9 $N = 4$ Super-Yang-Mills Theory

The largest supersymmetry that can be represented on a particle multiplet with spins ≤ 1 is $N = 4$. Correspondingly the $N = 4$ model [47] is called *maximally extended*. All $N = 4$ models must be constructed from gauge multiplets. This makes the particles necessarily massless and there will be no central charge.
Our interest in the $N = 4$ SYM is twofold. On the one hand, the theory is believed to be S-dual, one expects that the complete effective action, including all instanton and anti-instanton effects organize into an $SL(2, Z)$ invariant expression. On the other hand, not unrelated to the previous, we are motivated by the AdS/CFT correspondence.
On-shell the theory contains a gauge boson A_μ, 4 Majorana (or equivalently 4 Weyl) fermions which maybe grouped into two Dirac spinors, and 6 scalar fields. All fields transform according to the adjoint representation of the gauge group.

9.1 Scale invariance in 1 loop

Without knowledge of the precise action we can already calculate the $1 - loop$ β-function of this theory. Under scale transformations the Schwinger functional changes as

$$\begin{aligned} W_q[\lambda V, g] &= W_q(V, g) + \frac{\log \lambda}{16\pi^2} \int \mathrm{Tr}\left(6a_2^A(x) - 2a_2^{gh}(x) - 2a_2^{\psi}(x) + a_2^{A_\mu}(x)\right) \\ &= W_q(V, g) + \frac{\log \lambda}{16\pi^2} \int \mathrm{Tr}\left\{4a_2^A(x) - 2a_2^{\psi}(x) + a_2^{A_\mu}(x)\right\}, \end{aligned}$$

where the Seeley-deWitt coefficients have been given in [46]. We have used the field content of the theory that all fields transform under the adjoint representation. One finds, that the expression between the curly brackets vanishes. Thus the 1-loop action is scale invariant and the β-function and wave function renormalization vanishes,

$$\beta(g) = 0 \quad \text{and} \quad Z_3(g) = 0. \tag{9.1}$$

In [48] it was found that for the $N = 4$ theory the β-function remained zero up to three loops and that therefore there were no divergent graphs at all to that order. This led everyone to suspect that the theory may be a finite field theoretical model in four space-time dimensions and arguments were put forward to proof the finiteness to all orders in perturbation theory [49]. Some arguments are based on the relation between the trace anomaly in the energy-momentum tensor, the β-function and conformal invariance, other arguments used the explicit matching of bosonic and fermionic counting in the light-cone gauge and yet other arguments were based on the non-renormalisation theorem and the background gauge.

9.2 Kaluza-Klein-Reduction

The relevant models with extended supersymmetry are intimately related to Lagrangian field theories in more than four space-time dimensions. The idea of employing higher dimensions to understand the complexities of extended supersymmetry has been pioneered by J. Scherk [50]. As an exercise in getting familiar with higher dimensions, I shall show how $N=4$ supersymmetric Yang-Mills theories fit into a world in higher dimensions. We derive this theory by a sort of Kaluza-Klein reduction of a gauge theory in higher dimensions. After compactification all but four components of the vector potential become scalar fields. Since the $N=4$ SYM theory has 6 scalar fields and the gauge field has 4 components, this implies that we must reduce a gauge theory with a $4+6=10$-component gauge potential, that is a 10-dimensional gauge theory.

9.2.1 Spinors in 10 dimensions

In 10 dimension a Dirac spinor has 32 complex components which are four times the 4×4 real components of the 4 Majorana spinors in the $N=4$ theory in 4 dimensions. Hence we must assume that the spinors in 10 dimensions are both Weyl and Majorana. Such spinors do exists as we have discussed in chapter 3.
We start with the Dirac matrices γ^μ in 4-dimensional Minkowski space and give an explicit realization for the matrices $\Gamma^m, m=0,\dots,9$ in 10 dimensions: We make the ansatz

$$\Gamma_\mu = \Delta\otimes\gamma_\mu,\quad \Gamma_{3+a} = \Delta_a\otimes\gamma_5,\quad \mu=0,1,2,3,\quad a=1,\dots,6.$$

To see what are the condition on the Δ-factors such that

$$\{\Gamma_m,\Gamma_n\} = 2\eta_{mn},\qquad \eta=\mathrm{diag}(1,-1,\dots,-1) \tag{9.2}$$

holds, one uses that for $[A,B]=0$ on has

$$\{A\otimes C, B\otimes D\} = AB\otimes\{C,D\}\quad\text{and}\quad \{C\otimes A, D\otimes B\} = \{C,D\}\otimes AB. \tag{9.3}$$

and for $\{A,B\}=0$ one has

$$\{A\otimes C, B\otimes D\} = AB\otimes[C,D]\quad\text{and}\quad \{C\otimes A, D\otimes B\} = [C,D]\otimes AB. \tag{9.4}$$

Now it is easy to see that we must demand

$$\Delta^2 = \mathbb{1}_6,\quad [\Delta,\Delta_a]=0,\qquad \{\Delta_a,\Delta_b\} = -2\delta_{ab}\mathbb{1}_8.$$

in order for (9.2) to hold true. Since Γ^0 must be hermitean and the $\Gamma^{m>0}$ antihermitean and since $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_2$ is hermitean it also follows from

$$\left(A\otimes B\right)^\dagger = A^\dagger\otimes B^\dagger.$$

that

$$\Delta^\dagger = \Delta, \quad \Delta_a^\dagger = -\Delta_a.$$

Other useful identities which we shall sometimes need are

$$(A \otimes B)^T = A^T \otimes B^T \quad \text{and for vectors} \quad (x \otimes y, u \otimes v) = (x,u)(y,v). \tag{9.5}$$

