8 Supersymmetric Gauge Theories

In this chapter we study the various supersymmetric gauge gauge theories in flat 4-dimensional space time. For the simplest gauge theory, namely the $N = 1$ Abelian gauge theory containing a photon and a massless Majorana spinor, we explicitly show the invariance of the action, the closing of the supersymmetry algebra on the gauge invariant fields and 'construct' the supercharges. Then we repeat the same for $N = 1$ non-Abelian gauge theories. It follows a discussion of the $N = 2$ and $N = 4$ gauge theories. Recently there has been much progress on $N = 2$-gauge theories due to Witten and Seiberg [38]. The $N = 4$ gauge theories emerge as conformal field theories on the boundary of AdS in the celebrated AdS/CFT-correspondence [41]

8.1 $N = 1$ Abelian gauge theories

We have seen, that the $N = 1$ vector multiplet consists of a gauge field and a Majorana spinor. First we consider the simplest of all gauge theories, namely a $U(1)$ gauge theory. In an off-shell version and the Wess-Zumino gauge (see below) we need in addition to the vector and Majorana fields an uncharged pseudo-scalar field which later may be eliminated. Since a Majorana spinor is uncharged the Lagrangian density takes the simple form

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\,\psi + \tfrac{1}{2}\mathcal{G}^2, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial\!\!\!/ = \gamma^\mu\partial_\mu. \tag{8.1}$$

The dimensions of the fields are

$$[A_\mu] = L^{-1}, \quad [\psi] = L^{-3/2}, \quad [\mathcal{G}] = L^{-2},$$

and the supersymmetry parameter has dimension $[\alpha] = L^{1/2}$. Also recall that

$$\bar{\alpha}\psi,\ \bar{\alpha}\gamma^{\mu\nu}\psi,\ \bar{\alpha}\gamma_5\gamma^\mu\psi \text{ are hermitean} \quad \text{and} \quad \bar{\alpha}\gamma_5\psi,\ \bar{\alpha}\gamma^\mu\psi \text{ are antihermitean.}$$

We shall need the formulae in (3.54) in which we calculated whether $\bar{\alpha}\gamma^{(n)}\psi$ changes sign when one interchanges the Majorana spinors α and ψ.
Taking the dimensions and hermiticity properties of the various fields into account we could guess the following supersymmetry transformations

$$\delta_\alpha A_\mu = i\bar{\alpha}\gamma_\mu\psi, \qquad \delta_\alpha\psi = ipF^{\mu\nu}\Sigma_{\mu\nu}\alpha + iq\mathcal{G}\gamma_5\alpha, \qquad \delta_\alpha\mathcal{G} = r\bar{\alpha}\gamma_5\partial\!\!\!/\,\psi. \tag{8.2}$$

As earlier, the supersymmetry parameter α is a constant Majorana spinor which anti-commutes with itself and with ψ. We will fix the real parameters p, q, r such that the action is invariant. We need

$$\delta\bar{\psi} = \delta\psi^\dagger\gamma^0 = -ipF^{\mu\nu}\alpha^\dagger\Sigma^\dagger_{\mu\nu}\gamma^0 - iq\mathcal{G}\alpha^\dagger\gamma_5\gamma^0 = -ipF^{\mu\nu}\bar{\alpha}\Sigma_{\mu\nu} + iq\mathcal{G}\bar{\alpha}\gamma_5,$$

where we used $\Sigma^\dagger_{\mu\nu}\gamma^0 = \gamma^0\Sigma_{\mu\nu}$.
Next we wish to show that the Lagrangian is, up to surface terms, invariant under the above supersymmetry transformation. The variations of the bosonic terms in the Lagrangian density are

$$\delta_\alpha\big(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\big) = -iF^{\mu\nu}\bar{\alpha}\gamma_\nu\partial_\mu\psi \qquad \text{and} \qquad \delta_\alpha\big(\tfrac{1}{2}\mathcal{G}^2\big) = r\mathcal{G}\bar{\alpha}\gamma_5\partial\!\!\!/\,\psi.$$

Using that $\bar{\alpha}\gamma^\rho\gamma_5\psi$ is symmetric under exchange of α and ψ and that

$$\bar{\psi}\gamma^\rho\Sigma_{\mu\nu}\alpha = \bar{\alpha}\Sigma_{\mu\nu}\gamma^\rho\psi \tag{8.3}$$

yields for the variation of the fermionic term in the Lagrangian

$$\delta_\alpha\big(\tfrac{i}{2}\bar{\psi}\partial\!\!\!/\,\psi\big) = \tfrac{1}{2}q\,\partial_\rho(\bar{\alpha}\mathcal{G}\gamma_5\gamma^\rho\psi) - \tfrac{1}{2}p\,\partial_\rho\big(\bar{\alpha}\Sigma_{\mu\nu}F^{\mu\nu}\gamma^\rho\psi\big) - q\,\mathcal{G}\bar{\alpha}\gamma_5\partial\!\!\!/\,\psi + p\,F^{\mu\nu}\bar{\alpha}\Sigma_{\mu\nu}\partial\!\!\!/\,\psi. \tag{8.4}$$

Collecting the various terms it follows that one must choose $q = r$ such that the volume terms depending on the auxiliary field cancels, and this choice leads to

$$\delta_\alpha\mathcal{L} = -iF^{\mu\nu}\bar{\alpha}\gamma_\nu\partial_\mu\psi + pF^{\mu\nu}\bar{\alpha}\Sigma_{\mu\nu}\partial\!\!\!/\,\psi + \text{divergence}.$$

With the help of

$$\Sigma_{\mu\nu}\gamma_\rho = \tfrac{i}{2}\eta_{\mu\rho}\gamma_\nu - \tfrac{i}{2}\eta_{\nu\rho}\gamma_\mu - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5, \qquad \epsilon_{0123} = 1, \tag{8.5}$$

the second term on the right hand side can be recast into the form

$$F^{\mu\nu}\bar{\alpha}\Sigma_{\mu\nu}\partial\!\!\!/\,\psi = i\bar{\alpha}F^{\mu\nu}\gamma_\nu\partial_\mu\psi - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar{\alpha}\gamma^\sigma\gamma_5\partial^\rho\psi.$$

Clearly, we must take $p = 1$ in which case

$$\delta_\alpha\mathcal{L} = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar{\alpha}\gamma^\sigma\gamma_5\partial^\rho\psi + \text{divergence}.$$

