10 Superspaces

There are great advantages to constructing supersymmetric field theories in the superspace/superfield formalism, just as there are great advantages to constructing relativistic quantum field theories in a manifestly Lorentz covariant formalism. The following detour into superspace and superfield constructions pay off nicely when one constructs supersymmetric actions.

10.1 Minkowski spacetime as coset space

Let g be an arbitrary element of a group G which contains a subgroup H. We define equivalence classes in G: two elements g and g' are considered equivalent if they can be connected by a right multiplication with an element $h \in H$:

$$g' = g \circ h \quad \text{or} \quad g^{-1} \circ g' \in H. \tag{10.1}$$

This equivalence class is called the *coset* of g with respect to H. The set of all cosets is a manifold denoted by G/H. This way one obtains a fiber bundle with total space G, base manifold G/H and typical fiber H. A section $L(a)$, labeled by $\dim(G/H)$ parameters, parameterize the manifold if each coset contains exactly one of the L's. Once we have chosen a (local) section $L(a)$, each group element g can be uniquely decomposed into a product

$$g = L(a) \circ h.$$

A product of g with another group element, and in particular with $L(a)$, will define another L and h according to

$$g \circ L(a) = L(a') \circ h \quad \text{or} \quad g = L(a') \circ h \circ L^{-1}(a), \tag{10.2}$$

where a' and h are in general functions of both g and a. In particular, every Poincaré transformation can be uniquely decomposed as product of a translation and a Lorentz transformation,

$$(\Lambda, a) = (\mathbb{1}, a) \circ (\Lambda, 0)$$

and the Minkowski spacetime can be considered as coset manifold

$$\text{Minkowski spacetime} = \text{Poincaré group} \;/\; \text{Lorentz group}.$$

Choosing $g = (\Lambda, b)$ in (10.2) we find

$$(\Lambda, a) = (\mathbb{1}, \Lambda b + a) \circ (\Lambda, 0) \circ (\mathbb{1}, -b) \quad \text{such that} \quad a' = a + \Lambda b \quad \text{and} \quad h = \Lambda \tag{10.3}$$

in that equation. In section (4.3) we introduced the *representation*[1]

$$(\Lambda, a) \longrightarrow U(\Lambda, a) = \exp\big(-ia\hat{P} - \tfrac{i}{2}\omega\hat{M}\big)$$

of the Poincaré group. We recall the Poincaré algebra

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i\big(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\big) \\ [P_\rho, M_{\mu\nu}] &= i\big(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\big) \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{10.4}$$

Since $U(a)$ translates the argument of the quantum field,

$$U(a)\Phi(x)U^{-1}(a) = \Phi(x+a), \qquad U(a) = e^{-iaP},$$

we define

$$\Phi(x) = U(x)\Phi(0)U^{-1}(x). \tag{10.5}$$

From the relation (10.3) and representation property it follows that

$$U(a)U(x) = U(a+x) \quad \text{and} \quad U(\omega)U(x)U^{-1}(\omega) = U(e^\omega x), \quad U(\Lambda) \equiv U(\omega) = e^{-i\omega M/2}. \tag{10.6}$$

These composition rules for the unitary operators $U(a)$ and $U(\omega)$ also follow directly from the Poincaré algebra (10.4) which in particular implies

$$U(\omega)P_\mu U^{-1}(\omega) = \big(e^{-\omega}\big)_\mu^{\ \nu} P_\nu.$$

From these multiplication rules we may now easily extract the transformation properties of the quantum field (10.5) as follows:

$$\begin{aligned} U(a)\Phi(x)U^{-1}(a) &= U(a+x)\Phi(0)U^{-1}(a+x) = \Phi(a+x) \\ U(\omega)\Phi(x)U^{-1}(\omega) &= U(e^\omega x)U(\omega)\Phi(0)U^{-1}(\omega)U^{-1}(e^\omega x) = e^{i\omega\Sigma/2}\Phi(e^\omega x) \end{aligned}$$

where we assumed that

$$U(\omega)\Phi(0)U^{-1}(\omega) = e^{i\omega\Sigma/2}\Phi(0). \tag{10.7}$$

The $\Sigma_{\mu\nu}$ form some matrix representation of the algebra of the $M_{\mu\nu}$. For example, for Dirac spinors

$$\Sigma_{\mu\nu} = \tfrac{i}{2}\gamma_{\mu\nu}, \qquad S(e^{-i\omega\Sigma/2}) = e^\omega : \quad e^{i\omega\Sigma/2}\gamma^\rho e^{-i\omega\Sigma/2} = \big(e^\omega\big)^\rho_{\ \sigma}\gamma^\sigma.$$

The action of the Poincaré group on the quantum field is uniquely fixed once we knew how the Lorentz group acted on $\Phi(0)$, The Poincaré/Lorentz coset is not the most general example of a

[1] compared to the previous sections we change the signs of P_μ and $M_{\mu\nu}$. The (anti)commutators containing these generators must be changed accordingly

coset space. The translations form an invariant subgroup of the Poincaré group and it follows that the element h in (10.2) is independent of a. The infinitesimal version of this transformation rule reads

$$\delta_a \Phi = -i[aP, \Phi] \equiv i a^\mu r(P_\mu)\Phi, \quad \delta_\omega \Phi = -\tfrac{i}{2}[\omega M, \Phi] \equiv \tfrac{i}{2}\omega^{\mu\nu} r(M_{\mu\nu})\Phi$$

where we have defined the representations $r(P)$ and $r(M)$ of the infinitesimal translations and Lorentz boosts. One finds

$$r(P_\mu) = -i\partial_\mu \quad \text{and} \quad r(M_{\mu\nu}) = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}.$$

10.2 Superspace

Infinitesimal supersymmetry transformations are generated by the supercharges. Indeed in section 5 we have introduced the supercharges by the requirement that

$$\delta_\alpha \Phi = i[\bar{\alpha} Q, \Phi]$$

where Φ is any field of the theory. The $\bar{\alpha} Q$ fulfill commutation relations, contrary to the supercharges Q_α which satisfy anti-commutation relations. The spinor parameters α^α anti-commute with everything fermionic (including themselves) and commute with everything bosonic (including, of course, ordinary c-numbers. Hence with these parameters the supersymmetry algebra can be integrated to a group G, the *super-Poincaré group*, with typical group elements

$$U(a, \alpha, \omega) = \exp\big(- i(a, P) + i\bar{\alpha} Q - \tfrac{i}{2}(\omega, M)\big) \tag{10.8}$$

where we used the conventions in (2.4). Superspace is the coset space

Superspace = super-Poincaré group / Lorentz group.

The most commonly way to parameterize this 'manifold' is

$$U(a, \alpha) \equiv U(a, \alpha, \Lambda = \mathbb{1}) = \exp\left(-iaP + i\bar{\alpha} Q\right).$$

We rewrite this in terms of Weyl spinors. To calculate the product of group elements we need the Baker-Campbell-Hausdorff formula

$$e^A e^B = \exp\Big(\sum_{n=1}^\infty \frac{1}{n!} C_n(A, B)\Big) \tag{10.9}$$

where the C_n are multi-commutators of A and B:

$$C_1 = A + B, \quad C_2 = [A, B], \quad C_3 = \tfrac{1}{2}[[A, B], B] + \tfrac{1}{2}[A, [A, B]] \dots \tag{10.10}$$

We also need the various commutators of the super Poincaré algebra. These are the commutators of the Poincaré algebra plus the supersymmetry algebra

$$\{\bar{\beta} Q, \bar{\alpha} \bar{Q}\} = 2(\bar{\beta}\gamma^\mu \alpha) P_\mu.$$

With

$$[-bP+\bar\beta Q,-aP+\bar\alpha Q]=2\big(\bar\beta\gamma^\mu\alpha\big)P_\mu$$

we obtain the following composition rule for two sections in the supergroup

$$U(b^\mu,\beta)U(a^\mu,\alpha)=U\big(b^\mu+a^\mu-i\bar\beta\gamma^\mu\alpha,\beta+\alpha\big) \tag{10.11}$$

Let us calculate the conjugation of $U(a,\theta,\bar\theta)$ with a Lorentz transformation,

$$U(\omega)U(a,\alpha)U^{-1}(\omega)=\exp\Big(iU(\omega)\big(-aP+\alpha\bar Q\big)U^{-1}(\omega)\Big),\qquad U(\omega)=e^{-i\omega\Sigma/2}. \tag{10.12}$$

The conjugation of P_μ with the Lorentz boosts we have calculated above. To calculate the conjugation of the supercharges we use

$$[Q,M_{\mu\nu}]=\Sigma_{\mu\nu},$$

From the commutators we read find that

$$U(\omega)QU^{-1}(\omega)=e^{i\omega\Sigma/2}\,Q\quad\text{and}\quad U(\omega)\bar QU^{-1}(\omega)=\bar Q\,e^{-i\omega\Sigma/2}.$$

The transformations of the operators P,Q can be absorbed by the inverse transformations of the parameters (a,α) in (10.12). This way we find for the conjugation of $U(a,\alpha)$ with an arbitrary element of the Lorentz group

$$\exp\big(\tfrac{i}{2}\omega M\big)U(a,\alpha)=U(a',\alpha')\exp\big(\tfrac{i}{2}\omega M\big) \tag{10.13}$$

with transformed coordinates

$$a'^\mu=(e^\omega)^\mu{}_\nu a^\nu,\quad \alpha'=e^{-i\omega\Sigma/2}\alpha. \tag{10.14}$$

10.3 Representations on superfields

Similarly to $\Phi(x)=U(x)\Phi(0)U^{-1}(x)$ the action of the supersymmetry transformations on a superfield is defined via

$$\Phi(x,\alpha)=U(x,\alpha)\Phi(0,0)U^{-1}(x,\alpha). \tag{10.15}$$

For any group element the action on Φ is given by the coordinate transformation (10.13,10.14) in conjunction with

$$U(\omega)\Phi(0,0)U^{-1}(\omega)=\exp\big(\tfrac{i}{2}\omega\Sigma\big)\,\Phi(0,0).$$

More explicitly, from (10.12) and (10.13,10.7) we obtain

$$\begin{aligned} U(b,\beta)\Phi(x,\alpha)U^{-1}(b,\beta) &= \Phi(b+x-i\bar\beta\gamma^\mu\alpha,\alpha+\beta)\\ U(\omega)\Phi(x,\alpha)U^{-1}(\omega) &= e^{i\omega\Sigma}\Phi(e^\omega x,e^{-i\omega\Sigma/2}\alpha),\end{aligned} \tag{10.16}$$

the infinitesimal versions of which read

$$\delta\Phi = ib^\mu r(P_\mu)\Phi + i\bar\beta^\alpha r(Q_\alpha)\Phi \quad , \quad \delta\Phi = \tfrac{i}{2}\omega^{\mu\nu} r(M_{\mu\nu})\Phi \tag{10.17}$$

with

$$\begin{aligned} r(P_\mu) &= -i\partial_\mu \quad , \quad r(Q_\alpha) = -i\partial_\alpha - \gamma^\mu \alpha \partial_\mu \\ r(M_{\mu\nu}) &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu} - (\Sigma_{\mu\nu})^\alpha{}_\beta \alpha^\beta \partial_\alpha. \end{aligned} \tag{10.18}$$

Using the anticommuting properties of the α^α and

$$\partial_\alpha \alpha^\beta \equiv \frac{\partial}{\partial\alpha^\alpha}\alpha^\beta = \delta_\alpha^\beta$$

one can prove that the last term in (10.18) fulfills the Lorentz algebra. We rewrite the most relevant relations in terms of Weyl spinors. For that we recall our conventions for spinors and Dirac conjugate spinors:

$$\bar\alpha = \left(\theta^\alpha, \bar\theta_{\dot\alpha}\right), \quad Q = \begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix}, \quad \text{such that} \quad \bar\alpha Q = \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha} = \theta Q + \bar\theta \bar Q.$$

The representation of the supercharge reads as follows in the Weyl basis:

$$r(Q_\alpha) = -i\partial_\alpha - (\sigma^\mu\bar\theta)_\alpha \partial_\mu \quad , \quad r(\bar Q^{\dot\alpha}) = -i\bar\partial^{\dot\alpha} - (\tilde\sigma^\mu\theta)^{\dot\alpha}\partial_\mu \tag{10.19}$$

for the representation of the generators as differential operators. We have introduced

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \quad \bar\partial^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}}, \quad \text{such that} \quad \partial_\alpha\theta^\beta = \delta_\alpha^\beta, \quad \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta_{\dot\beta}^{\dot\alpha} \tag{10.20}$$

hold true. Because of the anticommuting property of the θ-parameters we must always anticommute θ or $\bar\theta$ to immediately behind the differentiation operator. We also insist on

$$\partial^\alpha = \frac{\partial}{\partial\theta_\alpha}, \quad \bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \quad \text{to satisfy} \quad \partial^\alpha\theta_\beta = \delta_\beta^\alpha, \quad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}, \tag{10.21}$$

which, for example, implies

$$\partial^\alpha\theta_\beta = \delta^\alpha_\beta = \varepsilon^{\alpha\sigma}\varepsilon_{\sigma\beta} = \epsilon^{\alpha\sigma}\epsilon_{\delta\beta}\delta^\delta_\sigma = -\epsilon^{\alpha\sigma}\epsilon_{\beta\delta}\partial_\sigma\theta^\delta = -\epsilon^{\alpha\sigma}\partial_\sigma\theta_\beta,$$

from which follows, that

$$\partial^\alpha = -\epsilon^{\alpha\sigma}\partial_\sigma. \tag{10.22}$$

We collect the basic notation and properties of N=1 superspace derivatives.