Since Δ commutes with all matrices we may choose it to be the identity,

$$\Delta = \mathbb{1}_8.$$

Note that the hermitean $i\Delta_a$ generate the Euclidean Clifford algebra in 6 dimensions and that the $[\Delta_a, \Delta_b]$ generate the group $spin(6)$. Earlier we have shown that in 6 Euclidean dimensions there is a Majorana representation. Hence we may choose Δ_a to be real and hence antisymmetric. To construct this Majorana representation we make for the Δ_a the following ansatz

$$\Delta_i = i\tau_1 \otimes \alpha_i \quad \text{and} \quad \Delta_{3+i} = i\tau_3 \otimes \tilde{\alpha}_i, \qquad i = 1,2,3,$$

so that

$$\{\alpha_i, \alpha_j\} = \{\tilde{\alpha}_i, \tilde{\alpha}_j\} = 2\delta_{ij}\mathbb{1}_4, \quad [\alpha_i, \tilde{\alpha}_j] = 0, \quad \alpha_i^\dagger = \alpha_i, \quad \tilde{\alpha}_i^\dagger = \tilde{\alpha}_i.$$

A possible solution is

$$\begin{aligned} \alpha_1 &= \tau_2 \otimes \tau_1, \quad \alpha_2 = \tau_0 \otimes \tau_2, \quad \alpha_3 = \tau_2 \otimes \tau_3 \\ \tilde{\alpha}_1 &= \tau_1 \otimes \tau_2, \quad \tilde{\alpha}_2 = -\tau_3 \otimes \tau_2, \quad \tilde{\alpha}_3 = \tau_2 \otimes \tau_0 \end{aligned}$$

These hermitean, imaginary and hence antisymmetric matrices obey

$$\alpha_i \alpha_j = \delta_{ij} + i\epsilon_{ijk}\alpha_k \quad \text{and} \quad \tilde{\alpha}_i \tilde{\alpha}_j = \delta_{ij} + i\epsilon_{ijk}\tilde{\alpha}_k, \quad [\alpha_i, \tilde{\alpha}_j] = 0.$$

They lead to *real and antisymmetric* Δ_a. With our earlier convention the hermitean $\Gamma_{11} = -\Gamma_0 \cdots \Gamma_9$ takes the form

$$\Gamma_{11} = \Gamma_* \otimes \gamma_5, \qquad \Gamma_* = -i\Delta_1 \cdots \Delta_6, \qquad \Gamma_*^\dagger = \Gamma_* = -\Gamma_*^T, \quad \gamma_5^\dagger = \gamma_5. \tag{9.6}$$

If we would choose a Majorana representation for the 4-dimensional γ_μ then they would be purely imaginary (see section 3.5) and so would be $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$. Since the Δ_a are real it follows that the Γ_m are purely imaginary as well and that the charge conjugation matrix is $\mathcal{C}_{10} = -\Gamma_0 = -\mathbb{1}_8 \otimes \gamma_0$. Hence we are lead to take the following antisymmetric charge conjugation matrix

$$\mathcal{C}_{10} = \mathbb{1}_8 \otimes \mathcal{C}_4 \tag{9.7}$$

where $\mathcal{C}_4$ is the antisymmetric charge conjugation matrix in 4 dimensions. Since the Δ_a are antisymmetric it is easily seen that

$$\mathcal{C}_{10}\Gamma_m\mathcal{C}_{10}^{-1} = -\Gamma_m^T. \tag{9.8}$$

If we choose the 4-dimensional Majorana representation, then $\mathcal{C}_4 = -\gamma_0$ and $\mathcal{C}_{10} = -\Gamma_0$ as required.

We see that the constants ϵ and η in the last table in section (??) are both 1, similarly as in 4 spacetime dimensions.
Let us first see how Majorana spinors look like. Since

$$\mathcal{B}_{10} = \mathcal{C}_{10}\Gamma_0^T = \mathbb{1}_8 \otimes \mathcal{C}_4\gamma_0^T = \mathbb{1}_8 \otimes \mathcal{B}_4$$

the Majorana condition reads

$$\Psi = \xi \otimes \chi = \Psi_c \Longleftrightarrow \xi \in \mathbb{R}^8,\ \chi_c = \chi, \tag{9.9}$$

and an arbitrary Majorana spinor in 10 space-time dimensions has the expansion

$$\Psi = \sum_{r=1}^{8} E_r \otimes \chi_r, \tag{9.10}$$

where the χ_r are Majorana spinors in 4 dimensions and the E_r form a (real) base in $\mathbb{R}^8$. A spinor has positive chirality if

$$\Psi = \Gamma_{11}\Psi = (\Gamma_* \otimes \gamma_5)\Psi. \tag{9.11}$$

To parameterize Weyl spinors we expand the factor ξ in $\Psi = \xi \otimes \chi$ in eigenvectors of Γ_*. First we take the 4 (necessarily complex) eigenvectors F_p of Γ_* with eigenvalue 1. For convenience we assume that the F_p are orthonormal. Since Γ_* is purely imaginary the vectors F_p^* have then eigenvalue -1 and together with the F_p form a basis. With this choice a Weyl-spinor with $\Gamma_{11} = 1$ has the expansion