Using the Bianchi identity $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu,\rho} = 0$ the first term on the right is seen to be a divergence

$$-\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar{\alpha}\gamma^\sigma\gamma_5\partial^\rho\psi = -\tfrac{1}{2}\partial^\rho\big(\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar{\alpha}\gamma^\sigma\gamma_5\psi\big)$$

so that under the transformations (8.2) the complete Lagrangian density (8.1) transforms into a divergence,

$$\delta_\alpha\mathcal{L} = -\tfrac{1}{2}\bar{\alpha}\partial^\rho\Big(F^{\mu\nu}\big(\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 + \Sigma_{\mu\nu}\gamma_\rho\big)\psi\Big) + \tfrac{q}{2}\bar{\alpha}\partial_\rho\big(\mathcal{G}\gamma_5\gamma^\rho\psi\big).$$

Now we use (8.5) and

$$\gamma_\rho\Sigma_{\mu\nu} = -\tfrac{i}{2}\eta_{\mu\rho}\gamma_\nu + \tfrac{i}{2}\eta_{\nu\rho}\gamma_\mu - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 \tag{8.6}$$

to further simplify this result. Using these identities finally yields the following rather simple expression for the variation of the Lagrangian density

$$\delta_\alpha \mathcal{L} = \bar\alpha \partial_\mu V^\mu, \quad V^\mu = \tfrac{1}{2}\gamma^\mu \Sigma_{\rho\sigma} F^{\rho\sigma}\psi + \tfrac{q}{2}\mathcal{G}\gamma_5\gamma^\mu\psi = \tfrac{1}{2}({}^*F^{\mu\nu}\gamma_5 - iF^{\mu\nu})\gamma_\nu\psi + \tfrac{q}{2}\mathcal{G}\gamma_5\gamma^\mu\psi \tag{8.7}$$

under the supersymmetry transformations

$$\delta_\alpha A_\mu = i\bar\alpha\gamma_\mu\psi, \qquad \delta_\alpha\psi = iF^{\mu\nu}\Sigma_{\mu\nu}\alpha + iq\mathcal{G}\gamma_5\alpha, \qquad \delta_\alpha\mathcal{G} = q\bar\alpha\gamma_5\partial\!\!\!/\,\psi. \tag{8.8}$$

Note that the real parameter q is not fixed. Only later, when we want to recover the known superalgebra are we forced to choose $q \in \{1, -1\}$.

8.1.1 The closing of the algebra

Now we are going to repeat what we have done for the Wess-Zumino model and calculate the commutators of two supersymmetry transformations. The commutator acting on the bosonic fields is easily computed. Using (8.3) we calculate

$$\begin{aligned}[\delta_{\alpha_1},\delta_{\alpha_2}]A_\mu &= -F^{\rho\sigma}\big(\bar\alpha_2\gamma_\mu\Sigma_{\rho\sigma}\alpha_1 - \bar\alpha_1\gamma_\mu\Sigma_{\rho\sigma}\alpha_2\big) = F^{\rho\sigma}\bar\alpha_2[\Sigma_{\rho\sigma},\gamma_\mu]\alpha_1 = 2i\bar\alpha_2\gamma^\nu\alpha_1 F_{\mu\nu}\\ &= -2i(\bar\alpha_2\gamma^\rho\alpha_1)\,\partial_\rho A_\mu + 2i\partial_\mu\big(A_\rho\bar\alpha_2\gamma^\rho\alpha_1\big).\end{aligned}$$

The first term is the expected infinitesimal translation of the vector field generated by the momentum operator. The second one we have not encountered in the Wess-Zumino-Model. It is just a pure gague with a field dependent gauge transformation with gauge parameter

$$\lambda = 2iA_\rho\bar\alpha_2\gamma^\rho\alpha_1.$$

Similarly, using (8.6), one finds

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\mathcal{G} = -2iq^2\bar\alpha_2\gamma^\rho\alpha_1\partial_\rho\mathcal{G} + iq\epsilon_{\mu\nu\rho\sigma}\bar\alpha_2\gamma^\sigma\alpha_1\partial^\rho F^{\mu\nu}.$$

The second term vanishes because of the Bianchi identity. If we take $q \in \{-1, 1\}$ then we find the by now familiar commutator

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\mathcal{G} = -2i(\bar\alpha_2\gamma^\rho\alpha_1)\,\partial_\rho\mathcal{G}.$$

To calculate the commutator of two transformations on the fermion field is a bit more trickier. First one obtains ($q^2 = 1$)

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = \Big(-2(\bar\alpha_1\gamma^\nu\partial^\mu\psi)\Sigma_{\mu\nu}\alpha_2 + i(\bar\alpha_1\gamma_5\gamma^\rho\partial_\rho\psi)\gamma_5\alpha_2\Big) - (\alpha_1 \leftrightarrow \alpha_2).$$

With the help of (5.12) one arrives at the following intermediate result

$$\begin{aligned}[\delta_{\alpha_1},\delta_{\alpha_2}]\psi &= \big(-\tfrac{i}{2}\gamma_\rho\partial\!\!\!/\,\psi - \Sigma_{\mu\nu}\gamma_\rho\gamma^\nu\partial^\mu\psi\big)(\bar\alpha_2\gamma^\rho\alpha_1)\\ &+ \big(2\Sigma_{\mu\nu}\gamma_{\rho\sigma}\gamma^\nu\partial^\mu\psi - i\gamma_{\rho\sigma}\partial\!\!\!/\,\psi\big)(\bar\alpha_2\gamma^{\rho\sigma}\alpha_1).\end{aligned}$$

A lengthy but straightforward calculation where one uses the Clifford algebra, the Lorentz algebra (3.12) and the relation (8.5) one ends up with the expected result

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = -2i(\bar\alpha_2\gamma^\rho\alpha_1)\,\partial_\rho\psi.$$

As we did for the chiral multiplet we introduce the supercharges

$$\delta_\alpha(..) = \bar{\alpha} Q(..).$$

Then the superalgebra reads

$$\{Q_\alpha, \bar{Q}^\beta\} = 2i(\gamma^\mu)_\alpha{}^\beta \partial_\mu - G_\alpha{}^\beta(A), \tag{8.9}$$

where $G_\alpha{}^\beta$ is the gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$ with $\lambda = 2i(\not{A})_\alpha{}^\beta$

8.1.2 Noether charge:

To find the Noether charge we must first calculate

$$\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} \delta_\alpha A_\nu + \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} \delta_\alpha \psi = -F^{\mu\nu} \delta_\alpha A_\nu - \tfrac{i}{2} \delta_\alpha \bar{\psi} \gamma^\mu \psi.$$

Using the above expressions for the supersymmetry transformations this can be written as (we take $q = 1$)

$$\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} \delta_\alpha A_\nu + \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} \delta_\alpha \psi = -\tfrac{3i}{2} F^{\mu\nu} \bar{\alpha} \gamma_\nu \psi + \tfrac{1}{2} {}^*F^{\mu\nu} \bar{\alpha} \gamma_\nu \gamma_5 \psi + \tfrac{1}{2} \mathcal{G} \bar{\alpha} \gamma_5 \gamma^\mu \psi.$$

Subtracting V^μ in (8.7) we find the conserved Noether current and Noether charge

$$J^\mu = -\big({}^*F^{\mu\nu} \gamma_5 + i F^{\mu\nu}\big) \gamma_\nu \psi \quad \text{and} \quad Q = -\int d^3x \,\big({}^*F^{0i} \gamma_5 - i F^{0i}\big) \gamma_i \psi. \tag{8.10}$$

With

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad ({}^*F^{\mu\nu}) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{pmatrix} \tag{8.11}$$

the gauge invariant Noether charges can be rewritten

$$Q = \int d^3x \,\big(i\pi_i - \gamma_5 B_i\big) \gamma_i \psi, \quad \text{where} \quad \pi_i = E_i$$

is the conjugate momentum field to A_i.