$$\begin{aligned} \partial^\alpha &= -\varepsilon^{\alpha\beta}\partial_\beta, \quad \bar\partial^{\dot\alpha} = -\varepsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\beta}, \qquad \partial_\alpha = -\varepsilon_{\alpha\beta}\partial^\beta, \qquad \bar\partial_{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\bar\partial^{\dot\beta} \\ \partial^\alpha\theta^\beta &= -\varepsilon^{\alpha\beta}, \qquad \partial_\alpha\theta_\beta = -\varepsilon_{\alpha\beta}, \qquad \bar\partial^{\dot\alpha}\bar\theta^{\dot\beta} = -\varepsilon^{\dot\alpha\dot\beta}, \qquad \bar\partial_{\dot\alpha}\bar\theta_{\dot\beta} = -\varepsilon_{\dot\alpha\dot\beta} \\ \partial_\alpha(\theta\theta) &= 2\theta_\alpha, \qquad \bar\partial_{\dot\alpha}(\bar\theta\bar\theta) = -2\bar\theta_{\dot\alpha}, \qquad \partial^2(\theta\theta) = 4 \qquad \bar\partial^2(\bar\theta\bar\theta) = 4 \end{aligned} \tag{10.23}$$

We can rewrite (10.19) as follows: From

$$\sigma_2\sigma_\mu\sigma_2 = \tilde{\sigma}_\mu^T \quad \text{or} \quad \sigma^\mu_{\alpha\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\alpha\beta}\tilde{\sigma}^{\mu\dot{\beta}\beta}$$

it follows that

$$r(\bar{Q}_{\dot{\alpha}}) \equiv \varepsilon_{\dot{\alpha}\dot{\beta}}\big(-i\bar{\partial}^{\dot{\beta}} - \tilde{\sigma}^{\mu\dot{\beta}\sigma}\varepsilon_{\sigma\rho}\theta^\rho\partial_\mu\big) = i\bar{\partial}_{\dot{\alpha}} + (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu.$$

Hence we arrive at the following equivalent form for the supercharges,

$$r(Q_\alpha) = -i\partial_\alpha - (\sigma^\mu\bar{\theta})_\alpha\partial_\mu \quad , \quad r(\bar{Q}_{\dot{\alpha}}) = i\bar{\partial}_{\dot{\alpha}} + (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu, \tag{10.24}$$

and these representations are mostly used in the literature. It is not difficult to see that

$$\{r(Q_\alpha), r(\bar{Q}_{\dot{\alpha}})\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu = 2\sigma^\mu_{\alpha\dot{\alpha}}r(P_\mu)$$

holds true, as required.

10.3.1 Component fields

The general scalar superfield $\Phi(x,\alpha) \equiv \Phi(x,\theta,\bar{\theta})$ is just a scalar function in $N=1$ rigid superspace. Its Taylor expansion in powers of θ and $\bar{\theta}$ is finite owing to the anticommuting property of these expansion parameters. For example, $\theta^\alpha(\theta\theta)$ vanishes. The coefficients in the expansion are local fields over Minkowski space. The following is already the most general superfield

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) \quad = \quad & C(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 M(x) + \bar{\theta}^{\,2}N(x) + (\theta\sigma^\mu\bar{\theta})\,V_\mu(x) \\ & +\theta^2\,\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^{\,2}\,\theta\psi(x) + \theta^2\bar{\theta}^{\,2}\,D(x).\end{aligned} \tag{10.25}$$

For example, we have used that $\bar{\theta}\tilde{\sigma}^\mu\theta V_\mu = -\theta\sigma^\mu\bar{\theta}V_\mu$ is not an independent term. This superfield contains as coefficient fields:

4 complex (pseudo) scalar fields	C, M, N, D
4 spinor fields	$\phi, \psi \in (\frac{1}{2}, 0), \quad (\chi, \lambda) \in (0, \frac{1}{2})$
1 Lorentz 4-vector field	V_μ

Altogether there are 16 (bosonic) + 16 (fermionic) field components.
The transformation laws for the components of a general $N=1$ superfield under supersymmetry transformation are calculated by comparing coefficients in the expansion

$$\begin{aligned}\delta\Phi(x,\theta,\bar{\theta}) \quad = \quad & \delta C(x) + \theta\delta\phi(x) + \bar{\theta}\delta\bar{\chi}(x) + \theta^2\delta M(x) + \bar{\theta}^{\,2}\delta N(x) + (\theta\sigma^\mu\bar{\theta})\,\delta V_\mu(x) \\ & +\theta^2\,\bar{\theta}\delta\bar{\lambda}(x) + \bar{\theta}^{\,2}\,\theta\delta\psi(x) + \theta^2\bar{\theta}^{\,2}\,\delta D(x).\end{aligned} \tag{10.26}$$

with

$$\begin{aligned}&\delta\Phi = i\zeta^\alpha r(Q_\alpha)\Phi + i\bar{\zeta}_{\dot{\alpha}}r(\bar{Q}^{\dot{\alpha}})\Phi \equiv \delta_\zeta\Phi + \delta_{\bar{\zeta}}\Phi \\ &\delta_\zeta = \zeta^\alpha\partial_\alpha - i(\zeta\sigma^\mu\bar{\theta})\partial_\mu \quad , \quad \delta_{\bar{\zeta}} = \bar{\zeta}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} - i(\bar{\zeta}\tilde{\sigma}^\mu\theta)\partial_\mu.\end{aligned} \tag{10.27}$$

Using the Fierz relations (8.90-8.90) one arrives after some lengthy but straightforward calculations at the following formulae for the supersymmetry transformations of the superfield:

$$\begin{aligned}
\delta_\zeta \Phi &= \zeta\phi + 2\theta\zeta M + \bar\theta(i\tilde\sigma^\mu\partial_\mu C - \tilde\sigma^\mu V_\mu)\zeta + \bar\theta^2(\zeta\psi + \tfrac{i}{2}\zeta\sigma^\mu\partial_\mu\bar\chi) \\
&\quad +(\theta\sigma^\mu\bar\theta)\big(\zeta\sigma_\mu\bar\lambda + \tfrac{i}{2}\partial_\nu\phi\sigma_\mu\tilde\sigma^\nu\zeta\big) \\
&\quad +i\theta^2\bar\theta\tilde\sigma^\mu\zeta\partial_\mu M + \bar\theta^2\theta(2\zeta D - \tfrac{i}{2}\sigma^\nu\tilde\sigma^\mu\zeta\partial_\mu V_\nu) + \tfrac{i}{2}\theta^2\bar\theta^2\zeta\sigma^\mu\partial_\mu\bar\lambda \qquad (10.28) \\
\delta_{\bar\zeta}\Phi &= \bar\zeta\bar\chi + 2\bar\theta\bar\zeta N + \theta(i\sigma^\mu\partial_\mu C + \sigma^\mu V_\mu)\bar\zeta + \theta^2(\bar\zeta\bar\lambda + \tfrac{i}{2}\bar\zeta\tilde\sigma^\mu\partial_\mu\phi) \\
&\quad +(\theta\sigma^\mu\bar\theta)\big(\psi\tilde\sigma_\mu\bar\zeta - \tfrac{i}{2}\partial_\nu\bar\chi\tilde\sigma_\mu\sigma^\nu\bar\zeta\big) \\
&\quad +i\bar\theta^2\theta\sigma^\mu\bar\zeta\partial_\mu N + \theta^2\bar\theta(2\bar\zeta D + \tfrac{i}{2}\tilde\sigma^\nu\sigma^\mu\bar\zeta\partial_\mu V_\nu + \tfrac{i}{2}\theta^2\bar\theta^2\bar\zeta\tilde\sigma^\mu\partial_\mu\psi
\end{aligned}$$

By comparing the two expressions (10.27) and (10.28) we obtain the following transformation rules for the component fields

$$\begin{aligned}
\delta C &= \zeta\phi + \bar\chi\bar\zeta && (10.29)\\
\delta\phi &= 2\zeta M + (i\sigma^\mu\partial_\mu C + \sigma^\mu V_\mu)\bar\zeta && (10.30)\\
\delta\bar\chi &= 2N\bar\zeta + (i\tilde\sigma^\mu\partial_\mu C - \tilde\sigma^\mu V_\mu)\zeta = 2N\bar\zeta + (\zeta\sigma^\mu\varepsilon)(i\partial_\mu C - V_\mu) && (10.31)\\
\delta M &= \bar\zeta(\bar\lambda + \tfrac{i}{2}\tilde\sigma^\mu\partial_\mu\phi) = \bar\lambda\bar\zeta - \tfrac{i}{2}\partial_\mu\phi\sigma^\mu\bar\zeta && (10.32)\\
\delta N &= \zeta\psi + \tfrac{i}{2}\zeta\sigma^\mu\partial_\mu\bar\chi && (10.33)\\
\delta V_\mu &= \zeta\sigma_\mu\bar\lambda + \psi\sigma_\mu\bar\zeta + \tfrac{i}{2}\partial_\nu\phi\sigma_\mu\tilde\sigma^\nu\zeta - \tfrac{i}{2}\partial_\nu\bar\chi\tilde\sigma_\mu\sigma^\nu\bar\zeta && (10.34)\\
\delta\bar\lambda &= 2D\bar\zeta + \tfrac{i}{2}\tilde\sigma^\nu\sigma^\mu\bar\zeta\partial_\mu V_\nu + i\tilde\sigma^\mu\zeta\partial_\mu M && (10.35)\\
\delta\psi &= 2D\zeta - \tfrac{i}{2}\sigma^\nu\tilde\sigma^\mu\zeta\partial_\mu V_\nu + i\sigma^\mu\bar\zeta\partial_\mu N && (10.36)\\
\delta D &= \tfrac{i}{2}\zeta\sigma^\mu\partial_\mu\bar\lambda + \tfrac{i}{2}\bar\zeta\tilde\sigma^\mu\partial_\mu\psi. && (10.37)
\end{aligned}$$

Note, that D transforms into a total derivative.

10.3.2 Real superfields:

So far we have not imposed any conditions on the superfield Φ. As a result the components of Φ form a highly reducible representation of $N = 1$ supersymmetry. We may easily half the degrees of freedom by imposing a reality condition on Φ which is consistent with supersymmetry. So let α, the $(1/2, 0)$ component of which is θ, and Q be Majorana spinors, such that

$$(\bar\alpha Q)^\dagger = (\theta Q + \bar Q\bar\theta)^\dagger = \bar Q\bar\theta + \theta Q$$

is hermitian and

$$\big(U(x,\theta,\bar\theta)\big)^\dagger = U^{-1}(x,\theta,\bar\theta)$$

is unitary. In this case

$$(\theta^\alpha)^* = \bar\theta^{\dot\alpha} \quad \text{and} \quad (\theta_\alpha)^* = \theta_{\dot\alpha}. \qquad (10.38)$$

Now we may impose the following *reality condition* on the superfield,

$$\big(\Phi(x,\theta,\bar\theta)\big)^\dagger = \Phi(x,\theta,\bar\theta), \tag{10.39}$$

and this condition is consistent with supersymmetry: If Φ is a real superfield, then the transformed superfield

$$\Phi(x',\theta',\bar\theta') = U(a,\zeta,\bar\zeta)\Phi(x,\theta,\bar\theta)U^{-1}(a,\zeta,\bar\zeta)$$

is real as well, provided the ζ satisfy the same reality conditions (10.38) as the coordinates θ of superspace. For a real superfield the component fields in (10.25) fulfill the conditions

$$C = C^\dagger, \quad D^\dagger = D, \quad M^\dagger = N, \quad V_\mu = V_\mu^\dagger, \quad \bar\chi_{\dot\alpha} = \phi^\dagger_\alpha \quad \text{and} \quad \bar\lambda_{\dot\alpha} = \psi^\dagger_\alpha. \tag{10.40}$$

The total number of component fields is now only $8+8$. The superfield (10.25) the components of which fulfill (10.40) is called the *real (general) superfield.*
We may rewrite a real superfield in the 4-component notation. For Majorana spinors we have

$$\begin{aligned}
&\bar\alpha\gamma_\mu\alpha = \bar\alpha\gamma_{\mu\nu}\alpha = 0, \quad \bar\alpha\alpha = \theta^2+\bar\theta^2, \quad \bar\alpha\gamma_5\alpha = \bar\theta^2-\theta^2\\
&\bar\alpha\gamma_\mu\gamma_5\alpha = \theta\sigma_\mu\bar\theta - \bar\theta\tilde\sigma_\mu\theta = 2\theta\sigma_\mu\bar\theta = -2\bar\theta\tilde\sigma_\mu\theta\\
&(\bar\alpha\alpha)(\bar\alpha\gamma_5\alpha) = (\bar\alpha\alpha)(\bar\alpha\gamma^\mu\alpha) = (\bar\alpha\alpha)(\bar\alpha\gamma_5\gamma^\mu\alpha) = 0.
\end{aligned} \tag{10.41}$$