$$\Psi = \sum_{p=1}^{4} \big(F_p \otimes P_+\chi_p + F_p^* \otimes P_-\chi_{4+p}\big), \tag{9.12}$$

where $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$ are the projections onto the spinors with fixed chirality. A Majorana-Weyl (MW) spinor in 10 dimensions must at the same time have the expansion (9.10). By using

$$\gamma_5\mathcal{B}_4 = -\mathcal{B}_4\gamma_5^* \quad \text{or} \quad \mathcal{B}_4 P_+^* = P_-\mathcal{B}_4, \quad \mathcal{B}_4 P_-^* = P_+\mathcal{B}_4$$

one finds that the Weyl-spinor (9.12) fulfills the Majorana conditions in 10 dimensions if $P_+\chi_p + P_-\chi_{4+p}$ is a Majorana spinor in 4 dimensions. We denote this spinor again by χ_p. This way we arrive at the following expansion for a MW-spinor

$$\Psi = \sum_{p=1}^{4} \big(F_p \otimes P_+\chi_p + F_p^* \otimes P_-\chi_p\big) \quad , \quad \bar{\Psi} = \sum_p \big(F_p^\dagger \otimes \bar{\chi}_p\, P_- + F_p^T \otimes \bar{\chi}_p\, P_+\big), \tag{9.13}$$

where the χ_p are Majorana-spinors in 4 space-time dimensions and $\bar{\chi} = \chi^\dagger\gamma_0$ denotes the Dirac conjugate of χ. Indeed, using the real basis

$$E_p = \frac{1}{\sqrt{2}}\big(F_p + F_p^*\big) \quad \text{and} \quad E_{4+p} = \frac{i}{\sqrt{2}}\big(F_p - F_p^*\big)$$

the above expansions can be rewritten as

$$\Psi = \frac{1}{\sqrt{2}} \sum_{p=1}^{4} \left\{ E_p \otimes \chi_p + \Gamma_* E_p \otimes \gamma_5 \chi_p \right\} \quad , \quad \bar{\Psi} = \frac{1}{\sqrt{2}} \sum_{p=1}^{4} \left\{ E_p^T \otimes \bar{\chi}_p - E_p^T \Gamma_* \otimes \bar{\chi}_p \gamma_5 \right\}. \tag{9.14}$$

Since $i\Gamma_*$ is real and $i\gamma_5\chi$ is a Majorana spinor if χ has this property, this expansion has the form (9.10), as required.

9.2.2 Reduction of Yang-Mills term

In 10 spacetime dimensions a gauge field and gauge coupling constant have the dimensions

$$[A_m] = L^{-4}, \quad [g_{10}] = L^3 \Longrightarrow [g_{10} A_m] = L^{-1}.$$

We may absorb the coupling constant in the gauge potential, $A_m \to g_{10} A_m$ such that the 10-dimensional coupling constant appears only in front of the the Yang-Mills action,

$$S_{YM} = -\frac{1}{4g_{10}^2} \int d^{10}x \, \mathrm{Tr} \, F_{mn} F^{mn}. \tag{9.15}$$

Now we perform the Kaluza-Klein reduction of the Yang-Mills action on $\mathbb{R}^4 \times T^6$. As internal space we choose the 6-dimensional torus T^6 with volume V_6. We write

$$A_m = (A_\mu, \Phi_a), \qquad m = 0, \ldots, 9; \quad a = 1, \ldots, 6, \tag{9.16}$$

and assume all fields are *independent of the internal coordinates* $x^4, \ldots, x^9$. We find the following decomposition of the field strength,

$$F_{\mu,3+a} = \partial_\mu \Phi_a - i[A_\mu, \Phi_a] \quad , \quad F_{3+a,3+b} = -i[\Phi_a, \Phi_b]$$

Inserting this into the 10-dimensional Yang-Mills action we find

$$S_{YM} \to \frac{1}{4g^2} \int d^4x \; \mathrm{Tr} \left(-F_{\mu\nu} F^{\mu\nu} + 2 \sum_a D^\mu \Phi_a D_\mu \Phi_a + \sum_{ab} [\Phi_a, \Phi_b]^2 \right), \tag{9.17}$$

where we took into account, that $\Phi_a = -\Phi^a$ and where the dimensionful coupling constant g_{10} and the *dimensionless coupling constant* g are related as

$$g^2 = g_{10}^2 / V_6. \tag{9.18}$$

Now we may rescale the fields to obtain after the Kaluza-Klein reduction the following Lagrangian in 4-dimensional Minkowski spacetime:

$$\mathcal{L}_{YM} \to \mathcal{L}_{YMH} = \mathrm{Tr} \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} \sum_a D^\mu \Phi_a D_\mu \Phi_a + \tfrac{1}{4} g^2 \sum_{ab} [\Phi_a, \Phi_b]^2 \right). \tag{9.19}$$

The covariant derivative is $D_\mu = \partial_\mu - ig\,\mathrm{ad}A_\mu$. Not unexpectedly we have gotten the action for a four-dimensional Yang-Mills-Higgs theory with 6 Higgs fields in the adjoint representation.