8.2 $N = 1$ Non-Abelian gauge theories

We consider $SU(N)$ gauge theories with a massless vector field A_μ a massless Majorana spinor ψ and an auxiliary field $\mathcal{G}$. All field transform according to the adjoint representation of the gauge group $SU(N)$,

$$A_\mu = A^a_\mu T_a, \quad \psi = \psi^a T_a \quad \text{and} \quad \mathcal{G} = \mathcal{G}^a T_a. \tag{8.12}$$

The real structure constants in

$$[T_a, T_b] = f_{ab}^{\ \ c} T_c \quad \text{and} \quad \mathrm{Tr}\, T_a T_b = \delta_{ab}$$

are antisymmetric with respect to all indices. The fields transform under gauge transformations as follows

$$A \to gAg^{-1} + igdg^{-1}, \quad \psi \to g\psi g^{-1} \quad \text{and} \quad \mathcal{G} \to g\mathcal{G}g^{-1}$$

the infinitesimal versions of which read

$$\delta A_\mu = D_\mu \lambda, \quad \delta\psi = i[\lambda, \psi] \quad \text{and} \quad \delta\mathcal{G} = i[\lambda, \mathcal{G}], \quad g \sim \mathbb{1} + i\lambda.$$

The covariant derivative of the spinor field is

$$D_\mu \psi = \partial_\mu \psi - i[A_\mu, \psi]$$

and the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

The Lagrangian density reads

$$\mathcal{L} = -\tfrac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{2}\mathrm{Tr}\,(\bar{\psi}\not{D}\psi) + \tfrac{1}{2}\mathrm{Tr}\,\mathcal{G}^2. \tag{8.13}$$

The supersymmetry transformations are gotten from those of the Abelian model if we only replace ordinary derivatives by covariant ones,

$$\delta_\alpha A_\mu = i\bar{\alpha}\gamma_\mu\psi, \qquad \delta_\alpha\psi = iF^{\mu\nu}\Sigma_{\mu\nu}\alpha + i\mathcal{G}\gamma_5\alpha, \qquad \delta_\alpha\mathcal{G} = \bar{\alpha}\gamma_5\not{D}\psi. \tag{8.14}$$

Note that for a Majorana spinor α the objects

$$i\Sigma_{\mu\nu}\alpha, \quad i\gamma_5\alpha, \quad \gamma_5\gamma^\mu\alpha \quad \text{and} \quad i\gamma^\mu\alpha$$

are Majorana spinors as well. To calculate the variation of the bosonic part of the Lagrangian density we use

$$\delta F_{\mu\nu} = i\bar{\alpha}\big(\gamma_\nu D_\mu - \gamma_\mu D_\nu\big)\psi$$

and obtain

$$-\tfrac{1}{4}\delta_\alpha \mathrm{Tr}\,\big(F_{\mu\nu}F^{\mu\nu}\big) = -i\mathrm{Tr}\,\big(F^{\mu\nu}\bar{\alpha}\gamma_\nu D_\mu\psi\big) \qquad \text{and} \quad \tfrac{1}{2}\delta_\alpha\big(\mathcal{G}^2\big) = \mathrm{Tr}\,\big(\mathcal{G}\bar{\alpha}\gamma_5\not{D}\psi\big). \tag{8.15}$$

For the variation of the Dirac term we need

$$\begin{aligned} \delta_\alpha\bar{\psi} &= -i\bar{\alpha}F^{\mu\nu}\Sigma_{\mu\nu} + i\bar{\alpha}\mathcal{G}\gamma_5 \\ \delta(D_\mu\psi) &= D_\mu\delta\psi - i[\delta A_\mu, \psi] = iD_\mu F^{\rho\sigma}\Sigma_{\rho\sigma}\alpha + iD_\mu\mathcal{G}\gamma_5\alpha + [\bar{\alpha}\gamma_\mu\psi, \psi]. \end{aligned}$$

The variation of the Dirac term is found similarly as in the Abelian case to be

$$\begin{aligned}\delta_\alpha\big(\mathrm{Tr}\,(\tfrac{i}{2}\bar\psi\not{D}\psi\big) &= \tfrac{1}{2}\partial_\rho\mathrm{Tr}\left(\mathcal{G}\bar\alpha\gamma_5\gamma^\rho\psi\right)-\mathrm{Tr}\left(\mathcal{G}\bar\alpha\gamma_5\not{D}\psi\right)\\ &- \tfrac{1}{2}\partial_\rho\mathrm{Tr}\left(F^{\mu\nu}\bar\alpha\Sigma_{\mu\nu}\gamma^\rho\psi\right)+\mathrm{Tr}\left(F^{\mu\nu}\bar\alpha\Sigma_{\mu\nu}\not{D}\psi\right)+\tfrac{i}{2}\mathrm{Tr}\left(\bar\psi\gamma^\mu[\bar\alpha\gamma_\mu\psi,\psi]\right)\end{aligned}\tag{8.16}$$

Using the Bianchi identity $D_{[\rho}F_{\mu\nu]}=0$ we find

$$\begin{aligned}-i\mathrm{Tr}\left(F^{\mu\nu}\bar\alpha\gamma_\nu D_\mu\psi\right)+\mathrm{Tr}\left(F^{\mu\nu}\bar\alpha\Sigma_{\mu\nu}\not{D}\psi\right) &= -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left(F^{\mu\nu}\bar\alpha\gamma^\sigma\gamma_5 D^\rho\psi\right)\\ &= -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\rho\mathrm{Tr}\left(F^{\mu\nu}\bar\alpha\gamma^\sigma\gamma_5\psi\right).\end{aligned}$$

and end up with

$$\begin{aligned}\delta_\alpha\mathcal{L} &= -\tfrac{1}{2}\bar\alpha\partial^\rho\,\mathrm{Tr}\left\{F^{\mu\nu}(\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5+\Sigma_{\mu\nu}\gamma_\rho\psi)\right\}+\tfrac{1}{2}\bar\alpha\,\partial_\rho\mathrm{Tr}\left(\mathcal{G}\gamma_5\gamma^\rho\psi\right)\\ &+\tfrac{i}{2}\mathrm{Tr}\left(\bar\psi\gamma^\mu[\bar\alpha\gamma_\mu\psi,\psi]\right).\end{aligned}\tag{8.17}$$