In the 4-component notation a real superfield takes the form

$$\begin{aligned}
\Phi(x,\alpha) \;&=\; C + \bar\alpha\phi + \tfrac12\bar\alpha\alpha M_1 + \tfrac{i}{2}\bar\alpha\gamma_5\alpha M_2 + \tfrac12(\bar\alpha\gamma^\mu\gamma_5\alpha)V_\mu + (\bar\alpha\alpha)(\bar\alpha\lambda) + \tfrac12(\bar\alpha\alpha)^2 D\\
&\qquad\text{where}\quad M_1 \equiv M+M^\dagger,\; M_2\equiv i(M-M^\dagger),
\end{aligned} \tag{10.42}$$

and

$$\phi = \begin{pmatrix}\phi_\alpha\\ \chi^{\dot\alpha}\end{pmatrix}, \quad \lambda = \begin{pmatrix}\psi_\alpha\\ \lambda^{\dot\alpha}\end{pmatrix} \quad\text{and}\quad \alpha = \begin{pmatrix}\theta_\alpha\\ \bar\theta^{\dot\alpha}\end{pmatrix}$$

are Majorana spinors. If β is the Majorana spinor whose positive chirality part is ζ, then

$$\delta_\beta\Phi(x,\alpha) = \Big(\bar\beta\frac{\partial}{\partial\bar\alpha} - i(\bar\beta\gamma^\mu\alpha)\partial_\mu\Big)\Phi. \tag{10.43}$$

We may read off the transformation laws for the components of a real superfield from (10.29-10.37) by imposing the reality constraints (10.40). Alternatively we may directly compute them by comparing coefficients in the expansion

$$\begin{aligned}
\delta\Phi(x,\alpha) \;&=\; \delta C + \bar\alpha\delta\phi + \tfrac12\bar\alpha\alpha\delta M_1 + \tfrac{i}{2}\bar\alpha\gamma_5\alpha\delta M_2\\
&\quad+\tfrac12(\bar\alpha\gamma^\mu\gamma_5\alpha)\delta V_\mu + (\bar\alpha\alpha)(\bar\alpha\delta\lambda) + \tfrac12(\bar\alpha\alpha)^2\delta D,
\end{aligned}$$

with the $\delta_\beta\Phi$ in (10.43) where one inserts the explicit parameterization (10.42) for the superfield. A straightforward calculation yields

$$\begin{array}{rcl}\big(\bar\beta\partial_{\bar\alpha} - i(\bar\beta\gamma^\mu\alpha)\partial_\mu\big)\Phi &=& \bar\beta\phi + \bar\beta\alpha M_1 + \bar\beta\gamma_5\alpha M_2 + (\bar\beta\gamma^\mu\gamma_5\alpha)V_\mu - i(\bar\beta\gamma^\mu\alpha)\partial_\mu C \\ &+& 2(\bar\beta\alpha)\bar\alpha\lambda + (\bar\alpha\alpha)\bar\beta\lambda - i(\bar\beta\gamma^\mu\alpha)\bar\alpha\partial_\mu\phi \\ &-& \frac{i}{2}(\bar\beta\gamma^\mu\alpha)(\bar\alpha\alpha)\partial_\mu M_1 + \frac12(\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma_5\alpha)\partial_\mu M_2 \\ &-& \frac{i}{2}(\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma^\nu\gamma_5\alpha)\partial_\mu V_\nu - i(\bar\beta\gamma^\mu\alpha)(\bar\alpha\alpha)\bar\alpha\partial_\mu\lambda + 2(\bar\beta\alpha)(\bar\alpha\alpha)D.\end{array} \tag{10.44}$$

To compare this result with (10.42) we use Fierz identities. For that we recall the Fierz identity (3.66), which for a Majorana spinor α reduces to

$$4\alpha\bar\alpha = -\bar\alpha\alpha + \gamma_5\gamma_\mu(\bar\alpha\gamma_5\gamma^\mu\alpha) - \gamma_5(\bar\alpha\gamma_5\alpha). \tag{10.45}$$

The second line in (10.44), i.e. the terms quadratic in α, can be rewritten as

$$\tfrac14(\bar\alpha\alpha)(2\bar\beta\lambda + i\bar\beta\partial\!\!\!/\,\phi) - \tfrac14(\bar\alpha\gamma_5\alpha)(2\bar\beta\gamma_5\lambda + i\bar\beta\gamma_5\partial\!\!\!/\,\phi) - \tfrac14(\bar\alpha\gamma^\mu\gamma_5\alpha)(2\bar\beta\gamma_5\gamma_\mu\lambda - i\bar\beta\gamma^\nu\gamma_5\gamma_\mu\partial_\nu\phi).$$

The last term in the equation maybe recast as follows:

$$\bar\beta\gamma^\nu\gamma_5\gamma_\mu\partial_\nu\phi = \partial_\nu\phi\gamma_\mu\gamma_5\gamma^\nu\beta.$$

Again using the Fierz identity for $\alpha\bar\alpha$ and the relations in the last line in (10.41) we obtain the identities

$$\begin{array}{rcl}(\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma_5\alpha) &=& (\bar\beta\gamma_5\gamma^\mu\alpha)(\bar\alpha\alpha) \\ -i(\bar\beta\gamma^\mu\alpha)(\bar\alpha\alpha)(\bar\alpha\partial_\mu\lambda) &=& \frac{i}{4}(\bar\alpha\alpha)^2\bar\beta\partial\!\!\!/\,\lambda \\ (\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma^\nu\gamma_5\alpha) &=& -(\bar\alpha\alpha)(\bar\alpha\gamma_5\gamma^\nu\gamma_\mu\beta),\end{array}$$

which allow us to recast the cubic and quartic in α terms in (10.44). Now we are ready to compare the coefficients in (10.42) and (10.44) and extract the following transformation rules for the coefficient functions:

$$\begin{array}{rcl}\delta C &=& \bar\beta\phi \\ \delta D &=& \frac{i}{2}\bar\beta\partial\!\!\!/\,\lambda \\ \delta V_\mu &=& \bar\beta\gamma_\mu\gamma_5\lambda - \bar\beta\gamma_5\partial_\mu\phi + \frac{i}{2}\bar\beta\gamma_5\gamma_\mu\partial\!\!\!/\,\phi \\ \delta M_1 &=& \bar\beta\lambda + \frac{i}{2}\bar\beta\partial\!\!\!/\,\phi \\ \delta M_2 &=& i\bar\beta\gamma_5\lambda - \frac12\gamma_5\partial\!\!\!/\,\phi \\ \delta\phi &=& \big(M_1 + iM_2\gamma_5 - \gamma_5\gamma^\mu V_\mu + i\partial\!\!\!/\,C\big)\beta \\ \delta\lambda &=& \big(2D + \frac{i}{2}\partial\!\!\!/\,M_1 + \frac12\gamma_5\partial\!\!\!/\,M_2 + \frac{i}{2}\gamma_5\gamma^\nu\partial\!\!\!/\,V_\nu\big)\beta\end{array} \tag{10.46}$$

This transformation laws agree with those in (10.29-10.37) after imposing the reality constraints (10.40). This is just a check for the correctness of derived formulae.
The real multiplet consisting of the scalars C, M_1, D, the pseudoscalar M_2, the vector V_μ and the Majorana spinors ϕ, λ is not an irreducible representation of the supersymmetry algebra as follows from the general theory about representations of the $N = 1$ supersymmetry algebra and the following considerations.

10.3.3 Sub-multiplets

It is not difficult to see that the fields

$$\tilde{D} = 2D + \tfrac{1}{2}\Box C \quad , \quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \qquad \text{and} \quad \chi = \lambda - \tfrac{i}{2}\partial\!\!\!/\,\phi$$

transform entirely among themselves and thus form a submultiplet dV with components

$$dV = (\chi, F_{\mu\nu}, \tilde{D}).$$

This multiplet is called *curl multiplet* and it is irreducible, contrary to V itself. The transformation laws are

$$\delta\chi = (\tilde{D} - \tfrac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}\gamma_5)\beta, \qquad \delta F_{\mu\nu} = \bar{\beta}\gamma_5(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu)\chi, \qquad \delta\tilde{D} = i\bar{\beta}\partial\!\!\!/\,\chi.$$

This is (almost) the transformation for the gauge multiplet we have studied in section (8.1.1). Hence we may take the corresponding calculations for the commutators of two supersymmetry transformations. The algebra closes, provided $F_{[\alpha\beta,\mu]} = 0$:

$$\begin{aligned}
[\delta_1,\delta_2]\tilde{D} &= 2i(\bar{\beta}_2\gamma^\beta\beta_1)\big(\partial_\beta\tilde{D} + \partial^\alpha\, {}^*F_{\alpha\beta}\big) = 2i(\bar{\beta}_2\gamma^\beta\beta_1)\partial_\beta\tilde{D}\\
[\delta_1,\delta_2]\chi &= 2i(\bar{\beta}_2\gamma^\beta\beta_1)\partial_\beta\chi\\
[\delta_1,\delta_2]F_{\mu\nu} &= 2i(\bar{\beta}_2\gamma^\beta\beta_1)\big(\partial_\nu F_{\mu\beta} + \partial_\mu F_{\beta\nu}\big) = 2i(\bar{\beta}_2\gamma^\beta\beta_1)\partial_\beta F_{\mu\nu}.
\end{aligned}$$

A different invariant submultiplet of V is the chiral multiplet

$$\partial V = (M_1, M_2;\ \chi = \lambda + \tfrac{i}{2}\partial\!\!\!/\,\phi;\ \partial_\mu V^\mu;\ \tilde{D} = 2D - \tfrac{1}{2}\Box C)$$

with the transformation laws

$$\begin{aligned}
&\delta M_1 = \bar{\beta}\chi \quad , \quad \delta M_2 = i\bar{\beta}\gamma_5\chi\\
&\delta(\partial_\mu V^\mu) = -\bar{\beta}\gamma_5\partial\!\!\!/\,\chi\\
&\delta\chi = (\tilde{D} + i\partial\!\!\!/\,M_1 + \gamma_5\partial\!\!\!/\,M_2 + i\gamma_5\partial_\mu V^\mu)\beta\\
&\delta\tilde{D} = i\bar{\beta}\partial\!\!\!/\,\chi
\end{aligned}$$

Both dV and ∂V are submultiplets of V, but their complements are not. Hence V is *reducible but not fully reducible* and this property is quit common in supersymmetry. To construct further multiplets it is useful to impose constraints on the superfield which are compatible with supersymmetry.

10.3.4 Covariant spinor derivatives

The problem we shall address here is to find conditions on the superfield Φ to reduce the number of degrees of freedom which they describe. The most interesting conditions involve covariant spinor derivatives which can be introduced in an elegant way. The associativity of group multiplication

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) \tag{10.47}$$

has consequences for the infinitesimal left and right action of the group: Let

$$g_1 \circ g_2 \sim g_2 + O_1^L g_2 \quad \text{and} \quad g_2 \circ g_3 \sim g_2 + O_3^R g_2.$$

Then the associative law (10.47) implies

$$\begin{aligned} (g_2 + O_1^L g_2) \circ g_3 &= g_1 \circ (g_2 + O_3^R g_2) \Longrightarrow \\ g_2 + O_3^R g_2 + O_1^L g_2 + O_3^R O_1^L g_2 &= g_2 + O_1^L g_2 + O_3^R g_2 + O_1^L O_3^R g_2, \end{aligned}$$

or that left and right-'derivatives' commute,

$$[O_1^L, O_3^R] = 0. \tag{10.48}$$

So far we have introduced the left action of one element of the supergroup on another

$$\begin{aligned} U(a^\mu, \beta) U(x^\mu, \alpha) &= U\big(a^\mu + x^\mu - i\bar{\beta}\gamma^\mu \alpha, \beta + \alpha\big) \\ &\sim \Big(\mathbb{1} + i b^\mu r(P_\mu) + i\beta^\alpha r(Q_\alpha)\Big) U(x, \alpha) \end{aligned}$$

Now we introduce the right action which has the form

$$\begin{aligned} U(x^\mu, \alpha) U(a^\mu, \beta) &= U\big(x^\mu + a^\mu + i\bar{\beta}\gamma^\mu \alpha, \bar{\beta} + \alpha\big) \\ &\sim \big(\mathbb{1} + a^\mu D_\mu + \bar{\beta}^\alpha D_\alpha\big) U(x, \alpha) \end{aligned}$$

It is easy to calculate the derivative operators D :

$$D_\mu = \partial_\mu, \quad D_\alpha = \partial_{\bar{\alpha}} + i(\gamma^\mu \alpha)\partial_\mu. \tag{10.49}$$

In terms of Weyl spinors these differential operators read

$$\begin{aligned} &\bar{\beta}^\alpha D_\alpha = \zeta^\alpha D_\alpha + \bar{\zeta}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \quad \Longrightarrow \\ &D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \quad , \quad \bar{D}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} + i\tilde{\sigma}^{\mu\dot{\alpha}\alpha} \theta_\alpha \partial_\mu \\ &D^\alpha = -\partial^\alpha - i\bar{\theta}_{\dot{\alpha}} \tilde{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \quad , \quad \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu. \end{aligned} \tag{10.50}$$

From our general discussion we know already that

$$[\bar{\beta}^\alpha r(Q_\alpha), \bar{\beta}'^\alpha D_\alpha] = 0$$

from which follows that the $r(Q_\alpha)$ must anticommute with the D_α

$$\{D, r(Q)\} = \{\bar{D}, r(Q)\} = \{D, r(\bar{Q})\} = \{\bar{D}, r(\bar{Q})\} = 0. \tag{10.51}$$

Since the operators D are invariant under supersymmetry transformations they are called *covariant derivatives*. The (anti)commutators of the D's with each other are found to be

$$\{D, D\} = \{\bar{D}, \bar{D}\} = [D, \partial_\mu] = [\bar{D}, \partial_\mu] = [\partial_\mu, \partial_\nu] = 0 \tag{10.52}$$

and

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i(\sigma^\mu)_{\alpha\dot{\beta}} \partial_\mu. \tag{10.53}$$

It follows that the (fermionic) derivative operators D commute with the covariant derivatives and satisfy the same anti-commutation relations as the supercharges. I leave the proof of these simple (anti)commutators to you. It is really very simple.