9.2.3 Reduction of the Dirac term

In 10 and 4 dimensions a spinor field has the dimension

$$[\Psi] = L^{-9/2} \quad \text{and} \quad [\chi] = L^{-3/2},$$

respectively. We start with the general expansions (9.13,9.14) for a 10-dimensional Majorana-Weyl spinor and its adjoint. We rescale the spinors such that the χ_p in

$$\Psi = \frac{1}{\sqrt{V_6}} \sum_{p=1}^{4} \big(F_p \otimes P_+ \chi_p + F_p^* \otimes P_- \chi_p \big) \quad , \quad \bar{\Psi} = \frac{1}{\sqrt{V_6}} \sum_{p=1}^{4} \big(F_p^\dagger \otimes \bar{\chi}_p P_- + F_p^T \otimes \bar{\chi}_p P_+ \big)$$

have the dimension of a spinorfield in 4-dimensional Minkowski spacetime. The spinor should be independent of the internal coordinates. Again we absorb the 10-dimensional gauge coupling constant in the gauge potential. We find

$$\begin{aligned} D_\mu \Psi &= \frac{1}{\sqrt{V_6}} \sum_p \big(F_p \otimes D_\mu P_+ \chi_p + F_p^* \otimes D_\mu P_- \chi_p \big), \qquad \mu = 0,1,2,3 \\ D_{3+a} \Psi &= -\frac{i}{\sqrt{V_6}} \sum_p \big(F_p \otimes [\Phi_a, P_+ \chi_p] + F_p^* \otimes [\Phi_a, P_- \chi_p] \big), \quad a = 1, \dots, 6. \end{aligned}$$

Now we may rewrite the Dirac term in 10 dimensions as follows:

$$\begin{aligned} \int d^{10}x \operatorname{Tr} \bar{\Psi} \Gamma^m D_m \Psi &= \int d^4x \operatorname{Tr} \bar{\chi}_p \not{D} \chi_p \\ &- i \int d^4x \operatorname{Tr} \Big\{ (\Delta_+^a)_{pq} \bar{\chi}_p P_+ [\Phi_a, \chi_q] - (\Delta_-^a \Gamma_*)_{pq} \bar{\chi}_p P_- [\Phi_a, \chi_q] \Big\}, \end{aligned}$$

where one should sum over the indices on the right. We have introduced

$$(\Delta_+^a)_{pq} = (F_p^*, \Delta^a F_q) \quad \text{and} \quad (\Delta_-^a)_{pq} = (F_p, \Delta^a F_q^*) \quad \text{with} \quad \Delta_+^a = (\Delta_-^a)^*. \tag{9.20}$$

Putting the various terms together we end up with the following $N = 4$ supersymmetric gauge theory in 4-dimensional Minkowski spacetime:

$$\begin{aligned} \mathcal{L} &= \operatorname{Tr} \Big(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} D^\mu \Phi_a D_\mu \Phi_a + \tfrac{1}{4} g^2 [\Phi_a, \Phi_b]^2 \Big) \\ &\quad + \tfrac{1}{2} \operatorname{Tr} \Big(i \bar{\chi}_p \not{D} \chi_p + g (\Delta_+^a)_{pq} \bar{\chi}_p P_+ [\Phi_a, \chi_q] - g (\Delta_-^a)_{pq} \bar{\chi}_p P_- [\Phi_a, \chi_q] \Big), \end{aligned} \tag{9.21}$$

where $a = 1, \dots, 6$ and $p = 1, \dots, 4$.

9.2.4 R-symmetry

In 10 dimensions a vector- and spinor field transform under Lorentz transformations as

$$A(x) \longrightarrow \Lambda_{10} A(\Lambda_{10}^{-1} x) \quad \text{and} \quad \Psi(x) \longrightarrow S_{10} \Psi(\Lambda_{10}^{-1} x), \quad \bar{\Psi}(x) \longrightarrow \bar{\Psi}(\Lambda_{10}^{-1} x) S_{10}^{-1},$$

where

$$S_{10}=\exp\left(\tfrac{1}{2}\omega_{mn}\Gamma^{mn}\right)\quad\text{and}\quad (\Lambda_{10})^m_{\ n}=\left(e^{\omega}\right)^m_{\ n},\qquad \Gamma_{mn}=\tfrac{1}{2}[\Gamma_m,\Gamma_n]$$

are the spinrotation and Lorentz transformation with parameter ω_{mn} in 10-dimensional spacetime. They are related via

$$S_{10}^{-1}\Gamma^m S_{10}=(\Lambda_{10})^m_{\ n}\Gamma^n.$$

The Lagrangian density is as scalar field such that the corresponding action is lorentz invariant. For example,

$$\left(\bar\Psi\Gamma^m A_m\Psi\right)(x)\longrightarrow\left(\bar\Psi S_{10}^{-1}\Gamma^m S_{10}(\Lambda_{10})^m_{\ n}A_n\psi\right)\left(\Lambda_{10}^{-1}x\right)=\left(\bar\Psi\Gamma^m A_m\Psi\right)\left(\Lambda_{10}^{-1}x\right).$$