If we can show, that the last term vanishes, then we have succeeded in showing, that the variation of $\mathcal{L}$ is a total divergence. It is not completely straightforward to show this, so let me give the proof. First we expand $\psi=\psi^aT_a,\ \bar\psi=\bar\psi^aT_a$ and calculate

$$+\tfrac{i}{2}\mathrm{Tr}\left(\bar\psi\gamma^\mu[\bar\alpha\gamma_\mu\psi,\psi]\right)=\tfrac{1}{2}f_{abc}(\bar\psi^a\gamma^\mu\psi^b)(\bar\alpha\gamma_\mu\psi^c).$$

Now, we use the Fierz relation (3.66)

$$4\psi^c\bar\psi^a=-(\bar\psi^a\psi^c)-\gamma_\rho(\bar\psi^a\gamma^\rho\psi^c)+\tfrac{1}{2}\gamma_{\rho\sigma}(\bar\psi^a\gamma^{\rho\sigma}\psi^c)+\gamma_5\gamma_\rho(\bar\psi^a\gamma_5\gamma^\rho\psi^c)-\gamma_5(\bar\psi^a\gamma_5\psi^c)$$

and insert it into

$$(\bar\psi^a\gamma^\mu\psi^b)(\bar\alpha\gamma_\mu\psi^c)=\bar\alpha\gamma_\mu\left(\psi^c\bar\psi^a\right)\gamma^\mu\psi^b.$$

Next we use the relations

$$\gamma_\mu\gamma_\rho\gamma^\mu=(2-d)\gamma_\rho,\quad \gamma_\mu\gamma_{\rho\sigma}\gamma^\mu=(d-4)\gamma_{\rho\sigma},\quad \gamma_\mu\gamma_5\gamma_\rho\gamma^\mu=(2-d)\gamma_\rho\gamma_5\tag{8.18}$$

and arrive at

$$\begin{aligned}(\bar\psi^a\gamma^\mu\psi^b)(\bar\alpha\gamma_\mu\psi^c) &= -(\bar\alpha\psi^b)(\bar\psi^a\psi^c)+\tfrac{1}{2}(\bar\alpha\gamma_\rho\psi^b)(\bar\psi^a\gamma^\rho\psi^c)\\ &-\tfrac{1}{2}(\bar\alpha\gamma_\rho\gamma_5\psi^b)(\bar\psi^a\gamma_5\gamma^\rho\psi^c)+(\bar\alpha\gamma_5\psi^b)(\bar\psi^a\gamma_5\psi^c).\end{aligned}\tag{8.19}$$

Note that all but the second term on the right hand side are symmetric in a and c. Hence

$$f_{abc}(\bar\psi^a\gamma^\mu\psi^b)(\bar\alpha\gamma_\mu\psi^c)=\tfrac{1}{2}f_{abc}(\bar\psi^a\gamma^\mu\psi^c)(\bar\alpha\gamma_\mu\psi^b)=-\tfrac{1}{2}f_{abc}(\bar\psi^a\gamma^\mu\psi^b)(\bar\alpha\gamma_\mu\psi^c)$$

which implies, that the expression on the left vanishes. Actually, the in a and c antisymmetric part of (8.19) can be rewritten as

$$(\bar\psi^a\gamma^\mu\psi^b)(\bar\alpha\gamma_\mu\psi^c)+(\bar\psi^b\gamma^\mu\psi^c)(\bar\alpha\gamma_\mu\psi^a)+(\bar\psi^c\gamma^\mu\psi^a)(\bar\alpha\gamma_\mu\psi^b)=0\quad\text{if}\quad\psi^a_c=\psi^a.$$

Hence the last term in (8.17) is absent and using (8.5) and (8.8) the result (8.17) further simplifies to

$$\delta_\alpha \mathcal{L} = \bar{\alpha}\partial_\mu V^\mu, \quad V^\mu = \tfrac{1}{2}\mathrm{Tr}\left\{({}^*F^{\mu\nu}\gamma_5 - iF^{\mu\nu})\gamma_\nu\psi\right\} + \tfrac{1}{2}\mathrm{Tr}\left(\mathcal{G}\gamma_5\gamma^\mu\psi\right) \tag{8.20}$$

To calculate the Noether current we need

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu A_{a\nu}}\delta_\alpha A_{a\nu} + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi_a}\delta_\alpha\psi_a = -\tfrac{3i}{2}\mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\gamma_\nu\psi\right) + \tfrac{1}{2}\,\mathrm{Tr}\left({}^*F^{\mu\nu}\bar{\alpha}\gamma_\nu\gamma_5\psi\right) + \tfrac{1}{2}\,\mathrm{Tr}\left(\mathcal{G}\bar{\alpha}\gamma_5\gamma^\mu\psi\right)$$

so that this current and the supercharge have the simple form

$$J^\mu = -\mathrm{Tr}\left({}^*F^{\mu\nu}\gamma_5 + iF^{\mu\nu}\right)\gamma_\nu\psi \quad \text{and} \quad Q = \int d^3x\,\mathrm{Tr}\left\{(i\pi_i - B_i\gamma_5)\gamma_i\psi\right\}. \tag{8.21}$$

The commutator of the supersymmetry transformations on the gauge potential is

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]A_\mu = 2i\bar{\alpha}_2(F_{\mu\nu}\gamma^\nu)\alpha_1 = -2i(\bar{\alpha}_2\gamma^\nu\alpha_1)\partial_\nu A_\mu + D_\mu\lambda, \quad \lambda = 2i(\bar{\alpha}_2\gamma^\nu\alpha_1)A_\nu.$$

Similarly as in the Abelian case we see, that the commutator of two supersymmetry transformations yields a translation plus an infinitesimal gauge transformation. The gauge parameter λ depends on the gauge field A_μ and the susy parameters α_i.
To calculate the commutator on the dummy field $\mathcal{G}$ one needs the relations

$$\bar{\alpha}\gamma_5\gamma^\rho\Sigma_{\mu\nu}\psi = \bar{\psi}\Sigma_{\mu\nu}\gamma^\rho\gamma_5\alpha \quad \text{and} \quad i\gamma_5[\gamma_\mu, \gamma_\nu] + \epsilon_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma = 0,$$

where we used the definition $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ and the identity (5.12). One finds after some lengthy but straightforward manipulations

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\mathcal{G} = -2i(\bar{\alpha}_2\gamma^\rho\alpha_1)\left(D_\rho\mathcal{G} - D_\sigma\,{}^*F^\sigma{}_\rho\right) = -2i(\bar{\alpha}_2\gamma^\rho\alpha_1)\partial_\rho\mathcal{G} + i[\lambda, \mathcal{G}],$$

where we used once again the Bianchi identity. Here λ it the above introduced field dependent gauge parameter. The commutator on the fermionic field becomes

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\psi = -2\Sigma_{\mu\nu}M\gamma^\nu D^\mu\psi + i\gamma_5 M\gamma_5 \not{D}\psi,$$

where M has been introduced in (5.12). Using the relations (3.12,8.5,5.12) and $2i\Sigma_{\mu\nu}\gamma^\nu = 3\gamma_\mu$ one obtains

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\psi = -2i(\bar{\alpha}_2\gamma^\rho\alpha_1)D_\rho\psi = -2i(\bar{\alpha}_2\gamma^\rho\alpha_1)\partial_\rho\psi + i[\lambda, \psi]. \tag{8.22}$$

Again we find that the superalgebra closes on the gauge invariant fields.