10.3.5 Projection operators

In subsequent sections we will consider superfields which satisfy certain constraints. The corresponding fields can be obtained from the most general superfield by projections. It is convenient to study the properties of these projection operators before applying them to superfields.
We begin by proving the following set of relations which all follow from the anticommutation relations derived above. We shall study properties of the covariant derivatives and of

$$D^2 = D^\alpha D_\alpha \quad \text{and} \quad \bar{D}^2 = \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}. \tag{10.54}$$

Note that

$$D^\alpha D^2 = \bar{D}\bar{D}^2 = 0.$$

The formulae will shall need later are:

$$\begin{aligned}
[D_\alpha, \bar{D}^2] &= -4i\sigma^\mu_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_\mu & (10.55)\\
[D^\alpha, \bar{D}^2] &= 4i\bar{D}_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}\partial_\mu & (10.56)\\
[\bar{D}_{\dot\alpha}, D^2] &= 4iD^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu & (10.57)\\
[\bar{D}^{\dot\alpha}, \bar{D}^2] &= -4i\tilde\sigma^{\mu\dot\alpha\alpha}D_\alpha\partial_\mu & (10.58)\\
[D^2, \bar{D}^2] &= -8i(D\sigma^\mu\bar{D})\partial_\mu + 16\Box & (10.59)\\
[\bar{D}^2, D^2] &= -8i(\bar{D}\tilde\sigma^\mu D)\partial_\mu + 16\Box & (10.60)\\
\bar{D}\tilde\sigma^\mu D &= -D\sigma^\mu\bar{D} - 4i\partial^\mu & (10.61)\\
D^\alpha\bar{D}^2D_\alpha &= \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha} & (10.62)
\end{aligned}$$

We prove only some of these identities starting with the anticommutation relation (10.53). We multiply this with $\bar{D}^{\dot\alpha}$ from the right and summing over $\dot\alpha$ one obtains

$$D_\alpha\bar{D}^2 + \bar{D}_{\dot\alpha}D_\alpha\bar{D}^{\dot\alpha} = -2i\sigma^\mu_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_\mu \tag{10.63}$$

Alternatively, we may multiply (10.53) with $\varepsilon^{\dot\alpha\dot\beta}$ and find

$$(D_\alpha\bar{D}_{\dot\alpha} + \bar{D}_{\dot\alpha}D_\alpha)\varepsilon^{\dot\alpha\dot\beta} = -(D_\alpha\bar{D}^{\dot\beta} + \bar{D}^{\dot\beta}D_\alpha) = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\varepsilon^{\dot\alpha\dot\beta}.$$

From this relation we conclude

$$\bar{D}_{\dot\beta}D_\alpha\bar{D}^{\dot\beta} = -\bar{D}^2D_\alpha + 2i\sigma^\mu_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_\mu$$

and this maybe used for the second term on the left in (10.63). This then yields the formula (10.55). The formula (10.56) follows from this one as follows

$$[D^\alpha, \bar{D}^2] = \varepsilon^{\alpha\beta}[D_\beta, \bar{D}^2] = -4i\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\alpha}\bar{D}_{\dot\beta}\partial_\mu = 4i\bar{D}_{\dot\beta}\sigma^{\mu\dot\beta\alpha}\partial_\mu,$$

where we used the idenities (8.85) which, for convenience, we recall:

$$\tilde\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\tilde\sigma^\mu_{\beta\dot\beta} \quad , \quad \sigma^\mu_{\alpha\dot\alpha} = \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\tilde\sigma^{\mu\dot\beta\beta}. \tag{10.64}$$

Let us also prove the identity (10.59):

$$[D^2, \bar{D}^2] \;=\; D^\alpha[D_\alpha, \bar{D}^2] + [D^\alpha, \bar{D}^2]D_\alpha = -4iD^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_\mu + 4i\bar{D}_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}D_\alpha\partial_\mu$$

$$
\begin{aligned}
&= -4i(D\sigma^\mu\bar{D})\partial_\mu + 4i\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\big(\{D_\alpha,\bar{D}_{\dot{\alpha}}\} - D_\alpha\bar{D}_{\dot{\alpha}}\big)\partial_\mu \\
&= -8i(D\sigma^\mu\bar{D})\partial_\mu + 8\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\partial_\mu = -8i(D\sigma^\mu\bar{D})\partial_\mu + 16\Box
\end{aligned}
$$

where in the last step we have used that

$$
\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu_{\alpha\dot{\alpha}} = \mathrm{Tr}\,(\tilde{\sigma}^\mu\sigma^\nu) = 2\eta^{\mu\nu}.
$$

Now we are ready to introduce various projection operators. These are

$$
\Pi_+ = -\frac{1}{16\Box}\bar{D}^2D^2, \quad \Pi_- = -\frac{1}{16\Box}D^2\bar{D}^2, \quad \Pi_T = \frac{1}{8\Box}\bar{D}_{\dot{\alpha}}D^2\bar{D}^{\dot{\alpha}} = \frac{1}{8\Box}D^\alpha\bar{D}^2D_\alpha. \tag{10.65}
$$

First we prove that the sum of these operators is the identity

$$
\Pi_+ + \Pi_- + \Pi_T = \mathbb{1}. \tag{10.66}
$$

This is shown as follows:

$$
\begin{aligned}
\Pi_+ + \Pi_- + \Pi_T &= \frac{1}{16\Box}\big(\bar{D}_{\dot{\alpha}}D^2\bar{D}^{\dot{\alpha}} + D^\alpha\bar{D}^2D_\alpha - \bar{D}^2D^2 - D^2\bar{D}^2\big) \\
&= \frac{1}{16\Box}\big\{(\bar{D}_{\dot{\alpha}}D^2 - D^2\bar{D}_{\dot{\alpha}})\bar{D}^{\dot{\alpha}} + (D^\alpha\bar{D}^2 - \bar{D}^2D^\alpha)D_\alpha\big\} \\
&\overset{(10.56,10.57)}{=} \frac{i}{4\Box}\big(D^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{D}^{\dot{\alpha}} + \bar{D}_{\dot{\alpha}}\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu D_\alpha\big) \\
&= \frac{i}{4\Box}\big(D\sigma^\mu\bar{D} + \bar{D}\tilde{\sigma}^\mu D\big)\partial_\mu \overset{(10.61)}{=} \frac{i}{4\Box}\big(-4i\partial^\mu\big)\partial_\mu = \mathbb{1}
\end{aligned}
$$

Now we show that these operators are idempotent:

$$
\begin{aligned}
\Pi_+\Pi_+ &= \frac{1}{16\Box}\bar{D}^2D^2\frac{1}{16\Box}\bar{D}^2D^2 = \big(\frac{1}{16\Box}\big)^2\bar{D}^2D^2\bar{D}^2D^2 \\
&\overset{(10.59)}{=} \big(\frac{1}{16\Box}\big)^2\bar{D}^2D^2\big\{D^2\bar{D}^2 + 8iD\sigma^\mu\bar{D}\partial_\mu - 16\Box\big\} = \frac{1}{16\Box}\bar{D}^2D^2 = \Pi_+.
\end{aligned}
$$

In the second to last step we used that $D^3 = D^4 = 0$. Similarly one proves the same property for the other two operators, such that

$$
\Pi_+^2 = \Pi_+ \quad , \quad \Pi_-^2 = \Pi_- \quad \text{and} \quad \Pi_T^2 = \Pi_T. \tag{10.67}
$$

Finally, since $D^3 = \bar{D}^3 = 0$ these operators project onto orthogonal subspaces:

$$
\begin{aligned}
\Pi_+\Pi_- &= \big(\frac{1}{16\Box}\big)^2\bar{D}^2D^2D^2\bar{D}^2 = 0 \\
\Pi_-\Pi_+ &= \big(\frac{1}{16\Box}\big)^2D^2\bar{D}^2\bar{D}^2D^2 = 0 \\
\Pi_+\Pi_T &= -\frac{1}{8\cdot 16\Box^2}\bar{D}^2D^2D^\alpha\bar{D}^2D_\alpha = 0
\end{aligned}
$$

From the very definition of the projection operators we conclude that Π_+ projects onto the kernel of $\bar{D}$ and Π_- projects on the kernel of Π_-:

$$\bar{D}_{\dot{\alpha}}\Pi_+ = 0 \quad \text{and} \quad D^\alpha \Pi_- = 0. \tag{10.68}$$

Fields $\Phi = \Pi_+\Phi$ and $\Phi^\dagger = \Pi_-\Phi^\dagger$ are called *(anti)chiral superfields.*

10.4 Chiral superfields

Like all covariant derivatives, D and $\bar{D}$ can be used to impose covariant conditions on superfields. The most prominent such conditions are those for a *chiral superfield* Φ which we already discussed in the last subsection,

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{10.69}$$

and an *anti-chiral superfield* $\bar{\Phi}$ with

$$D_\alpha \Phi^\dagger = 0. \tag{10.70}$$

Since $\{D, \bar{D}\} \sim \partial_\mu$ a superfield cannot be both chiral and anti-chiral except if it is a constant field. These conditions are first-order differential equations and can easily be solved:

$$\Phi(x,\theta,\bar{\theta}) = \exp(i\theta\not{\partial}\bar{\theta})\Phi(x,\theta) \quad \text{and} \quad \Phi^\dagger(x,\theta,\bar{\theta}) = \exp(-i\theta\not{\partial}\bar{\theta})\Phi^\dagger(x,\bar{\theta}), \qquad \not{\partial} = \sigma^\mu\partial_\mu. \tag{10.71}$$

That these are the general solutions follows at once from

$$\bar{D}_{\dot{\alpha}} \exp(i\theta\not{\partial}\bar{\theta}) = \exp(i\theta\not{\partial}\bar{\theta})\bar{\partial}_{\dot{\alpha}} \quad \text{and} \quad D_\alpha \exp(-i\theta\not{\partial}\bar{\theta}) = \exp(-i\theta\not{\partial}\bar{\theta})\partial_\alpha. \tag{10.72}$$

The exponentials in (10.71) just shift the x-coordinate of the superfield and the explicit form of a chiral and anti-chiral superfield reads

$$\Phi = \Phi(x^\mu + i\theta\sigma^\mu\bar{\theta}, \theta) \quad \text{and} \quad \Phi^\dagger = \Phi(x^\mu - i\theta\sigma^\mu\bar{\theta}, \bar{\theta}). \tag{10.73}$$

The Taylor expansion of the fields $\Phi(x,\theta)$ and $\bar{\Phi}(x,\bar{\theta})$ in (10.71) is particularly simple

$$\Phi(x,\theta) = A + \sqrt{2}\theta\psi + \theta^2 F \quad \text{and} \quad \Phi^\dagger(x,\bar{\theta}) = A^\dagger + \sqrt{2}\bar{\psi}\bar{\theta} + \bar{\theta}^2 F^\dagger. \tag{10.74}$$

Actually when dealing with chiral superfields it is more convenient to pass to a different parametrization of superfields.

10.4.1 Alternative parametrizations of superfields

First we introduce different parametrizations for the elements of the supergroup as compared to (10.8), namely

$$
\begin{aligned}
U_1(a,\theta,\bar\theta) &= \exp\big(-i(a,P)+i\theta Q\big)\exp\big(i\bar\theta\bar Q\big)\\
U_2(a,\theta,\bar\theta) &= \exp\big(-i(a,P)+i\bar\theta\bar Q\big)\exp\big(i\theta Q\big),
\end{aligned}
\tag{10.75}
$$

where we made us of the conventions in (2.4). Using the group multiplication law (10.11) we can relate U and U_1, U_2:

$$
\begin{aligned}
U_1(x,\theta,\bar\theta) &= U(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \Longleftrightarrow U(x,\theta,\bar\theta) = U_1(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)\\
U_2(x,\theta,\bar\theta) &= U(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \Longleftrightarrow U(x,\theta,\bar\theta) = U_2(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)
\end{aligned}
\tag{10.76}
$$

and correspondingly the associated superfields (10.15)

$$
\begin{aligned}
\Phi_1(x,\theta,\bar\theta) &= \Phi(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \quad , \quad \Phi(x,\theta,\bar\theta) = \Phi_1(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)\\
\Phi_2(x,\theta,\bar\theta) &= \Phi(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \quad , \quad \Phi(x,\theta,\bar\theta) = \Phi_2(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)
\end{aligned}
\tag{10.77}
$$

Let us calculate how $U(a,\zeta,\bar\zeta)$ acts on Φ_1 and Φ_2, repectively:

$$
\begin{aligned}
&U(a,\zeta,\bar\zeta)\,\Phi_1(x,\theta,\bar\theta)\,U^{-1}(a,\zeta,\bar\zeta) = U(a,\zeta,\bar\zeta)\,\Phi(x-i\theta\sigma\bar\theta,\theta,\bar\theta)\,U^{-1}(a,\zeta,\bar\zeta)\\
&\overset{(10.11)}{=} \Phi(a+x-i\theta\sigma\bar\theta - i\zeta\sigma\bar\theta - i\bar\zeta\tilde\sigma\theta,\theta+\zeta,\bar\theta+\bar\zeta)\\
&\overset{(10.77)}{=} \Phi_1(a+x+2i\theta\sigma\bar\zeta + i\zeta\sigma\bar\zeta,\theta+\zeta,\bar\theta+\bar\zeta)
\end{aligned}
$$

and similarly

$$
U(a,\zeta,\bar\zeta)\,\Phi_2(x,\theta,\bar\theta)\,U^{-1}(a,\zeta,\bar\zeta) = \Phi_2(a+x-2i\zeta\sigma\bar\theta - i\zeta\sigma\bar\zeta,\theta+\zeta,\bar\theta+\bar\zeta).
$$