When we reduce the theory to $\mathbb{R}^4$ we must require that the fields do not depend on the internal coordinates. Clearly this condition is not compatible with the 10 dimensional Lorentz invariance. Only those Lorentz transformations survive which do not mix the coordinates on $\mathbb{R}^4$ with those in the internal space, hence

$$SO(1,9)\longrightarrow SO(1,3)\times SO(6)\quad\text{or}\quad \Lambda_{10}\longrightarrow\begin{pmatrix}\Lambda & 0\\ 0 & R\end{pmatrix},$$

where Λ is a 4-dimensional Lorentz transformation and $R\in O(6)$[1]. With our choice for the Γ_m the generators of the corresponding spin transformations in 10 dimensions read

$$\Gamma_{\mu\nu}=\mathbb{1}_8\otimes\gamma_{\mu\nu}\quad,\quad\Gamma_{3+a,3+b}=\Delta_{ab}\otimes\mathbb{1}_4 \tag{9.22}$$

The Δ_{pq} generate the $spin(6)\sim su(4)$ subalgebra of $spin(1,9)$. Since the $\Gamma_{\mu\nu}$ act trivially on the first factor in the decomposition $\Psi=\xi\otimes\chi$ and the other generators in (9.22) act trivially on the second factor, the remaining spin rotations after the dimensional reduction decompose as follows:

$$\begin{array}{ll} S_{10}=S_6\otimes S, & S_6:\xi\longrightarrow S_6\xi,\quad S:\chi\longrightarrow S\chi\\ S_6^{-1}\Delta^a S_6=R^a_{\ b}\Delta^b, & S^{-1}\gamma^\mu S=\Lambda^\mu_{\ \nu}\gamma^\nu,\end{array}$$

where S is a spin rotation acting on spinors in 4-dimensional Minkowski spacetime. To summarize, the remaining Lorentz transformation act on the various fields as follows

$$\begin{array}{llll} S_6: & \Phi_a(x)\longrightarrow R_a^{\ b}\Phi_b(x), & \Psi(x)\longrightarrow S_6\xi\otimes\chi, & A_\mu(x)\to A_\mu(x)\\ S: & \Phi_a(x)\longrightarrow\Phi_a(\Lambda^{-1}x), & \Psi(x)\longrightarrow\xi\otimes S\chi(\Lambda^{-1}x), & A_\mu(x)\longrightarrow\Lambda_\mu^{\ \nu}A_\nu(\Lambda^{-1}x).\end{array}$$

We see that the spin rotations S_6 and the associated $O(6)$-rotations R have become *internal symmetries* which rotate the fields without changing the coordinates (on $\mathbb{R}^4$).
In the reduced 4-dimensional theory the first factor in $\Psi=\xi\otimes\chi$ is absent and we should reinterpret the transformation $\Psi\to S_6\xi\otimes\chi$ as transformation of the 4-dimensional spinor χ. To find this transformation we note that the real spin rotations S_6 commute with Γ_* such that

$$S_6F_p=U_{qp}F_q\quad\text{and}\quad S_6F_p^*=U_{qp}^*F_q^*,\qquad U,U^*\in SU(4).$$

[1] actually, only $\det\Lambda\cdot\det R=1$ is required.

These relations between S_6 and U or U^* are just the isomorphisms $spin(6) \sim SU(4)$. Therefore, under the R-symmetry the spinors transform as

$$P_+\chi_p \longrightarrow \sum_q U_{pq}P_+\chi_q \quad , \quad P_-\chi_p \longrightarrow \sum_q U^*_{pq}P_-\chi_q$$

and the corresponding transformations of the Dirac conjugates read

$$\bar\chi_p P_+ \longrightarrow \sum_q \bar\chi_q P_+(U^T)_{qp} \quad , \quad \bar\chi_p P_- \longrightarrow \sum_q \bar\chi_q P_- U^\dagger_{qp}.$$

Note that for Majorana spinors χ_p the linear combinations

$$\sum_q \left(U_{pq}P_+\chi_q + U^*_{pq}P_-\chi_p\right)$$

are Majorana spinors as well.
Now it is not difficult to prove that the action (9.21) of the $N=4$ extended SYM-theory is invariant under R-symmetry transformations. For the terms in (9.21) containing no fermions the invariance is manifest. The Diracterm is also invariant: for example

$$\sum_p \bar\chi_p P_- \not{D}\chi_p \longrightarrow \sum_{pqr} \bar\chi_q P_- U^\dagger_{qp}\not{D} U_{pr}\chi_r = \sum_q \bar\chi_q P_- \not{D}\chi_q$$

is left invariant by $SU(4)$-rotations of the fermions with positive chirality. To show the invariance of the Yukawa terms is a bit more difficult. First we calculate

$$\begin{aligned}
(\Delta^a_+)_{pq}\bar\chi_p P_+[\Phi_a,\chi_q] \quad &\longrightarrow \quad (U^T\Delta^a_+U)_{pq}\bar\chi_p P_+[R_a{}^b\Phi_b,\chi_q] \\
(\Delta^a_-)_{pq}\bar\chi_p P_-[\Phi_a,\chi_q] \quad &\longrightarrow \quad (U^\dagger\Delta^a_-U^*)_{pq}\bar\chi_p P_-[R_a{}^b\Phi_b,\chi_q].
\end{aligned}$$

Now we use $S_6^{-1}\Delta^a S_6 = R^a{}_b\Delta^b$ and the definitions in (9.20) and these give rise to the following isomorphisms between $SO(6)$ and $SU(4)$:

$$U^T\Delta^a_+U = R^a{}_b\Delta^b_+ \quad \text{and} \quad U^\dagger\Delta^a_-U^* = R^a{}_b\Delta^b_-.$$

These relations imply that the Yukawa terms are both invariant under R-transformations.

