have started with the following definition of the Dirac conjugate in Euclidean space,

$$\bar\psi = \psi^\dagger A, \quad A = \gamma_0\gamma_1$$

then Majorana spinors would have $\eta = \epsilon = 1$, so that

$$\mathcal{C}^T = -\mathcal{C} \quad \text{and} \quad \gamma_\mu^T = -\mathcal{C}^{-1}\gamma_\mu\mathcal{C}.$$

A possible representation is

$$\gamma_0 = \sigma_1, \quad \gamma_1 = \sigma_2 \Longrightarrow \mathcal{C} = \sigma_2, \quad \gamma_* = i\gamma_0\gamma_1 = \sigma_3, \quad A = i\sigma_3, \quad B = -\sigma_1.$$

Now the Majorana condition reads

$$\psi_c = B\psi^* = \psi \Longrightarrow \psi_2 = -\psi_1^*, \quad \text{or} \quad \psi = \begin{pmatrix}\psi_1\\ -\psi_1^*\end{pmatrix}, \quad \bar\psi = (i\psi_1^*, i\psi_1).$$

For Majorana spinors the symmetry properties are

$$\bar\psi\chi = \bar\chi\psi, \quad \bar\psi\gamma_\mu\chi = -\bar\chi\gamma_\mu\psi \quad \text{and} \quad \bar\psi\gamma_*\chi = -\bar\chi\gamma_*\psi$$

and the hermiticity poperties become

$$\bar\psi\chi \quad \text{is hermitean and} \quad \bar\psi\gamma_*\chi,\ \bar\psi\gamma_*\chi \quad \text{are antihermitean.}$$

Finally, the Fierz identity reads

$$\psi\bar\chi = -\tfrac{1}{2}\bar\chi\psi - \tfrac{1}{2}\gamma_\mu(\bar\chi\gamma_\mu\psi) - \tfrac{1}{2}\gamma_*(\bar\chi\gamma_*\psi).$$

For Majorana spinors further Fierz identities are easily derived from these properties:

$$(\bar{\alpha}\psi)(\bar{\alpha}\psi) = (\bar{\psi}\alpha)(\bar{\alpha}\psi) = -\tfrac{1}{2}(\bar{\alpha}\alpha)(\bar{\psi}\psi) \quad , \quad (\bar{\psi}\gamma^\mu\alpha)\psi = -\tfrac{1}{2}(\bar{\psi}\psi)\gamma^\mu\alpha.$$

A hermitean superfield has the form

$$\Phi(x,\alpha) = A(x) + i\bar{\alpha}\psi(x) + \tfrac{i}{2}(\bar{\alpha}\alpha)F(x)$$

with real fields A, F and Majorana spinorfield ψ. As supercharges we may take

$$Q = -i\frac{\partial}{\partial\bar{\alpha}} - \gamma^\mu\alpha\partial_\mu \quad \text{and} \quad \bar{Q} = i\frac{\partial}{\partial\alpha} + \bar{\alpha}\gamma^\mu\partial_\mu = Q^T\sigma_1.$$

One finds the following nontrivial anticommutation relations

$$\{Q,\bar{Q}\} = -2i(\gamma^\mu\partial_\mu).$$

The superderivatives are

$$D = -i\frac{\partial}{\partial\bar{\alpha}} + (\gamma^\mu\alpha)\partial_\mu \quad , \quad \bar{D} = i\frac{\partial}{\partial\alpha} - (\bar{\alpha}\gamma^\mu)\partial_\mu.$$

From the supersymmetry variation of the superfield we read off the following transformation rules for the component fields

$$\delta A = i\bar{\beta}\psi, \quad \delta\psi = (F + \partial\!\!\!/ A)\beta \quad \delta F = i\bar{\beta}\partial\!\!\!/\psi.$$

The supersymmetric, lorentzinvariant and *unstable* Euclidean action is gotten from the density

$$\begin{aligned} \tfrac{i}{2}\bar{D}\Phi D\Phi = &= \tfrac{i}{4}\bar{\psi}\psi + \tfrac{i}{2}(\bar{\alpha}\psi)F - \tfrac{i}{2}\bar{\alpha}\gamma^\mu\psi\partial_\mu A + \tfrac{i}{2}\bar{\alpha}\alpha\mathcal{L}_0 \\ \mathcal{L}_0 &= -\tfrac{1}{2}\partial_\mu A\partial^\mu A - \tfrac{i}{4}\bar{\psi}\partial\!\!\!/\psi + \tfrac{i}{4}\partial_\mu\bar{\psi}\gamma^\mu\psi + \tfrac{1}{2}F^2. \end{aligned}$$

Again we switch on an interaction by adding a superpotential

$$W(\Phi) = g\Phi + \tfrac{1}{3}\lambda\Phi^3,$$

which has the component expansion

$$W(\Phi) = W(A) + i\bar{\alpha}\psi W'(A) + \tfrac{i}{2}\bar{\alpha}\alpha\mathcal{L}_1 \quad \text{with} \quad \mathcal{L}_1 = gF + \lambda(A^2F - iA\bar{\psi}\psi).$$

The interacting theory has density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$.
The auxiliary field fulfils the algebraic equation

$$F = -g - \lambda A^2.$$

Eliminating it we obtain

$$-\mathcal{L} = \tfrac{1}{2}(\partial A)^2 + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \tfrac{1}{2}(g + \lambda A^2)^2 + i\lambda A\bar{\psi}\psi.$$

Hence, on the classical level we have constructed a $N = 1$ supersymmetric and stable Wess-Zumino model. Recall that here $\bar{\psi}$ is not $\psi^\dagger$ but rather $\psi^\dagger A$. Again we conclude that in the off-shell formulation the Euclidean model is unstable. It becomes stable only after the elimination of the auxiliary field F.