The infinitesimal transformation of Φ_1 and Φ_2 read

$$
\begin{aligned}
\delta\Phi_1 &= a^\mu\partial_\mu\Phi_2 + \Big(\bar\zeta_{\dot\alpha}\big(\bar\partial^{\dot\alpha} - 2i\tilde\sigma^{\mu\dot\alpha\alpha}\theta_\alpha\partial_\mu\big) + \zeta^\alpha\partial_\alpha\Big)\Phi_2\\
\delta\Phi_2 &= a^\mu\partial_\mu\Phi_2 + \Big(\zeta^\alpha\big(\partial_\alpha - 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\big) + \bar\zeta_{\dot\alpha}\partial^{\dot\alpha}\Big)\Phi_2
\end{aligned}
\tag{10.78}
$$

from which we read off how the supercharges act on these superfields:

$$
\begin{aligned}
&\Phi_1: \quad r_1(Q_\alpha) = -i\partial_\alpha, \quad r_1(\bar Q^{\dot\alpha}) = -i\bar\partial^{\dot\alpha} - 2(\tilde\sigma^\mu\theta)^{\dot\alpha}\partial_\mu, \quad r_1(\bar Q_{\dot\alpha}) = i\bar\partial_{\dot\alpha} + 2(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\\
&\Phi_2: \quad r_2(Q_\alpha) = -i\partial_\alpha - 2(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \quad r_2(\bar Q^{\dot\alpha}) = -i\bar\partial^{\dot\alpha}, \quad r_2(\bar Q_{\dot\alpha}) = i\bar\partial_{\dot\alpha}.
\end{aligned}
\tag{10.79}
$$

Of course, these transformed supercharges fulfil the correct anticommutation relations

$$
\{r(Q_\alpha), r(\bar Q_{\dot\alpha})\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu.
$$

Along with these transformed supercharges we obtain transformed covariant derivatives:

$$
\begin{aligned}
D^{(1)}_\alpha &= \partial_\alpha + 2i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \quad , \quad \bar D^{(1)}_{\dot\alpha} = -\bar\partial_{\dot\alpha}\\
D^{(2)}_\alpha &= \partial_\alpha \quad , \quad \bar D^{(2)}_{\dot\alpha} = -\bar\partial_{\dot\alpha} - 2i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu.
\end{aligned}
\tag{10.80}
$$

As one may guess from the relations (10.72) the operators in the 3 parametrisations for the superfield are related as

$$O^{(1)} = e^{-i\theta\not{\partial}\bar\theta}\, O\, e^{i\theta\not{\partial}\bar\theta} \quad \text{and} \quad O^{(2)} = e^{i\theta\not{\partial}\bar\theta}\, O\, e^{-i\theta\not{\partial}\bar\theta}.$$

In the parametrization (1) a chiral superfield fulfils the simple condition

$$D_{\dot\alpha}\Phi_1 = -\partial_{\dot\alpha}\Phi_1 = 0$$

and hence has the simple expansion (10.74), namely

$$\begin{aligned} \Phi_1(x,\theta) &= A(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) \Longrightarrow \\ \Phi(x,\theta,\bar\theta) &= A(x+i\theta\sigma\bar\theta) + \sqrt{2}\theta\psi(x+i\theta\sigma\bar\theta) + \theta^2 F(x+\theta\sigma\bar\theta). \end{aligned} \tag{10.81}$$

The last expansion can be simplified:

$$\begin{aligned} \Phi(x,\theta,\bar\theta) &= A(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu A(x) - \tfrac{1}{2}(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu A(x) \\ &\quad + \sqrt{2}\theta\psi(x) + i\sqrt{2}(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\psi(x)) + \theta^2 F(x) \\ &\overset{(8.93)}{=} \Big(A + i(\theta\sigma^\mu\bar\theta)\partial_\mu A - \tfrac{1}{4}\theta^2\bar\theta^2\Box A + \sqrt{2}\theta\psi + i\sqrt{2}(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\psi) + \theta^2 F\Big)(x). \end{aligned}$$

The second to last term can be rewritten by using

$$(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\psi) = \theta^\beta\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\psi_\beta = \tfrac{1}{2}\varepsilon^{\alpha\beta}\theta^2\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\psi_\beta = -\tfrac{1}{2}\theta^2(\partial_\mu\psi\sigma^\mu\bar\theta)$$

so that the chiral superfield has the following representation

$$\Phi = \Big(A + \sqrt{2}\theta\psi + \theta^2 F + i(\theta\sigma^\mu\bar\theta)\partial_\mu A + \frac{i}{\sqrt{2}}\theta^2(\bar\theta\tilde\sigma^\mu\partial_\mu\psi) - \tfrac{1}{4}\theta^2\bar\theta^2\Box A\Big)(x). \tag{10.82}$$

Now we may compare this expansion with the expansion (10.25) for a general superfield to make the following identifications:

$$\begin{array}{lllll} C \to A, & M \to F, & N \to 0, & D \to -\tfrac{1}{4}\Box A, & V_\mu \to i\partial_\mu A \\ \phi_\alpha \to \sqrt{2}\,\psi_\alpha, & \bar\chi \to 0, & \bar\lambda^{\dot\alpha} \to \dfrac{i}{\sqrt{2}}(\tilde\sigma^\mu\partial_\mu\psi)^{\dot\alpha}, & \psi(x) \to 0 & \end{array}$$

We may use this identification in the supersymmetry transformations (10.29-10.37) for the general $N=1$ superfield and end up with

$$\delta A = \sqrt{2}\,\zeta\psi, \quad \delta\psi_\alpha = \sqrt{2}\,\zeta_\alpha F + i\sqrt{2}\,(\sigma^\mu\bar\zeta)_\alpha\partial_\mu A, \quad \delta F = i\sqrt{2}\,\bar\zeta\tilde\sigma^\mu\partial_\mu\psi. \tag{10.83}$$

It is evident from these transformations that the fields (A,ψ,F) constitute an irreducible representation of the supersymmetry algebra. The supersymmetry algebra is realized linearly and offshell. If we would eliminate F, then supersymmetry would be realized non-linearly and on-shell, as we have discussed previously. The transformations (10.83) are just the supersymmetry transformation of the fields in the Wess-Zumino model (in the chiral basis). For later use we note, that for a chiral superfield the F-term transforms into a total derivative. The commutator of two transformations are

$$[\delta_1,\delta_2]\ldots = 2i(\bar\alpha_2\gamma^\mu\alpha_1)\partial_\mu\ldots\,,$$

as expected. The α_i are Majorana spinors parameter with $(1/2, 0)$ component ζ_i.
We could repeat the same reasoning for antichiral fields fulfilling $D_\alpha \Phi^\dagger = 0$. We would find

$$\Phi^\dagger = A^*(z) + \sqrt{2}\bar{\theta}\bar{\psi}(z) + \bar{\theta}^2 F^*(z), \qquad z = x - i\theta\sigma^\mu\bar{\theta} \tag{10.84}$$

instead of

$$\Phi = A(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y), \qquad y = x + i\theta\sigma^\mu\bar{\theta}, \tag{10.85}$$

as we have found for a chiral field.

10.5 Invariant action for scalar superfields

There cannot exist a supersymmetric Lagrangian density since $\delta\mathcal{L} = 0$ would imply that

$$[\delta_1, \delta_2]\mathcal{L} \sim \partial\mathcal{L} = 0,$$

or that $\mathcal{L}$ must be constant. Even if $\mathcal{L}$ is not supersymmetric, the action is still supersymmetric if $\delta\mathcal{L}$ is a derivative which would not contribute to δS. In general $\mathcal{L}$ can be written as a sum of terms, each of which is some component of a superfield that is constructed out of elementary superfields and their covariant derivatives. The transformation rules (10.29-10.37) show that for a general $N = 1$ superfield the only component whose variation is a derivative is the D-component, since only the last two terms of the supercharges

$$r(Q_\alpha) = -i\partial_\alpha - (\sigma^\mu\bar{\theta})_\alpha\partial_\mu \quad \text{and} \quad r(\bar{Q}_{\dot{\alpha}}) = i\bar{\partial}_{\dot{\alpha}} + (\theta\sigma^\mu)_\alpha\partial_\mu$$

contribute to the variation of the D-terms. For D to be a scalar, the superfield itself must be a scalar. Thus for a general scalar superfield only the integral of the D-term is a good candidate for an invariant action

$$S = \int d^4x \Phi|_D. \tag{10.86}$$

But no satisfactory action can be gotten this way without special conditions on the superfield. For a general superfield Φ the only sort of kinematic action S_0 that is bilinear in Φ and $\Phi^\dagger$ and involves no more than two derivatives is of the form

$$S_0 \sim \int d^4x \left[\Phi^\dagger\Phi\right]_D. \tag{10.87}$$

Earlier we have seen, that for a (anti)chiral superfield the F-term transforms into a total derivative,

$$\delta F = i\sqrt{2}\bar{\zeta}\tilde{\sigma}^\mu\partial_\mu\psi,$$

so that this term is another good candidate for an invariant action

$$S = \int d^4x\, \Phi^{\text{chiral}}|_F \quad , \quad S = \int d^4x\, \Phi^{\dagger\text{antichiral}}|_F \tag{10.88}$$

For a renormalizable theory the chiral field in (10.88) will be the product of elementary chiral fields. Hence to make further progress we need to study the product of superfields.

10.5.1 Products of superfields

Let Φ_1 and Φ_2 be two superfields, that is two field on superspace which transform as in (10.16). Then $\Phi = \Phi_1\Phi_2$ transforms the same way as Φ_1 and Φ_2 do and hence is a superfield as well. This follows from

$$\begin{aligned} U\Phi(x,\alpha)U^{-1} &= U\Phi_1(x,\alpha)U^{-1}U\Phi_2(x,\alpha)U^{-1} = \Phi_1(x',\alpha')\Phi_2(x',\alpha') = \Phi(x',\alpha'), \\ \text{where} \quad & U = U(b,\beta), \quad x' = x + b - i\bar\beta\gamma\alpha, \quad \alpha' = \alpha + \beta. \end{aligned}$$

Equivalently, the infinitesimal supersymmetry transformation of the product field is

$$\delta\Phi = i[(\bar\beta Q), \Phi_1\Phi_2] = \delta\Phi_1\Phi_2 + \Phi_1\delta\Phi_2.$$

Clearly, the product of chiral superfields is a chiral superfield and the product of anti-chiral superfields is a anti-chiral superfield. For example, let

$$\begin{aligned} \Phi_i &= A_i(y) + \sqrt{2}\theta\psi_i(y) + \theta^2 F_i(y), \quad y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta \\ \Phi_i^\dagger &= A_i^*(z) + \sqrt{2}\bar\theta\bar\psi_i(z) + \bar\theta^2 F_i^*(z), \quad z^\mu = x^\mu - i\theta\sigma^\mu\bar\theta \end{aligned} \tag{10.89}$$

be a collection of chiral and antichiral superfields. Then

$$\begin{aligned} (\Phi_i\Phi_j)(x,\theta,\bar\theta) &= (A_iA_j)(y) + \sqrt{2}\theta(A_i\psi_j + A_j\psi_i)(y) + \theta^2(A_iF_j + A_jF_i - \psi_i\psi_j)(y), \\ (\Phi_i^\dagger\Phi_j^\dagger)(x,\theta,\bar\theta) &= (A_i^*A_j^*)(z) + \sqrt{2}\bar\theta(A_i^*\bar\psi_j + A_j^*\bar\psi_i)(z) + \bar\theta^2(A_i^*F_j^* + A_j^*F_i^* - \bar\psi_i\bar\psi_j)(z), \end{aligned} \tag{10.90}$$

where we made use of the Fierz identity (8.90). More interesting for our purpose is the product of a chiral and antichiral superfield:

$$\begin{aligned} \Phi_i^\dagger(z)\Phi_j(y) &= A_i^*A_j + \sqrt{2}A_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_iA_j + \theta^2A_i^*F_j + \bar\theta^2F_i^*A_j \\ &\quad + 2(\bar\theta\bar\psi_i)(\theta\psi_j) + \sqrt{2}\bar\theta^2\theta\psi_jF_i^* + \sqrt{2}\theta^2\bar\theta\bar\psi_iF_j + \theta^2\bar\theta^2F_i^*F_j, \end{aligned} \tag{10.91}$$

where the argument of (A,ψ,F) is y and that of $(A^*,\bar\psi,F^*)$ is z. Not all terms in this expansion are or interest to us. Since $\Phi^\dagger\Phi$ ia a general superfield we concentrate on the D term which is a good candidate for the Lagrangian density of an invariant action. We use (see 10.81):

$$\begin{aligned} A(y) &= A(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu A(x) - \tfrac{1}{4}\theta^2\bar\theta^2\Box A(x) \\ A^*(z) &= A^*(x) - i(\theta\sigma^\mu\bar\theta)\partial_\mu A^*(x) - \tfrac{1}{4}\theta^2\bar\theta^2\Box A^*(x) \\ \theta\psi(y) &= \theta\psi(x) + i(\theta\sigma^\mu\bar\theta)\big(\theta\partial_\mu\psi(x)\big) \\ \bar\theta\bar\psi(z) &= \bar\theta\bar\psi(x) - i(\theta\sigma^\mu\bar\theta)\big(\bar\theta\partial_\mu\bar\psi(x)\big) \\ \theta^2F(y) &= \theta^2F(x) \\ \bar\theta^2F^*(z) &= \bar\theta^2F^*(x) \end{aligned} \tag{10.92}$$