9.3 Supersymmetry transformations in $d=10$

The susy transformations in 10 dimensions are

$$\delta_\alpha A_m = i\bar\alpha\Gamma_m\Psi, \quad \delta_\alpha\Psi = iF^{mn}\Sigma_{mn}\alpha \quad \text{and} \quad \delta_\alpha\bar\Psi = -i\bar\alpha F^{mn}\Sigma_{mn}. \tag{9.23}$$

The calculation parallel those for the $N=1$ theory in 4 dimensions. In particular

$$\begin{aligned}
-\tfrac{1}{4}\delta_\alpha \mathrm{Tr}\left(F_{mn}F^{mn}\right) \quad &= \quad -i\mathrm{Tr}\left(F^{mn}\bar\alpha\Gamma_n D_m\Psi\right) \qquad (9.24)\\
\delta_\alpha \not{D}\psi \quad &= \quad i(D^pF^{mn})\Gamma_p\Sigma_{mn}\alpha + \Gamma^m[(\bar\alpha\Gamma_m\Psi,\Psi] \\
\delta_a \mathrm{Tr}\left(\tfrac{i}{2}\bar\Psi\not{D}\Psi\right) \quad &= \quad \tfrac{1}{2}\mathrm{Tr}\Big(F^{mn}\bar\alpha\Sigma_{mn}\Gamma_p D^p\Psi - (D^pF^{mn})\bar\Psi\Gamma_p\Sigma_{mn}\alpha + i\bar\Psi\Gamma^m[(\bar\alpha\Gamma_m\Psi),\Psi]\Big).
\end{aligned}$$

Since we have $\epsilon = \eta = 1$ (as we had in 4 dimensions) it follows that

$$\bar\Psi\Gamma_p\Sigma_{mn}\alpha = \bar\alpha\Sigma_{mn}\Gamma_p\Psi$$

which implies

$$\begin{aligned}\delta_a \mathrm{Tr}\left(\tfrac{i}{2}\bar\Psi \not{D}\Psi\right) &= -\tfrac{1}{2}\partial_p \mathrm{Tr}\left(F^{mn}\bar\alpha\Sigma_{mn}\Gamma_p\Psi\right) + \mathrm{Tr}\left(F^{mn}\bar\alpha\Sigma_{mn}\Gamma_p D^p\Psi\right) \\ &\quad +\tfrac{i}{2}\mathrm{Tr}\left(\bar\Psi\Gamma^m[(\bar\alpha\Gamma_m\Psi),\Psi]\right).\end{aligned} \tag{9.25}$$

To continue we use

$$\begin{aligned}\Sigma_{mn}\Gamma_p &= \tfrac{i}{2}\eta_{mp}\Gamma_n - \tfrac{i}{2}\eta_{np}\Gamma_m - \tfrac{i}{2}\Gamma_{mnp} \\ \Gamma_p\Sigma_{mn} &= -\tfrac{i}{2}\eta_{mp}\Gamma_n + \tfrac{i}{2}\eta_{np}\Gamma_m - \tfrac{i}{2}\Gamma_{mnp}\end{aligned} \tag{9.26}$$

The term (9.24) and the second term in (9.25) add up to

$$\begin{aligned}\mathrm{Tr}\, F^{mn}\bar\alpha\big(\Sigma_{mn}\Gamma_p - \tfrac{i}{2}\eta_{mp}\Gamma_n + \tfrac{i}{2}\eta_{np}\Gamma_m\big)D^p\Psi &= -\tfrac{i}{2}\mathrm{Tr}\left(F^{mn}\bar\alpha\Gamma_{mnp}D^p\Psi\right) \\ &= -\tfrac{i}{2}\partial^p \mathrm{Tr}\left(F^{mn}\bar\alpha\Gamma_{mnp}\Psi\right),\end{aligned} \tag{9.27}$$

where we have used the Bianchi-identity in the last step. To prove that the 10-dimensional action is invariant we need to show that the last term in (9.25) vanishes. For that we expand the Majorana field Ψ in terms of an orthonormal base T_a of the Lie algebra and find

$$\tfrac{i}{2}\mathrm{Tr}\left(\bar\Psi\Gamma^m[(\bar\alpha\Gamma_m\Psi),\Psi]\right) = \tfrac{1}{2}f_{abc}(\bar\Psi^a\Gamma^m\Psi^b)(\bar\alpha\Gamma_m\Psi^c) \tag{9.28}$$