8.3 $N = 2$ Supersymmetric Gauge Theory (Seiberg-Witten)

In this section we will be considering theories with $N = 2$ spacetime supersymmetries. Realistic (i.e. chiral) models of particle interactions have only $N = 1$ supersymmetry. There are theoretical reasons for discussing $N = 2$, the main being that the dynamics of this theory is under much better control and this allows one to make statements about the spectrum which are valid non-perturbatively. This model may possess a non-vanishing central charge and, as we have seen, the

Bogomol'nyi bound applies. Magnetic monopoles in $N=2$ supersymmetric theories were first discussed in [39].
The rather simple $N=2$ model has played in important role in recent developments about confinement in 4 dimensional asymptotically free gauge theories. Seiberg and Witten have got an analytic expression for the low energy effective action (the leading two terms in a derivative expansion) of this theory [38]. On shell this model contains a vector field A_μ, two Majorana spinors χ_1, χ_2, which can be combined to a Dirac spinor ψ, a scalar field A and a pseudo-scalar field B. All fields transform according to the adjoint representation of a gauge group. Note that a Dirac spinor can always be written as a linear combination of two Majorana spinors,

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \quad , \quad \psi_c = \frac{1}{\sqrt{2}}(\chi_1 - i\chi_2) \\ \chi_1 &= \frac{1}{\sqrt{2}}(\psi + \psi_c) \quad , \quad \chi_2 = \frac{1}{i\sqrt{2}}(\psi - \psi_c). \end{aligned} \tag{8.23}$$

The same decomposition exists for the supersymmetry parameter

$$\alpha = \frac{1}{\sqrt{2}}(\epsilon_1 + i\epsilon_2) \quad \text{and} \quad \alpha_c = \frac{1}{\sqrt{2}}(\epsilon_1 - i\epsilon_2). \tag{8.24}$$

It follows, that

$$i\bar\psi \not{D}\psi = \tfrac{i}{2}\sum(\bar\chi_i \not{D}\chi_i) - \tfrac{1}{2}\partial_\rho(\bar\chi_1\gamma^\rho\chi_2) \quad \text{and} \quad \bar\alpha M\psi + \bar\alpha_c M\psi_c = \sum \bar\epsilon_i M\chi_i \tag{8.25}$$

for any matrix M. Now we could go back to the $N=1$ vector multiplet and take the following transformation rule for the gauge potential,

$$\delta_\alpha A_\mu = i\sum \bar\epsilon_i\gamma_\mu\chi_i = i\bar\alpha\gamma_\mu\psi + i\bar\alpha_c\gamma_\mu\psi_c. \tag{8.26}$$

By dimensional arguments there are no other terms which are linear combinations of the χ_i contracted with the supersymmetry parameters.

8.3.1 Action and field equations

We just add the Wess-Zumino Lagrangian (5.19) for massless spin-0 fields, with $\not\partial$ replaced by $\not{D}$ and A, B in the adjoint representation, to the gauge model Lagrangian (8.13). If A, B are in the adjoint representation, then the self interaction term in (5.19), namely

$$-g\bar\psi(A - i\gamma_5 B)\psi - \tfrac{1}{2}g^2\,(A^2 + B^2)^2$$

should be replaced by something like

$$-g\mathrm{Tr}\left(\bar\psi[A,\psi]\right) + ig\mathrm{Tr}\left(\bar\psi\gamma_5[B,\psi]\right) + \tfrac{1}{2}g^2\,\mathrm{Tr}\left([A,B]^2\right).$$

Hence we would guess that the Lagrangian of the $N=2$ vector multiplet on shell should have the form

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mathrm{Tr}\,(D_\mu A)^2 + \tfrac{1}{2}\mathrm{Tr}\,(D_\mu B)^2 + \tfrac{1}{2}g^2\,\mathrm{Tr}\left([A,B]^2\right) \\
&\quad + i\,\mathrm{Tr}\,\bar\psi \not{D}\psi - g\,\mathrm{Tr}\left(\bar\psi[A,\psi]\right) + ig\,\mathrm{Tr}\left(\bar\psi\gamma_5[B,\psi]\right)
\end{aligned}
\tag{8.27}
$$

Up to a surface term, the second line can be rewritten as

$$
\mathcal{L}_\psi = \mathrm{Tr}\left\{\tfrac{i}{2}\bar\chi_i\not{D}\chi_i - ig\bar\chi_1[A,\chi_2] - g\bar\chi_1\gamma_5[B,\chi_2]\right\}. \tag{8.28}
$$

Expanding every field as $A = A^a T_a$ such that

$$
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu, \quad (D_\mu A)^a = \partial_\mu A^a + g f^a{}_{bc} A^b_\mu A^c, \dots,
$$

we obtain for the component fields

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \tfrac{1}{2}(D_\mu A)^a(D^\mu A)_a + \tfrac{1}{2}(D_\mu B)^a(D^\mu B)_a - \tfrac{1}{2}g^2 f^a{}_{bc} f_{apq}\left(A^b A^p B^c B^q\right) \\
&\quad + i\bar\psi^a\gamma^\mu\left(\partial_\mu\psi_a + g f_{abc}A^b_\mu\psi^c\right) + ig f_{abc}A^a\bar\psi^b\psi^c + g f_{abc}B^a\bar\psi^b\gamma_5\psi^c.
\end{aligned}
\tag{8.29}
$$

The model contains only ONE COUPLING CONSTANT g: the self-coupling of the scalar fields and the Yukawa couplings are determined by g. There is a potential term in the Lagrangian but it has an exact flat direction whenever $[A,B]=0$. Also it is classically scale invariant and this scale invariance will be spontaneously broken by having a non-zero expectation value for the scalar fields. The corresponding Euler-Lagrange equations are the following set of hyperbolic field equations: The variation with respect to the gauge potential yields the YANG-MILLS EQUATIONS

$$
D_\mu F^{\mu\nu} = ig[A, D^\nu A] + ig[B, D^\nu B] - g[\bar\psi, \gamma^\nu\psi], \quad \text{where} \quad [\bar\psi, M\psi] \equiv i f^a{}_{bc}\bar\psi^b M\psi^c T_a. \tag{8.30}
$$