To calculate the D-term we only need consider terms in (10.91) which contain as many powers of θ as of $\bar{\theta}$. Hence the only non-vanishing D-terms come from

$$\begin{aligned}
A_i^*(z)A_j(y)|_D &= \tfrac{1}{2}\big(\partial_\mu A_i^*\partial^\mu A_j - \tfrac{1}{2}A_i^*\Box A_j - \tfrac{1}{2}\Box A_i^* A_j\big) = \partial_\mu A_i^*\partial^\mu A_j - \tfrac{1}{4}\Box(A_i^* A_j) \\
\big(\bar{\theta}\bar{\psi}_i(z)\big)\big(\theta\psi_j(y)\big) &= i\theta\sigma^\mu\bar{\theta}\big\{(\bar{\theta}\bar{\psi}_i)(\theta\partial_\mu\psi_j) - (\theta\psi_j)(\bar{\theta}\partial_\mu\bar{\psi}_i)\big\} = \tfrac{i}{4}(\partial_\mu\psi_j\sigma^\mu\bar{\psi}_i) - \tfrac{i}{4}(\psi_j\sigma^\mu\partial_\mu\bar{\psi}_i) \\
\theta^2\bar{\theta}^2 F_i(z)^* F_j(y) &= \theta^2\bar{\theta}^2 F_i(x)F_j^*(x).
\end{aligned}$$

and we end up with

$$\Phi_i^\dagger(z)\Phi_j(y)\big|_D = \partial_\mu A_i^*\partial^\mu A_j - \tfrac{1}{4}\Box(A_i^* A_j) + \tfrac{i}{2}(\partial_\mu\psi_j\sigma^\mu\bar{\psi}_i) - \tfrac{i}{2}(\psi_j\sigma^\mu\partial_\mu\bar{\psi}_i) + F_i(x)F_j^*(x). \tag{10.93}$$

For $i=j$ this can be rewritten as

$$\Phi_i^\dagger(z)\Phi_i(y)\big|_D = \partial_\mu A_i^*\partial^\mu A_i - \tfrac{i}{2}\bar{\psi}_i\partial\!\!\!/\,\psi_i + |F_i(x)|^2 + \text{surface term}, \tag{10.94}$$

where ψ_i is the Majorana spinor with $(1/2,0)$-componennt ψ_i. Summing over i this just becomes the Lagrangian density for the well-studied free *Wess-Zumino* model[2]. Let us finally calculate the D-terms of the product of 3 chiral or 3 antichiral fields

$$\begin{aligned}
(\Phi_i\Phi_j\Phi_k)(y) &= \big(A_iA_jA_k + \sqrt{2}(A_iA_j\theta\psi_k + \text{cycl}) + \theta^2\big((A_iA_jF_k - A_k\psi_i\psi_j) + \text{cycl}\big)\big)(y) \\
(\Phi_i^\dagger\Phi_j^\dagger\Phi_k^\dagger)(z) &= \big(A_i^*A_j^*A_k^* + \sqrt{2}(A_i^*A_j^*\bar{\theta}\bar{\psi}_k + \text{cycl}) + \bar{\theta}^2\big((A_i^*A_j^*F_k^* - A_k^*\bar{\psi}_i\bar{\psi}_j) + \text{cycl}\big)\big)(z).
\end{aligned} \tag{10.95}$$

The D-terms of these composite chiral fields will enter the Lagrangian densities for scalar superfields.

10.5.2 Invariant actions

The most general supersymmetric, renormalizable Lagrangian, involving only N scalar superfields is given by

$$\begin{aligned}
\mathcal{L} &= \Phi_i^\dagger\Phi_i|_D + W|_F + W^\dagger|_F, \quad \text{where} \\
W(\Phi) &= g_i\Phi_i + \tfrac{1}{2}m_{ij}\Phi_i\Phi_j + \tfrac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k
\end{aligned} \tag{10.96}$$

is called the superpotential. The mass matrix m_{ij} and the λ_{ijk} are symmetric in their indices. We have already argued that the corresponding action is invariant under supersymmetry transformations. Let us see what are the dimensions of the various fields and coupling constants. We shall see that the polynomial $W(\Phi)$ may not contain quartic or higher powers of the superfield in order for the corresponding model to be perturbatively renormalizable. From the explicit form of the supercharges we take

$$[\theta] = [\bar{\theta}] = L^{1/2}.$$

A spinorfield has dimensions $L^{-3/2}$ and hence

[2] The different sign for the Dirac term originates in the different signs of $[\delta_1, \delta_2]$ here and in earlier chapters.

$$
\begin{aligned}
[\Phi] = [\Phi^\dagger] = L^{-1} \quad &\Longrightarrow \quad [\Phi^\dagger\Phi] = L^{-2}, \quad [\Phi^\dagger\Phi|_D] = L^{-4} \\
[\Phi^n] = L^{-n} \quad &\Longrightarrow \quad [\Phi^n|_F] = [\Phi^{\dagger n}|_F] = L^{-n-1}.
\end{aligned}
\tag{10.97}
$$

We read off the following dimensions of the coupling constant and component fields

$$
[m] = L^{-2}, \quad [\lambda] = L^0, \quad [A] = L^{-1} \quad \text{and} \quad [F] = L^{-1}.
$$

If we would allow for powers of Φ higher than the third in the above Lagrangian density, then the supersymmetry model would not be perturbatively renormalizable.
Now we collect our previous results to express $\mathcal{L}$ in (10.96) in component fields:

$$
\begin{aligned}
\mathcal{L} \;=\;& \partial_\mu A_i^* \partial^\mu A_i - \tfrac{i}{2}\bar{\psi}_i \partial\!\!\!/ \psi_i + |F_i(x)|^2 \\
&+\Big(g_i F_i + m_{ij}(A_i F_j - \tfrac{1}{2}\psi_i\psi_j) + \lambda_{ijk}\big(A_i A_j F_k - A_k \psi_i \psi_j\big)\Big) + \Big(\text{h.c.}\Big)
\end{aligned}
\tag{10.98}
$$

In the first line ψ is regarded as Majorana spinor and in the second line as Weyl spinor. This is just the action of the Wess-Zumino model which we have studied in detail in chapter 5. As we have seen there, we could eliminate the auxiliary fields F_i to arrive at the more familiar action for the on-shell model containing N complex fields F_i and N Majorana spinors ψ_i.

10.5.3 Superspace integration

There in an elegant way to extract the D and F terms from a superfield which is based on an integration calculus on superspace. We begin with the Berezin integral for a single Grassmann parameter θ:

$$
\int d\theta\ \theta = 1 \quad , \quad \int d\theta = 0 \quad , \quad \int d\theta\ f(\theta) = f_1 \tag{10.99}
$$

where we have used the fact that an arbitrary function of a single Grassmann parameter θ has the Taylor series expansion $f(\theta) = f_0 + \theta f_1$. We demand that the $d\theta$ anticommute,

$$
\{d\theta, d\theta\} = \{d\theta, \theta\} = 0.
$$

We note three facts which follow from the definitions in (10.99).

- The Berezin integration is translationally invariant:

$$
\int d(\theta+\xi)\ f(\theta+\xi) = \int d\theta\ f(\theta) \quad , \quad \int d\theta\ \frac{d}{d\theta} f(\theta) = 0. \tag{10.100}
$$

- The Berezin integration is equivalent to differentiation:

$$
\frac{d}{d\theta} f(\theta) = f_1 = \int d\theta\ f(\theta) \tag{10.101}
$$

- We can define a Grassmann delta function by

$$\int d\theta f(\theta)\delta(\theta) = f(0) \Longrightarrow \delta(\theta) \equiv \theta \tag{10.102}$$

These results are easily generalized to the case of the N=1 superspace coordinates θ^α, $\overline{\theta}_{\dot\alpha}$: All $\theta_\alpha, \theta_\alpha, d\bar\theta^{\dot\alpha}, \bar\theta^{\dot\alpha}$ anticommute so that for example

$$\int d\theta_1 \int d\theta_2 \theta_1\theta_2 = -\int d\theta_1 (\int d\theta_2\theta_2)\theta_1 = -\int d\theta_1\theta_1 = -1$$

and the integral of an arbitary function

$$f = f^{(0)} + \theta_1 f^{(1)} + \theta_2 f^{(2)} + \theta_1\theta_2 f^{(3)}$$

is equal to

$$\int d\theta_1 d\theta_2 f = -f^{(3)}.$$

The volume elements in superspace are given by

$$d^2\theta = -\tfrac{1}{4} d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta} \quad , \quad d^2\bar\theta = -\tfrac{1}{4} d\bar\theta_{\dot\alpha} d\bar\theta_{\dot\beta} \varepsilon^{\dot\alpha\dot\beta} \quad , \quad d^4\theta = d^2\theta\, d^2\bar\theta. \tag{10.103}$$

Using this notation and the spinor summation convention, we have the following identities:

$$\int d^2\theta\, \theta\theta = 1 \quad , \quad \int d^2\bar\theta\, \bar\theta\bar\theta = 1. \tag{10.104}$$

We prove the second to last identity:

$$\int d^2\theta\, \theta\theta = -\tfrac{1}{4}\int d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta}\, \theta\theta = \tfrac{1}{2}\int d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta} \theta^1\theta^2 = -\int d\theta^1 d\theta^2 \theta^1\theta^2 = 1.$$

The delta-function is defined by

$$\int d^2\theta f(\theta)\delta^2(\theta) = f(0) \Longrightarrow \delta^2(\theta) = \theta\theta \quad \text{and similarly} \quad \delta^2(\bar\theta) = \bar\theta\bar\theta.$$

Now we consider a general superfield with component expansion given by (10.25). If we integrate Φ with $d^4\theta$ we just obtain the D-component of the superfield,

$$\begin{aligned} \int d^4\theta \Phi(x,\theta,\bar\theta) &= \int d^4\theta \Big(f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + (\theta\sigma^\mu\bar\theta)\, V_\mu(x) \\ &\qquad + (\theta\theta)(\bar\theta\bar\lambda(x)) + (\bar\theta\bar\theta)(\theta\psi(x)) + (\theta\theta)(\bar\theta\bar\theta)\, D(x)\Big) \\ &= \int d^2\theta d^2\bar\theta (\theta\theta)(\bar\theta\bar\theta) D(x) = d(x). \end{aligned}$$

Analogouly, for a chiral field (10.85) we may project onto the F component as follows:

$$\int d^4\theta \Phi(y,\theta)\delta(\bar\theta) = \int d^2\theta \Phi(x,\theta) = F(x).$$

This shows, that the action of a chiral field giving rise to a renormalizable model can be written as follows

$$S = \int d^4x\, d^4\theta\, \Phi_i^\dagger \Phi_i + \int d^4x\, d^4\theta \delta^2(\bar\theta) \left(g_i \Phi_i + \tfrac{1}{2} m_{ij} \Phi_i \Phi_j + \tfrac{1}{3} \lambda_{ijk} \Phi_i \Phi_j \Phi_k + \text{h.c.} \right) \tag{10.105}$$

10.6 Vector-superfields and susy-gauge transformations

After having constructed the supersymmetric Lagrangian describing spin-0 and spin-1/2 particles we want to construct models containing spin-1 particles. Ultimately we are interested in supersymmetric Yang-Mills theories in the superfield formulation. In component fields we have already investigated these theories earlier on.
We start with a real superfield V satisfying the reality condition

$$V(x,\theta,\bar\theta) = V^\dagger(x,\theta,\bar\theta) \tag{10.106}$$

Earlier we have shown that such a field has the expansion (10.42). It is convenient to make the replacements

$$\lambda \longrightarrow \lambda + \tfrac{i}{2} \partial\!\!\!/\, \phi \quad \text{and} \quad D \longrightarrow D - \tfrac{1}{4} \Box C$$

in that formula, such that

$$\begin{aligned} V(x,\alpha) \;&=\; C + \bar\alpha\phi + \tfrac{1}{2}\bar\alpha\alpha M_1 + \tfrac{i}{2}\bar\alpha\gamma_5\alpha M_2 + \tfrac{1}{2}(\bar\alpha\gamma^\mu\gamma_5\alpha)V_\mu \\ &\quad + (\bar\alpha\alpha)\bar\alpha(\lambda + \tfrac{i}{2}\partial\!\!\!/\,\phi) + \tfrac{1}{2}(\bar\alpha\alpha)^2(D - \tfrac{1}{4}\Box C) \end{aligned} \tag{10.107}$$

with real C, D, M_1, M_2 and V_μ and Majorana spinors ϕ and λ. The component V_μ lends its name to the entire multiplet. Again the D term of V transforms into a spacetime derivative, see (10.43),

$$\delta D = \tfrac{i}{2}\bar\beta \partial\!\!\!/\, \lambda$$

which makes it a good candidate for the Lagrangian density giving rise to an invariant action. Examples of vector-superfields are

$$V = \Phi^\dagger\Phi, \quad V = \Phi + \Phi^\dagger,$$

where Φ is a chiral superfield. For Φ in (10.82) we find

$$\begin{aligned} \Phi + \Phi^\dagger \;&=\; A_1 + \sqrt{2}\bar\alpha\psi + \tfrac{1}{2}\bar\alpha\alpha F_1 + \tfrac{i}{2}\bar\alpha\gamma_5\alpha F_2 + \tfrac{1}{2}(\bar\alpha\gamma^\mu\gamma_5\alpha)\partial_\mu A_2 \\ &\quad + \frac{i}{\sqrt{2}}(\bar\alpha\alpha)(\bar\alpha\partial\!\!\!/\,\psi) - \tfrac{1}{8}(\bar\alpha\alpha)^2\Box A_1 \end{aligned} \tag{10.108}$$

where we introduced the real fields

$$A_1 = A + A^*, \quad A_2 = i(A - A^*), \quad F_1 = F + F^* \quad \text{and} \quad F_2 = i(F - F^*).$$

If we now transform the real superfield as

$$V(x,\alpha) \longrightarrow V'(x,\alpha) = V(x,\alpha) + \Phi(x,\alpha) + \Phi^\dagger(x,\alpha) \qquad (10.109)$$

then the component field V_μ is gauge tansformed with gauge parameter A_2:

$$V'_\mu = V_\mu + \partial_\mu A_2. \qquad (10.110)$$

Following Wess and Zumino [51] one therefore calls the transformation (10.109) the supersymmetric extension of a gauge transformation. Under this transformation the components of V transform as

$$\begin{array}{l} C' = C + A_1, \quad M'_1 = M_1 + F_1, \quad M'_2 = M_2 + F_2, \quad D' = D \\ V'_\mu = V_\mu + \partial_\mu A_2, \quad \phi' = \phi + \sqrt{2}\psi, \quad \lambda' = \lambda \end{array} \qquad (10.111)$$

We see that the λ and D components of V are super-gauge invariant, that is they are invariant under the transformation (10.109). As we already mentioned above, the field V_μ transforms as an abelian gauge potential so that the corresponding field strength

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

is also super-gauge invariant. The important conclusion is the following: the D term of a real superfield transforms into a spacetime derivative under supersymmetry transformations and at the same time is super-gauge invariant. Thus this term is a good candidate for a supersymmetric Lagrangian which is super-gauge invariant.