To proceed we need some Fierz relations in 10 dimensions: The relation (3.64) becomes

$$\begin{aligned}32\Psi\bar\chi &= -\bar\chi\psi - \Gamma_m(\bar\chi\Gamma^m\Psi) + \tfrac{1}{2!}\Gamma_{mn}(\bar\chi\Gamma^{mn}\Psi) + \tfrac{1}{3!}\Gamma_{mnp}(\bar\chi\Gamma^{mnp}\Psi) - \tfrac{1}{4!}\Gamma_{mnpq}(\bar\chi\Gamma^{mnpq}\Psi) \\ &\quad -\tfrac{1}{5!}\Gamma_{mnpqr}(\bar\chi\Gamma^{mnpqr}\Psi) - \tfrac{1}{4!}\Gamma_{11}\Gamma_{mnpq}(\bar\chi\Gamma_{11}\Gamma^{mnpq}\Psi) - \tfrac{1}{3!}\Gamma_{11}\Gamma_{mnp}(\bar\chi\Gamma_{11}\Gamma^{mnp}\Psi) \\ &\quad +\tfrac{1}{2!}\Gamma_{11}\Gamma_{mn}(\bar\chi\Gamma_{11}\Gamma^{mn}\Psi) + \Gamma_{11}\Gamma_m(\bar\chi\Gamma_{11}\Gamma^m\Psi) - \Gamma_{11}(\bar\chi\Gamma_{11}\Psi)\end{aligned}$$

We used (3.10) and

$$\epsilon_{m_1\ldots m_a n_1\ldots n_b}\epsilon^{m_1\ldots m_a p_1\ldots p_b} = -a!\,b!\,\delta^{[p_1}_{[n_1}\ldots\delta^{p_b]}_{n_b]}.$$

Next we use the identities

$$\begin{aligned}&\Gamma_s\Gamma_m\Gamma^s = (2-d)\Gamma_m \quad , \quad \Gamma_s\Gamma_{mn}\Gamma^s = (d-4)\Gamma_{mn} \quad , \quad \Gamma_s\Gamma_{mnp}\Gamma^s = (6-d)\Gamma_{mnp} \\ &\Gamma_s\Gamma_{mnpq}\Gamma^s = (d-8)\Gamma_{mnpq} \quad , \quad \Gamma_s\Gamma_{mnpqr}\Gamma^s = (10-d)\Gamma_{mnpqr}\end{aligned}$$

and

$$\begin{aligned}&\Gamma_s\Gamma_{11}\Gamma_m\Gamma^s = -(2-d)\Gamma_{11}\Gamma_m \quad , \quad \Gamma_s\Gamma_{11}\Gamma_{mn}\Gamma^s = -(d-4)\Gamma_{11}\Gamma_{mn} \\ &\Gamma_s\Gamma_{11}\Gamma_{mnp}\Gamma^s = -(6-d)\Gamma_{11}\Gamma_{mnp} \quad , \quad \Gamma_s\Gamma_{11}\Gamma_{mnpq}\Gamma^s = -(d-8)\Gamma_{11}\Gamma_{mnpq}\end{aligned}$$

Taking into account that for Majorana spinors

$$(\bar\Psi^a\Gamma^{(n)}\Psi^c) = (\bar\Psi^c\Gamma^{(n)}\Psi^a) \quad \text{for} \quad n = 0,3,4,7,8$$

and for Weyl spinors

$$(\bar\Psi^a\Gamma^{(n)}\Psi^c)=0 \quad \text{for} \quad n=0,2,4,6,8,10$$

we may rewrite (twice) the term on the right hand side of (9.28) as

$$\begin{aligned} f_{abc}(\bar\Psi^a\Gamma^m\Psi^b)(\bar\alpha\Gamma_m\Psi^c) &= \tfrac{1}{32}f_{abc}\Big(8(\bar\alpha\Gamma_m\Psi^b)(\bar\Psi^a\Gamma^m\Psi^c)+8(\bar\alpha\Gamma_{11}\Gamma_m\Psi^b)(\bar\Psi^a\Gamma_{11}\Gamma^m\Psi^c)\Big) \\ &= \tfrac{1}{2}f_{abc}(\bar\alpha\Gamma_m\Psi^b)(\bar\Psi^a\Gamma^m\Psi^c)=-\tfrac{1}{2}f_{abc}(\bar\Psi^a\Gamma^m\Psi^b)(\bar\alpha\Gamma_m\Psi^c) \end{aligned}$$

which proves that the left hand side vanishes. Hence we end up with

$$\delta_\alpha\mathcal{L}=\bar\alpha\partial_m V^m, \qquad V^m=-\tfrac{i}{2}\mathrm{Tr}\,(F^{mn}\Gamma_n\Psi)-\tfrac{i}{4}\mathrm{Tr}\,(F_{pq}\Gamma^{pqm}\Psi) \tag{9.29}$$

Since

$$\frac{\partial\mathcal{L}}{\partial(\partial_m A_{an})}\delta_\alpha A_{an}+\frac{\partial\mathcal{L}}{\partial(\partial_m\Psi_a)}\delta_\alpha\Psi_a=-\tfrac{3i}{2}\mathrm{Tr}\left(F^{mn}\bar\alpha\Gamma_n\Psi\right)+\tfrac{i}{4}\mathrm{Tr}\left(F_{pq}\bar\alpha\Gamma^{pqm}\Psi\right)$$

the Noether current takes the simple form

$$J^m=-i\mathrm{Tr}\,(F^{mn}\Gamma_n\Psi)+\tfrac{i}{2}\mathrm{Tr}\,(F_{pq}\Gamma^{pqm}\Psi) \tag{9.30}$$