The variations with respect to the scalar and pseudo-scalar field implies the KLEIN-GORDON EQUATIONS

$$
\begin{aligned}
D_\mu D^\mu A &= -g[\bar\psi,\psi] \ + \ g^2[B,[B,A]] \\
D_\mu D^\mu B &= -ig[\bar\psi,\gamma_5\psi] + g^2[A,[A,B]],
\end{aligned}
\tag{8.31}
$$

and the variation of the Dirac-spinor field yields the DIRAC EQUATION:

$$
i\not{D}\psi = g[A - i\gamma_5 B, \psi]. \tag{8.32}
$$

A particular class of solutions is obtained if we set

$$
\psi = 0 \quad \text{and} \quad B = 0,
$$

in which case the field equations for the remaining fields reduce to

$$
D_\mu F^{\mu\nu} = ig[A, D^\nu A] \quad \text{and} \quad D_\mu D^\mu A = 0. \tag{8.33}
$$

These reduced equations admit MAGNETIC MONOPOLE SOLUTIONS. The mass of the monopoles are proportional to the asymptotic value of $|A|$ and inversely proportional to the gauge coupling

constant g. These monopoles are relevant for a field theoretic understanding of the central charge of the $N=2$ gauge theory.
If we now further set $A=0$ then we get the pure Yang-Mills field equations

$$D_\mu F^{\mu\nu}=0 \tag{8.34}$$

which in the Euclidean sector possess (anti)selfdual INSTANTON SOLUTIONS. Later we shall see, that instantons break half of the supersymmetry: the instanton background is left invariant by a subset of the $N=2$ supersymmetry transformations which generate a $N=1$ supersymmetry.
To calculate the Hamiltonian density we note that the momenta conjugate to the vector, scalar and fermion fields are

$$\pi_i=-F^{0i}=E_i,\quad \pi_A=D_0A,\quad \pi_B=D_0B\quad\text{and}\quad \pi_\psi=i\psi^\dagger.$$

When discussing the Hamiltonian structure of gauge theories it is advantageous to choose the Weyl gauge $A_0=0$ for which

$$\pi_i=\dot A_i,\quad \pi_A=\dot A,\quad \pi_B=\dot B\quad\text{and}\quad \pi_\psi=i\psi^\dagger.$$

The Hamiltonian density is gotten as Legendre transform from the Lagrangian density and it reads

$$\begin{aligned}\mathcal{H} \;=\; \mathrm{Tr}\Big\{&\tfrac12(\vec\pi^2+\vec B^2+\pi_A^2+\pi_B^2)+\tfrac12(\vec D A)^2+\tfrac12(\vec D B)^2-i\psi^\dagger\gamma^0\gamma^i D_i\psi\\ &+g\bar\psi[A,\psi]-ig\bar\psi\gamma_5[B,\psi]-\frac{g^2}{2}[A,B]^2\Big\}.\end{aligned}\tag{8.35}$$

This density is (formally) hermitean and the bosonic part is bounded below.

8.3.2 Supersymmetry transformations and invariance of S

Now we shall fix the variations of the remaining fields A,B and ψ such that $\mathcal{L}$ is invariant up to surface terms. The calculations are almost the same as in the previous cases and hence I shall not give many details here. We start from the general formulae

$$\begin{aligned}\delta D_\mu A=D_\mu\delta A-ig[\delta A_\mu,A]\quad &,\quad \delta D_\mu B=D_\mu\delta B-ig[\delta A_\mu,B]\\ \delta F_{\mu\nu}=D_\mu\delta A_\nu-D_\mu\delta A_\mu\quad &,\quad \delta \not{D}\psi=\not{D}\delta\psi-ig[\delta A_\mu,\gamma^\mu\psi]\end{aligned}$$

and deduce the following transformation rules for the various terms in the Lagrangian density:

$$\begin{aligned}\tfrac14\delta\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) &= \mathrm{Tr}\,(F^{\mu\nu}D_\mu\delta A_\nu)\\ \tfrac12\delta\mathrm{Tr}\,(D_\mu A D^\mu A) &= \mathrm{Tr}\,\{D_\mu A(D_\mu\delta A-ig[\delta A_\mu,A])\}\\ \tfrac12\delta\mathrm{Tr}\,(D_\mu B D^\mu B) &= \mathrm{Tr}\,\{D_\mu B(D_\mu\delta B-ig[\delta A_\mu,B])\}\\ \delta\mathrm{Tr}\,([A,B]^2) &= 2\mathrm{Tr}\,\{[A,B]([\delta A,B]+[A,\delta B])\}\\ \tfrac{i}{2}\delta\mathrm{Tr}\,(\bar\chi\not{D}\chi) &= \tfrac12\mathrm{Tr}\,\{i\delta\bar\chi\not{D}\chi+i\bar\chi\not{D}\delta\chi+g\bar\chi[\delta A_\mu,\gamma^\mu\chi]\}\\ \delta\mathrm{Tr}\,(\bar\chi_1[A,\chi_2]) &= \mathrm{Tr}\,\{\delta\bar\chi_1[A,\chi_2]+\bar\chi_1[A,\delta\chi_2]+\bar\chi_1[\delta A,\chi_2]\}\\ \delta\mathrm{Tr}\,(\bar\chi_1\gamma_5[B,\chi_2]) &= \mathrm{Tr}\,\{\delta\bar\chi_1\gamma_5[B,\chi_2]+\bar\chi_1\gamma_5[B,\delta\chi_2]+\bar\chi_1\gamma_5[\delta B,\chi_2]\}\end{aligned}$$

To fix the transformation properties of the scalar fields we first study the terms in $\delta\mathcal{L}$ which are TRILINEAR IN THE DIRAC FIELD:

$$g\mathrm{Tr}\left\{\tfrac{1}{2}\bar\chi_i[\delta A_\mu,\gamma^\mu\chi_i]-i\bar\chi_1[\delta A,\chi_2]-\bar\chi_1\gamma_5[\delta B,\chi_2]\right\}$$

Using the explicit form of δA_μ and that $\mathrm{Tr}\,(\bar\chi[\bar\epsilon\gamma_\mu\chi,\gamma^\mu\chi])=0$ we obtain

$$=gf_{abc}\left\{\tfrac{1}{2}(\bar\chi_1^a\gamma^\mu\chi_1^b)(\bar\epsilon_2\gamma_\mu\chi_2^c)+\tfrac{1}{2}(\bar\chi_2^a\gamma^\mu\chi_2^b)(\bar\epsilon_1\gamma_\mu\chi_1^c)-(\chi_1^a\chi_2^b)\delta A^c+i(\bar\chi_1^a\gamma_5\chi_2^b)\delta B^c\right\}$$