10.6.1 The Wess-Zumino gauge

To achieve the socalled Wess-Zumino gauge one chooses the chiral superfield in (10.109) such that

$$C' = M'_1 = M'_2 = \phi' = 0.$$

This gauge can always be attained as is easily seen from the transformation rules (10.111) for the component fields. This is not a complete gauge fixing, since we do not fix A_2 wich causes the gauge transformation of V_μ. In this gauge the real superfield simplifies considerably,

$$\begin{array}{rcl} V_{WZ}(x,\alpha) & = & \frac{1}{2}(\bar{\alpha}\gamma^\mu\gamma_5\alpha)V_\mu + (\bar{\alpha}\alpha)(\bar{\alpha}\lambda) + \frac{1}{2}(\bar{\alpha}\alpha)^2 D \\ & = & (\theta\sigma^\mu\bar{\theta})V_\mu + (\theta\theta)(\bar{\theta}\bar{\lambda}) + (\bar{\theta}\bar{\theta})(\theta\lambda) + (\theta\theta)(\bar{\theta}\bar{\theta})D \end{array} \qquad (10.112)$$

and contains the gauge potentail V_μ, its supersymmetric partner λ and an auxiliary field D. After this partial gauge fixing the number of degrees of freedom reduces from $8+8$ to $4+4$. Recalling the supersymmetry transformations (10.43) of the components of a real superfield we see immediately that the Wess-Zumino gauge breaks supersymmetry. For example, $M_i = 0$ is not left invariant under susy transformations. When one chooses the Wess-Zumino gauge and performes a supersymmetry transformation then one needs a compensating gauge transformation to bring the superfield back into the Wess-Zumino gauge.

For constructing sensible actions out of V we need the exponential of V_{WZ}. Due to the anticommuting character of α the only non-vanishing power (bigger one) is two,

$$V_{WZ}^2 = \tfrac{1}{4}(\bar{\alpha}\gamma^\mu\gamma_5\alpha)(\bar{\alpha}\gamma^\nu\gamma_5\alpha)V_\mu V_\nu = \tfrac{1}{4}(\bar{\alpha}\alpha)^2 V_\mu V^\mu.$$

Then we have

$$\begin{aligned} \exp\left(V_{WZ}\right) &= 1 + \tfrac{1}{2}(\bar{\alpha}\gamma^\mu\gamma_5\alpha)V_\mu + (\bar{\alpha}\alpha)(\bar{\alpha}\lambda) + \tfrac{1}{2}(\bar{\alpha}\alpha)^2(D + \tfrac{1}{4}V_\mu V^\mu) \\ &= 1 + (\theta\sigma^\mu\bar{\theta})V_\mu + (\theta\theta)(\bar{\theta}\bar{\lambda}) + (\bar{\theta}\bar{\theta})(\theta\lambda) + (\theta\theta)(\bar{\theta}\bar{\theta})(D + \tfrac{1}{4}V_\mu V^\mu) \end{aligned} \tag{10.113}$$

We continue by defining the analog of a field strength for a general real superfield by

$$W_\alpha = -\tfrac{1}{4}(\bar{D}\bar{D})D_\alpha V \quad \text{and} \quad \bar{W}_{\dot{\alpha}} = -\tfrac{1}{4}(DD)\bar{D}_{\dot{\alpha}}V, \tag{10.114}$$

where we switched to the Weyl basis. Since $D^3 = \bar{D}^3 = 0$ these fields are chiral or antichiral. More precisely, W_α is a chiral $(1/2, 0)$ field and $\bar{W}_{\dot{\alpha}}$ is a antichiral $(0, 1/2)$ field. Less obvious is the susy-gauge invariance of these field strengths. Since $D_\alpha\Phi^\dagger = 0$ we have

$$W'_\alpha \equiv -\tfrac{1}{4}(\bar{D}\bar{D})D_\alpha V' = -\tfrac{1}{4}(\bar{D}\bar{D})D_\alpha V - \tfrac{1}{4}(\bar{D}\bar{D})D_\alpha\Phi.$$

Since Φ is chiral, $\bar{D}_{\dot{\alpha}}\Phi = 0$, the last term vanishes,

$$\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}D_\alpha\Phi = \bar{D}_{\dot{\alpha}}\{\bar{D}^{\dot{\alpha}}, D_\alpha\}\Phi = -2i\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\alpha\dot{\beta}}\bar{D}_{\dot{\alpha}}\partial_\mu\Phi = -2i\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{D}_{\dot{\alpha}}\Phi = 0.$$

We used the anticommutation relation (10.53,10.52). This proves that W_α is susy-gauge invariant. The invariance of $\bar{W}_{\dot{\alpha}}$ is proven very similarly.
The explicit expression for these field strength is rather involved. I refer to [10] for a detailed derviation. One finds

$$\begin{aligned} W_\alpha &= \lambda_\alpha(y) + 2D(y)\theta_\alpha + \tfrac{i}{2}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) - i(\theta\theta)(\sigma^\mu\partial_\mu\bar{\lambda}(y))_\alpha \\ \bar{W}^{\dot{\alpha}} &= \bar{\lambda}^{\dot{\alpha}}(z) + 2D(z)\bar{\theta}^{\dot{\alpha}} + \tfrac{i}{2}(\tilde{\sigma}^{\mu\nu}\bar{\theta})^{\dot{\alpha}}F_{\mu\nu}(z) - i(\bar{\theta}\bar{\theta})(\tilde{\sigma}^\mu\partial_\mu\lambda(z))^{\dot{\alpha}}, \end{aligned} \tag{10.115}$$

where the $\sigma^{\mu\nu}$ and $\tilde{\sigma}^{\mu\nu}$ have been defined in (7.2). Note that these (anti)chiral superfields only contain the gauge invariant component fields D, λ and $F_{\mu\nu}$. Furthermore, with (10.62) if follows immediately, that

$$\bar{D}^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \tag{10.116}$$

holds true.

10.6.2 Invariant actions

We have seen that the field strengths W_α and $\bar{W}^{\dot{\alpha}}$ are (anti) chiral and gauge invariant spinorial superfields which are linear in the field strength tensor $F_{\mu\nu}$. Then $W^\alpha W_\alpha$ and $\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$ are (anti)chiral and gauge invariant scalar superfields which are quadratic in $F_{\mu\nu}$. Hence, their F terms are good candidates for gauge and Lorentz invariant Lagrangian densities for a supersymmetric gauge theory. First observe that

$$
\begin{aligned}
W^\alpha W_\alpha &= \theta\theta\big(4D^2 - 2i\lambda\sigma^\mu\partial_\mu\bar\lambda\big) + 2i\theta\sigma^{\mu\nu}\theta F_{\mu\nu}D - \tfrac{1}{4}(\sigma^{\mu\nu}\theta)^\sigma(\sigma^{\alpha\beta}\theta)_\sigma F_{\mu\nu}F_{\alpha\beta} + \dots \\
\bar W_{\dot\alpha}\bar W^{\dot\alpha} &= \bar\theta\bar\theta\big(4D^2 - 2i\bar\lambda\tilde\sigma^\mu\partial_\mu\lambda\big) + 2i\bar\theta\tilde\sigma^{\mu\nu}\bar\theta F_{\mu\nu}D - \tfrac{1}{4}(\tilde\sigma^{\mu\nu}\bar\theta)_{\dot\sigma}(\tilde\sigma^{\alpha\beta}\bar\theta)^{\dot\sigma} F_{\mu\nu}F_{\alpha\beta} + \dots ,
\end{aligned}
$$

where $\dots$ stands for terms which do not contribute to the F-terms. Now we use that

$$
\begin{aligned}
\theta\sigma^{\mu\nu}\theta + \bar\theta\tilde\sigma^{\mu\nu}\bar\theta &= \bar\alpha\gamma^{\mu\nu}\alpha = 0 \\
(\sigma^{\mu\nu}\theta)(\sigma^{\alpha\beta}\theta) + (\tilde\sigma^{\mu\nu}\bar\theta)(\tilde\sigma^{\alpha\beta}\bar\theta) &= \alpha^\dagger\gamma^{\mu\nu}\gamma^0\gamma^{\alpha\beta}\alpha = -\bar\alpha\gamma^{\mu\nu}\gamma^{\alpha\beta}\alpha \\
= \theta(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha} + i\epsilon^{\mu\nu\alpha\beta})\theta &+ \bar\theta(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha} - i\epsilon^{\mu\nu\alpha\beta})\bar\theta
\end{aligned}
$$

For the sum of the F-terms we obtain

$$
\begin{aligned}
W^\alpha W_\alpha|_F + \bar W_{\dot\alpha}\bar W^{\dot\alpha}|_F &= 8D^2 - 2i\lambda\sigma^\mu\partial_\mu\bar\lambda - 2i\bar\lambda\tilde\sigma^\mu\partial_\mu\lambda - F_{\mu\nu}F^{\mu\nu} \\
&= 8D^2 - 2i\bar\lambda\gamma^\mu\partial_\mu\lambda - F_{\mu\nu}F^{\mu\nu}
\end{aligned}
\tag{10.117}
$$

This is the action for a pure supersymmetric abelian gauge theory with an uncharged gaugino spinor λ. The field $D(x)$ is an auxiliary field which can be eliminated with the help of the equations of motion, see section 8.
We may rewrite the action belonging to (10.114) as superspace integral as follows

$$
\begin{aligned}
S &= \tfrac{1}{4}\int d^4x \int d^4\theta\Big(W^\alpha W_\alpha \delta^2(\bar\theta) + \bar W_{\dot\alpha}\bar W^{\dot\alpha}\delta^2(\theta)\Big) \\
&= \int d^4x\big(2D^2 - \tfrac{i}{2}\bar\lambda\gamma^\mu\partial_\mu\lambda - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\big).
\end{aligned}
\tag{10.118}
$$

10.6.3 Minimal coupling

We study the simple gauge group $U(1)$. The gauge transformation of a scalar superfield under global $U(1)$ transformations is

$$
\Phi_i' = e^{iq_i\Lambda}\Phi_i, \tag{10.119}
$$

where q_i is the $U(1)$-charge of the superfield Φ_i. Clearly, the kinetic term $\Phi_i^\dagger\Phi_i|_D$ is invariant under these global phase transformations. However, a general superpotential $W(\Phi)$ in (10.96) will not be invariant. For W to be invariant we must demand

$$
\begin{aligned}
g_i &= 0 && \text{if} \quad q_i \neq 0 \\
m_{ij} &= 0 && \text{if} \quad q_i + q_j \neq 0 \\
\lambda_{ijk} &= 0 && \text{if} \quad q_i + q_j + q_k \neq 0.
\end{aligned}
\tag{10.120}
$$

Next we wish to make this global symmetry local in which case the gauge parameter λ in (10.119) will depend on spacetime. But then the transformed Φ will in general not be a chiral superfield. For Φ' to be chiral if Φ is chiral we must demand that λ is a chiral superfield, so that

$$\Phi_i' = e^{iq_i\Lambda}\Phi_i, \quad \bar{D}_{\dot\alpha}\Lambda = 0 \quad \Longrightarrow \quad \Phi_i'^{\dagger} = \Phi_i^\dagger e^{-iq_i\Lambda^\dagger}, \quad D^\alpha\Lambda^\dagger = 0. \tag{10.121}$$