9.4 Supersymmetry transformation of reduced theory

After we have gotten the supersymmetry transformations for the $N=1$-model in 10 spacetime dimensions we are now in the position to derive the symmetry transformations of the $N=4$ extended SYM-theory in 4 dimensions. To that goal we insert the expansion (9.13) for a Majorana-Weyl spinor into the supersymmetry transformations (9.23)

$$\begin{aligned} \delta_\alpha A_\mu &= i\bar\alpha(\mathbb{1}_8\otimes\gamma_\mu)\Psi \quad , \quad \delta_\alpha\Phi_a=i\bar\alpha(\Delta_a\otimes\gamma_5)\Psi \\ \delta_\alpha\Psi &= \Big(iF_{\mu\nu}\mathbb{1}_8\otimes\Sigma^{\mu\nu}+D_\mu\Phi_a(\Delta^a\otimes\gamma^\mu\gamma_5)+\frac{g_{10}}{2i}[\Phi_a,\Phi_b]\Delta^{ab}\otimes\mathbb{1}_4\Big)\,\alpha \end{aligned}$$

as well as the corresponding expansion for the supersymmetry parameter α:

$$\alpha=\sum_{p=1}^{4}\left(F_p\otimes P_+\theta_p+F_p^*\otimes P_-\theta_p\right\} \quad , \quad \bar\alpha=\sum_{p=1}^{4}\left\{F_p^\dagger\otimes\bar\theta_pP_-+F_p^T\otimes\bar\theta_pP_+\right\}.$$

Using (9.13) we find for the variations of the vector potential and scalar fields in 4 dimensions

$$\delta_\alpha A_\mu=i\sum_{p=1}^{4}\bar\theta_p\gamma_\mu\chi_p \quad \text{and} \quad \delta_\alpha\Phi_a=i\sum_p(\Delta_a)_{pq}\bar\theta_p\gamma_5\chi_q+i\sum_{pq}(\Delta_a\Gamma_*)_{pq}\bar\theta_p\chi_q, \tag{9.31}$$

where we did not make the rescaling by $\sqrt{V_6}$ explicit. To get the susy variation of the 4 Majorana spinors is a bit trickier. We insert into the above formula for

$$\delta_\alpha\Psi=\sum_p\left(F_p\otimes P_+\delta_\alpha\chi_p+F_p^*\otimes P_-\delta_\alpha\chi_p\right)$$

the expansion for α and compare coefficients. After one introduces

$$(\Delta_+^{ab})_{pq} \equiv (F_p, \Delta^{ab} F_q) \quad \text{and} \quad (\Delta_-^{ab})_{pq} \equiv (F_p^*, \Delta^{ab} F_q^*) \quad \text{with} \quad \Delta_+^{ab} = (\Delta_-^{ab})^*,$$

the variations of the 4-dimensional Majorana spinors can be written as

$$\begin{aligned}
\delta\chi_p &= iF_{\mu\nu}\Sigma^{\mu\nu}\theta_p + \not{D}\Phi_a \sum_q \Big((\Delta_+^a)_{pq} P_+\theta_q - (\Delta_-^a)_{pq} P_-\theta_q\Big) \\
&\qquad + \frac{g}{2i}[\Phi_a, \Phi_b] \sum_q \Big((\Delta_+^{ab})_{pq} P_+\theta_q + (\Delta_-^{ab})_{pq} P_-\theta_q\Big).
\end{aligned} \tag{9.32}$$

The Noether current for the extended YM theory can be gotten from the current (9.30) by dimensional reduction. The 4 Noether currents are linear in the χ_p, the field strength and its dual but also linear in the covariant derivatives of the scalar fields Φ_a. Their explicit form reads

$$\begin{aligned}
J_p^\mu &= -\mathrm{Tr}\left({}^*F^{\mu\nu}\gamma_5 + iF^{\mu\nu}\right)\gamma_\nu\chi_p \\
&\quad -i\mathrm{Tr}\Big\{D_\alpha\Phi_a\Big((\Delta_+^a)_{pq} P_+\gamma^\alpha\gamma^\mu\chi_q - (\Delta_-^b)_{pq} P_-\gamma^\alpha\gamma^\mu\chi_q\Big)\Big\} \\
&\quad -\tfrac{1}{2}\mathrm{Tr}\Big\{\Phi_a, \Phi_b]\Big((\Delta_+^{ab})_{pq} P_-\gamma^\mu\chi_q + (\Delta_-^{ab})_{pq} P_+\gamma^\mu\chi_q\Big)\Big\}.
\end{aligned} \tag{9.33}$$

From these 4 currents one can get the 4 supercharges of the $N = 4$ supersymmetric Yang-Mills theory. They fulfill the anticommutation relations (6.75) with $N = 4$.
At the beginning of the section we have argued that $N = 4$ super-YM is scale invariant since the β-function vanishes to all orders in perturbation theory. Thus the theory should be conformally invariant and the supersymmetry algebra maybe extended to the $N = 4$ superconformal algebra. This enlarged symmetry leads to stringent conditions for the spectrum of the theory. We would need to discuss the representations of the superconformal algebras to understand the spectrum of $N = 4$ super-YM. When one tries to argue in favor of the AdS-CFT correspondence one needs this spectrum. Unfortunately, at this point I must refer to the literature, since time is running out and there are other important topics we must discuss. In the next chapter we turn to the superspace formalism.