We rearrange the first two terms with the help of the Fierz identities

$$\begin{aligned}
\tfrac{1}{2}(\bar\epsilon_1\gamma_\mu\chi_1^a)(\bar\chi_2^b\gamma^\mu\chi_2^c)-\tfrac{1}{2}(\bar\epsilon_1\gamma_5\gamma_\mu\chi_1^a)(\bar\chi_2^b\gamma_5\gamma^\mu\chi_2^c) &= -(\bar\chi_1^a\chi_2^b)(\bar\epsilon_1\chi_2^c)+(\bar\chi_1^a\gamma_5\chi_2^b)(\bar\epsilon_1\gamma_5\chi_2^c)\\
\tfrac{1}{2}(\bar\epsilon_2\gamma_\mu\chi_2^b)(\bar\chi_1^a\gamma^\mu\chi_1^c)-\tfrac{1}{2}(\bar\epsilon_2\gamma_5\gamma_\mu\chi_2^b)(\bar\chi_1^a\gamma_5\gamma^\mu\chi_1^c) &= -(\bar\chi_1^a\chi_2^b)(\bar\epsilon_2\chi_1^c)+(\bar\chi_1^a\gamma_5\chi_2^b)(\bar\epsilon_2\gamma_5\chi_1^c)
\end{aligned}$$

The second terms on the left hand sides do not contribute, since they are symmetric in b,c or a,c, respectively. Now we can read off how the scalar fields must transform such that the terms trilinear in $\delta\mathcal{L}$ chancel:

$$\begin{aligned}
\delta_\alpha A &= -(\bar\epsilon_1\chi_2-\bar\epsilon_2\chi_1)=i(\bar\alpha\psi-\bar\alpha_c\psi_c)\\
\delta_\alpha B &= i(\bar\epsilon_1\gamma_5\chi_2-\bar\epsilon_2\gamma_5\chi_1)=\bar\alpha\gamma_5\psi-\bar\alpha_c\gamma_5\psi_c.
\end{aligned}\tag{8.36}$$

Again, the variations of A and B can only be linear combinations of the χ_i contracted with the supersymmetry parameters. Let us see, whether we can arrange for a transformation of the Majorana spinors such the terms trilinear in the bosonic fields A,B chancel in $\delta\mathcal{L}$. Such term can only arise from

$$g\mathrm{Tr}\Big\{g[A,B]\big([\delta A,B]+[A,\delta B]\big)-i\delta\bar\chi_1[A,\chi_2]-i\bar\chi_1[A,\delta\chi_2]-\delta\bar\chi_1\gamma_5[B,\chi_2]-\bar\chi_1\gamma_5[B,\delta\chi_2]\Big\}.$$

So let us assume, that

$$\delta\chi_i=g[A,B]\gamma_5\epsilon_i+\dots,\qquad \delta\bar\chi_i=g\bar\epsilon_i\gamma_5[A,B]+\dots$$

Then the last four terms in the second to last equation read

$$g\mathrm{Tr}\left\{[A,B]\left\{-i[A,\bar\epsilon_1\gamma_5\chi_2-\bar\epsilon_2\gamma_5\chi_1]+[\bar\epsilon_1\chi_2-\bar\epsilon_2\chi_1,B]\right\}\right\}\tag{8.37}$$

and exactly cancel the first two terms. From our experience with the Wess-Zumino model we also expect terms

$$\delta\chi_i=-i\not{D}(A-i\gamma_5B)\kappa_i+\dots,\qquad \delta\bar\chi_i=i\bar\kappa_i\not{D}(A+i\gamma_5B)+\dots\tag{8.38}$$

which are fixed by considering the contributions which are quadratic in the scalar fields. These contributions can only come from

$$\begin{aligned}
\delta\mathcal{L} &= -ig\mathrm{Tr}\left\{D_\mu A[\delta A_\mu,A]+D_\mu B[\delta A_\mu,B]\right\}+\tfrac{i}{2}\mathrm{Tr}\left\{\delta\bar\chi_i\not{D}\chi_i+\bar\chi_i\not{D}\delta\chi_i\right\}\\
&\quad -ig\mathrm{Tr}\left\{\delta\bar\chi_1[A,\chi_2]+\bar\chi_1[A,\delta\chi_2]\right\}-g\mathrm{Tr}\left\{\delta\bar\chi_1\gamma_5[B,\chi_2]+\bar\chi_1\gamma_5[B,\delta\chi_2]\right\}+\dots
\end{aligned}$$

In the first two terms we insert the variation of the gauge potential, in the following two terms the result (8.37) and in the last two terms the expected results (8.38). This the leads to

$$=\tfrac{i}{2}g\left\{[A,B]D_\mu(\bar\epsilon_i\gamma_5\gamma^\mu\chi_i)-D_\mu[A,B](\bar\epsilon_i\gamma_5\gamma^\mu\chi_i)\right\}$$

$$+g\{[A,D_\mu A]+[B,D_\mu B]\}\{\bar\epsilon_i\gamma^\mu\chi_i-(\bar\kappa_1\gamma^\mu\chi_2-\bar\kappa_2\gamma^\mu\chi_1)\}$$
$$+ig\{[A,D_\mu B]+[D_\mu A,B]\}\{\bar\kappa_1\gamma_5\gamma^\mu\chi_2-\bar\kappa_2\gamma_5\gamma^\mu\chi_1)\}+\dots,$$

where ... indicates the terms which are not quadratic in A,B. Clearly, we must choose $\kappa_1=\epsilon_2$ and $\kappa_2=-\epsilon_1$ such that the second line vanishes and this becomes a total divergence

$$\tfrac{i}{2}\partial_\mu \mathrm{Tr}\,\{[A,B]\bar\epsilon_i\gamma_5\gamma^\mu\chi_i\}.$$

Collecting the various contribution we conclude, that

$$\begin{aligned}\delta_\alpha\psi &= iF^{\mu\nu}\Sigma_{\mu\nu}\alpha-\not{D}(A-i\gamma_5 B)\alpha+g\,[A,B]\gamma_5\alpha\\ \delta_\alpha\bar\psi &= -i\bar\alpha F^{\mu\nu}\Sigma_{\mu\nu}-\bar\alpha\not{D}(A+i\gamma_5 B)+g\,\bar\alpha\gamma_5[A,B].\end{aligned}\tag{8.39}$$