The kinetic term is not invariant under these transformations of the chiral and antichiral superfields. But we have already argued that the transformations

$$V(x,\theta,\bar\theta) \longrightarrow V'(x,\theta,\bar\theta) - i\big(\Lambda(x,\theta,\bar\theta) - \Lambda^\dagger(x,\theta,\bar\theta)\big)$$

are generalizations of the gauge transformation and are called susy gauge transformations, see (10.109) with $\lambda = i\Phi$. One sees at once that

$$\mathcal{L}_{kin} = \Phi_i^\dagger e^{q_i V}\Phi_i|_D \tag{10.122}$$

is gauge invariant. The coupling (10.122) of the matter superfield Φ to the gauge potential V is the supersymmetric extension of the principle of minimal coupling in ordinary gauge theories as we shall demonstrate now. This is most easily done in the Wess-Zumino gauge. With (10.113) we obtain

$$\begin{aligned} \Phi^\dagger e^{qV}\Phi|_D &= \Phi^\dagger\Phi|_D + q\big(\theta\sigma^\mu\bar\theta V_\mu\Phi^\dagger\Phi\big)|_D + q\big(\theta\theta\bar\theta\bar\lambda(x)\Phi^\dagger\Phi\big)|_D \\ &+ q\big(\bar\theta\bar\theta\theta\lambda(x)\Phi^\dagger\Phi\big)|_D + q\big\{\theta\theta\bar\theta\bar\theta\big(D(x) + \tfrac{q}{4}V_\mu(x)V^\mu(x)\big)\Phi^\dagger\Phi\big\}|_D, \end{aligned} \tag{10.123}$$

where $\Phi = \Phi(y,\theta)$ and $\Phi^\dagger = \Phi^\dagger(z,\bar\theta)$ are the chiral and antichiral fields in (10.89). The first term on the right is the kinetic term without coupling to the gauge field and has been calculated in (10.93). To calculate the remaining terms we use (10.91) and the Fierz identies (8.90-8.96). The second term can be rewritten as

$$\begin{aligned} q\big(\theta\sigma^\mu\bar\theta V_\mu\Phi^\dagger\Phi\big)|_D &= q(\theta\sigma^\mu\bar\theta V_\mu)\big(iA^*\bar\theta\sigma^\nu\theta\partial_\nu A - i\theta\sigma^\nu\bar\theta\partial_\nu A^* A + 2\bar\theta\bar\psi\theta\psi\big)|_D \\ &= \tfrac12 q\big\{i(A^*\partial_\mu A - \partial_\mu A^* A) - \bar\psi\tilde\sigma^\mu\psi\big\}V_\mu. \end{aligned}$$

The third term on the right becomes

$$q\big(\theta\theta\,\bar\theta\bar\lambda(x)\,\Phi^\dagger\Phi\big)|_D = q\sqrt2\theta\theta\,\bar\theta\bar\lambda\,\bar\theta\bar\psi A|_D = -\frac{q}{\sqrt2}\,(\bar\lambda\bar\psi)A$$

and analogously the fourth term

$$q\big(\bar\theta\bar\theta\,\theta\lambda(x)\,\Phi^\dagger\Phi\big)|_D = q\sqrt2\bar\theta\bar\theta\,\theta\lambda\,\theta\psi A^*|_D = -\frac{q}{\sqrt2}\,(\lambda\psi)A^*.$$

Finally, the last term in (10.123) is simply

$$q\big\{\theta\theta\bar\theta\bar\theta\big(D + \tfrac{q}{4}V_\mu V^\mu\big)\Phi^\dagger\Phi\big\}|_D = q\big(D + \tfrac{q}{4}V_\mu V^\mu\big)A^*A.$$

Adding up the various terms we obtain the following gauge invariant kinetic term (in the WZ-gauge):

$$\begin{aligned} \Phi^\dagger e^{qV}\Phi|_D &= \partial_\mu A^*\partial^\mu A - \tfrac14\Box(A^*A) + \tfrac{i}{2}q(A^*\partial_\mu A - \partial_\mu A^*A)V_\mu + \frac{q^2}{4}V_\mu V^\mu A^*A \\ &+ \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi - \tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi - \tfrac12 q\bar\psi\tilde\sigma^\mu\psi V_\mu + |F|^2 - \frac{q}{\sqrt2}\big(\bar\lambda\bar\psi A + \lambda\psi A^*\big) + qD|A|^2 \end{aligned}$$

After introducing the covariant derivatives

$$D_\mu A = (\partial_\mu - \tfrac{i}{2} q V_\mu) A \quad \text{and} \quad D_\mu \psi = (\partial_\mu - \tfrac{i}{2} q V_\mu)\psi \tag{10.124}$$

the gauge invariant kinetic term reads

$$\Phi^\dagger e^{qV}\Phi|_D \;=\; (D_\mu A)^* D^\mu A - i\psi\sigma^\mu D^*_\mu \bar\psi + |F|^2 + q|A|^2 D - \frac{q}{\sqrt{2}}\big(\bar\lambda\bar\psi A + \lambda\psi A^*\big). \tag{10.125}$$

The minimal coupling not only couples the charged field A and its superpartner ψ to the gauge field, it also couples these fields to the gaugino λ which is the superpartner of V_μ.

10.6.4 Super-QED

Here we extend ordinary electrodynamics to a supersymmetric theory. We need at least a photon field and an electron field. Off shell the photon is accompanied by the photino and an auxiliary field and each chirality of the electron by a scalar and an auxiliary field. Hence, besides the gauge multiplett we need two scalar superfields Φ_1 and Φ_2 which transform under $U(1)$ gauge transformations as

$$\Phi_1' = e^{ie\Lambda}\Phi_1 \quad \text{and} \quad \Phi_2' = e^{-ie\Lambda}\Phi_2. \tag{10.126}$$

Later on the two spin-$\frac{1}{2}$ components of these chiral superfields will be identified with the electron. Gauge invariance requires the condition (10.120) on the constants, which in the present case read

$$g_1 = g_2 = 0, \quad m_{11} = m_{22} = 0, \quad \lambda_{ijk} = 0 \quad \text{and} \quad m_{12} = m_{21} \equiv m \neq 0.$$

By using this contraints we obtain the following supersymmetric and $U(1)$-invariant action

$$\begin{aligned} S_{SED} \;=\; \int d^4x \int d^4\theta \Big(& \tfrac{1}{4} W^\alpha W_\alpha \delta^2(\bar\theta) + \bar W_{\dot\alpha} \bar W^{\dot\alpha} \delta^2(\theta) + \Phi_1^\dagger e^{eV}\Phi_1 + \Phi_2^\dagger e^{-eV}\Phi_2 \\ & -m\Phi_1\Phi_2\delta^2(\bar\theta) - m\Phi_1^\dagger\Phi_2^\dagger\delta^2(\theta)\Big) \end{aligned} \tag{10.127}$$

Note that each term is separately supersymmetric and gauge invariant. Using the previous results (10.118,10.125,10.93) we may rewrite this action in terms of the component fields and obtain

$$\begin{aligned} S_{SED} \;=\; \int d^4x \Big(& -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \tfrac{i}{2}\bar\lambda\gamma^\mu\partial_\mu\lambda + 2D^2 + |F_1|^2 + |F_2|^2 + |D_\mu A_1|^2 + |D_\mu A_2|^2 \\ & -i\psi_1\sigma^\mu D^*_\mu\bar\psi_1 - i\psi_2\sigma^\mu D^*_\mu\bar\psi_2 + eD\big\{|A_1|^2 - |A_2|^2\big\} \\ & -2^{-1/2}e\big\{\bar\lambda(\bar\psi_1 A_1 - \bar\psi_2 A_2) + \lambda(\psi_1 A_1^* - \psi_2 A_2^*)\big\} \\ & -m\big\{A_1F_2 + A_1^*F_2^* + A_2F_1 + A_2^*F_1^* - \psi_1\psi_2 - \bar\psi_1\bar\psi_2\big\}\Big) \end{aligned} \tag{10.128}$$

Note that the covariant derivative act differently on the different fields, since the fields in Φ_2 have opposite charge to those in Φ_1. We may eliminate the auxiliary fields by their purely algebraic equations of motions

$$
\begin{aligned}
F_1: &\quad F_1^* - mA_2 = 0 \\
F_2: &\quad F_2^* - mA_1 = 0 \\
D: &\quad 4D + e\big(|A_1|^2 - |A_2|^2\big) = 0.
\end{aligned}
\tag{10.129}
$$

After eliminating the auxiliary fields in S we obtain

$$
\begin{aligned}
S_{SED} \;=\; \int d^4x \Big(& -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{i}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda + |D_\mu A_1|^2 + |D_\mu A_2|^2 - m^2(|A_1|^2 + |A_2|^2) \\
& -i\psi_1\sigma^\mu D_\mu^*\bar{\psi}_1 - i\psi_2\sigma^\mu D_\mu^*\bar{\psi}_2 + m(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2) \\
& -\frac{e}{\sqrt{2}}\big\{\bar{\lambda}(\bar{\psi}_1 A_1 - \bar{\psi}_2 A_2) + \lambda(\psi_1 A_1^* - \psi_2 A_2^*)\big\} - \frac{e^2}{8}(|A_1|^2 - |A_2|^2)^2\Big).
\end{aligned}
\tag{10.130}
$$

To compare this model with QED it is convenient to introduce the Dirac spinor

$$
\psi = \begin{pmatrix} \psi_{1\alpha} \\ \bar{\psi}_2^{\dot{\alpha}} \end{pmatrix} \Longrightarrow \psi^\dagger = (\,\bar{\psi}_{1\dot{\alpha}}, \psi_2^\alpha\,) \quad \text{and} \quad \bar{\psi} = (\,\psi_2^\alpha, \bar{\psi}_{1\dot{\alpha}}\,)\,.
$$

Up to surface terms in S the terms containing derivatives of the ψ_i can be rewritten as

$$
\begin{aligned}
&\psi_1\sigma^\mu D_\mu^*\bar{\psi}_1 + \psi\sigma^\mu D_\mu^*\bar{\psi}_2 = -D_\mu^*\bar{\psi}_{1\dot{\alpha}}\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{1\alpha} + \psi_2^\alpha\sigma^\mu_{\alpha\dot{\alpha}}D_\mu^*\bar{\psi}_2^{\dot{\alpha}} \\
&= \bar{\psi}_1\tilde{\sigma}^\mu D_\mu\psi_1 + \psi_2\sigma^\mu D_\mu^*\bar{\psi}_2 = \bar{\psi}\not{D}\psi,
\end{aligned}
$$

where we have used that ψ_1 and ψ_2 have opposite charge,

$$
D_\mu\psi_1 = (\partial_\mu - i\tfrac{e}{2}V_\mu)\psi_1 \quad \text{and} \quad D_\mu^*\bar{\psi}_2 = (\partial_\mu - i\tfrac{e}{2}V_\mu)\bar{\psi}_2.
$$

We conclude that the second line in (10.130) may be written as

$$
-\bar{\psi}(i\not{D} - m)\psi \tag{10.131}
$$

which is just the Lagrangian for a massive electron field of mass m and charge $-e$, minimally coupled to the abelian gauge potential V_μ.

10.6.5 Fayet-Ilioopulos term

We could add the supersymmetric and gauge invariant Fayet-Iliopoulos term [52]

$$
S_{FI} = \kappa \int d^4x d^4\,\theta V = \kappa \int D \tag{10.132}
$$

to the action S_{SED} in (10.127) with the result that it induces spontaneous supersymmetry breaking. The whole construction has been generalized to the non-Abelian case in [53]. Adding S_{FI} to S_{SED} changes the third equation for the auxiliary field D in (10.129) into

$$
4D + \kappa + e(|A_1|^2 + |A_2|^2) = 0.
$$

After eliminating the auxiliary fields we again obtain the action (10.130), wherein the last term is replaced by

$$-\frac{e^2}{8}\big(|A_1|^2-|A_2|^2\big)^2 \longrightarrow -\frac{1}{8}\big(e|A_1|^2-e|A_2|^2+\kappa\big)^2.$$

Let us have a closer look at the scalar potential of this socalled *Fayet-Iliopoulos* model. It is

$$\begin{aligned} V(A_1,A_2) &= m^2\big(|A_1|^2+|A_2|^2\big)+\frac{1}{8}\big(e|A_1|^2-e|A_2|^2+\kappa\big)^2\big) \\ &= \frac{\kappa^2}{8}+(m^2+\tfrac{1}{4}e\kappa)|A_1|^2+(m^2-\tfrac{1}{4}e\kappa)|A_2|^2+\frac{e^2}{8}\big(|A_1|^2-|A_2|^2\big)^2. \end{aligned}$$

Now we must distinguish between two cases.

• $4m^2 > e\kappa$:

In this case all coefficients of the quadratic and quartic terms are positive and the absolute minimum of V is attained at

$$A_1=A_2=0 \Rightarrow V_{\min}=\frac{\kappa}{8}>0, \quad D=-\frac{\kappa}{4},\ F_i=0,$$

indicating that supersymmetry is broken. Both scalar fields have real masses and vanishing vacuum expectation values which shows that (at the classical level) the $U(1)$-gauge symmetry is unbroken. The model contains the following particles

$$\begin{array}{ll} A_1: \quad M_1^2=m^2+\tfrac{1}{4}e\kappa, & A_2: \quad M_2^2=m^2-\tfrac{1}{4}e\kappa \\ \psi=(\psi_1,\psi_2): \quad M=m & \\ \lambda, V_\mu: \quad \text{massless.} & \end{array}$$

The mass splitting inside the matter multiplett establishes that supersymmetry has been broken. Note that only the mass of the scalars is splitted by the supersymmetry breaking. The massless photino field λ is just the Goldstino field appearing in the process of supersymmetry breaking

• $4m^2 < e\kappa$:

Now the minimum of the potential is at a finite value of the matter fields,

$$|A_1|=0 \quad \text{and} \quad |A_2|^2\equiv v^2=\frac{e\kappa-4m^2}{e^2} \Longrightarrow V(0,v)=\frac{m^2}{e^2}(e\kappa-2m^2)>0.$$

This indicates that both supersymmetry and gauge symmetry are spontaneously broken. To study the mass spectrum we expand the Lagrangian about the minimum. For that we shift

$$A_2(x)\longrightarrow v+A_2(x).$$

We insert this into the Lagrangian and expand up to second order... (comes later).

10.7 Non-Abelian Gauge Theories

To be written

10.8 Spontaneous Symmetrybreaking

To be written