Let us summarize our findings (what has not been proven here, we leave as an exercise). The action (8.27) is invariant with respect to the following supersymmetry transformations:

$$\begin{aligned}\delta_\alpha A_\mu &= i(\bar\alpha\gamma_\mu\psi+\bar\alpha_c\gamma_\mu\psi_c),\quad \delta_\alpha A=i(\bar\alpha\psi-\bar\alpha_c\psi_c),\quad \delta_\alpha B=\bar\alpha\gamma_5\psi-\bar\alpha_c\gamma_5\psi_c\\ \delta_\alpha\psi &= iF^{\mu\nu}\Sigma_{\mu\nu}\alpha-\not{D}(A-i\gamma_5 B)\alpha+g\,[A,B]\gamma_5\alpha.\end{aligned}\tag{8.40}$$

Note that a constant A-field background is left invariant by an arbitrary supersymmetry transformation. A constant scalar field does not 'break supersymmetry'.
We have already done most of the calculation which show that S is invariant. The explicit result reads

$$\begin{aligned}\delta\mathcal{L} &= \partial_\mu\big(\bar\alpha V^\mu+\bar\alpha_c V_c^\mu\big)\\ V^\mu &= \mathrm{Tr}\,\{\big(ig[A,B]\gamma_5\gamma^\mu+2D_\nu(A-iB\gamma_5)\Sigma^{\nu\mu}+{}^*F^{\mu\nu}\gamma_5\gamma_\nu\big)\psi\}\\ V_c^\mu &= -i\mathrm{Tr}\,\{\big(D^\mu(A-iB\gamma_5)+F^{\mu\nu}\gamma_\nu\big)\psi_c\}\end{aligned}$$

To construct the Noether current we need to calculate

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\delta A_\nu+\dots=\mathrm{Tr}\{-F^{\mu\nu}\delta A_\nu+i\bar\psi\gamma^\mu\delta\psi+D^\mu A\delta A+D^\mu B\delta B\}$$

and subtract $\bar\alpha V^\mu+\bar\alpha_c V_c^\mu$. The result is the following hermitean Noether current:

$$\begin{aligned}J^\mu &= \bar\alpha\{{}^*F^{\mu\nu}\gamma_\nu\gamma_5-iF^{\mu\nu}\gamma_\nu+i\not{D}(A+iB\gamma_5)\gamma^\mu+ig[A,B]\gamma^\mu\gamma_5\}\psi\\ &+ \bar\psi\{{}^*F^{\mu\nu}\gamma_\nu\gamma_5+iF^{\mu\nu}\gamma_\nu-i\gamma^\mu\not{D}(A-iB\gamma_5)+ig[A,B]\gamma^\mu\gamma_5\}\alpha\end{aligned}$$

The corresponding Noether charge reads

$$\begin{aligned}Q &= \bar\alpha\int d^3x(R+S)\psi+\int d^3x\bar\psi(R-S)\alpha\qquad\text{with}\\ R &= \gamma_i B_i\gamma_5+i\gamma^0\gamma_i D_i(A-iB\gamma_5)+i\gamma^0\gamma_5[A,B]\quad\text{and}\quad S=i\gamma_i\pi_i+i\pi_A+\pi_B\gamma_5\end{aligned}$$

Here $\pi_i=E_i$ is the momentum conjugate to A_i, $\pi_A=D_0A$ the momentum conjugate to A and $\pi_B=D_0B$ the momentum conjugate to B. One should compare these charges of the $N=2$ vector multiplet with the Noether charges (5.28) of the Wess-Zumino model and the charges after (8.11) of the $N=1$ vector multiplet.

Exercise: Check, that the commutator of two supersymmetry transformations is

$$[\delta_{\alpha_1},\delta_{\alpha_2}]A_\mu = a^\rho\partial_\rho A_\mu + D_\mu\lambda \qquad [\delta_{\alpha_1},\delta_{\alpha_2}]\psi = a^\rho\partial_\rho\psi + i[\lambda,\psi]$$
$$[\delta_{\alpha_1},\delta_{\alpha_2}]A = a^\rho\partial_\rho A + i[\lambda,A] \qquad [\delta_{\alpha_1},\delta_{\alpha_2}]B = a^\rho\partial_\rho B + i[\lambda,B]$$

where

$$a^\rho = -2i(\bar\alpha_2\gamma^\rho\alpha_1 - \bar\alpha_1\gamma^\rho\alpha_2)$$

is the parameter for the infinitesimal translation generated by the commutator of two susy transformations and

$$\lambda = 2i\big\{(\bar\alpha_2\gamma^\rho\alpha_1 - \bar\alpha_1\gamma^\rho\alpha_2)A_\rho - \bar\alpha_2(A-i\gamma_5 B)\alpha_1 + \bar\alpha_1(A-i\gamma_5 B)\alpha_2\big\}$$

the field dependent gauge parameter.

8.3.3 Chiral basis

We rewrite the previous results in terms of Weyl fermions

$$\psi = \begin{pmatrix}\lambda_\alpha\\ \bar\lambda^{\dot\alpha}\end{pmatrix} \qquad \bar\psi = (\,\lambda^\alpha\,,\bar\lambda_{\dot\alpha}\,) \quad \text{and} \quad \phi = (A+iB)/\sqrt{2}.$$

Using the conventions in the appendix we see that

$$\begin{aligned}
\bar\psi \not{D}\psi &= \lambda^\alpha\,\sigma_{\mu\alpha\dot\alpha}D^\mu\bar\lambda^{\dot\alpha} + \bar\lambda_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}D_\mu\lambda_\alpha \equiv \lambda(\sigma D)\bar\lambda + \bar\lambda(\tilde\sigma D)\lambda\\
\bar\psi[A,\psi] &= \lambda^\alpha\,[A,\lambda_\alpha\,] + \bar\lambda_{\dot\alpha}[A,\bar\lambda^{\dot\alpha}] = \lambda[A,\lambda] + \bar\lambda[A,\bar\lambda]\\
\bar\psi\gamma_5[B,\psi] &= -\lambda^\alpha\,[B,\lambda_\alpha\,] + \bar\lambda_{\dot\alpha}[B,\bar\lambda^{\dot\alpha}] = -\lambda[B,\lambda] + \bar\lambda[B,\bar\lambda]\\
[\phi,\phi^\dagger] &= -i[A,B]
\end{aligned}$$

In terms of these Weyl spinors the action reads

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}\,(D_\mu\phi D^\mu\phi^\dagger) - \tfrac{1}{2}g^2\,\mathrm{Tr}\left([\phi,\phi^\dagger]^2\right)\\
&\quad + \mathrm{Tr}\left\{i\lambda(\sigma D)\bar\lambda + i\bar\lambda(\tilde\sigma D)\lambda - \sqrt{2}g\lambda[\phi,\lambda] - \sqrt{2}\bar\lambda[\phi^\dagger,\bar\lambda]\right\}
\end{aligned} \tag{8.41}$$

To rewrite the susy-transformations we also set

$$\alpha = \begin{pmatrix}\theta_\alpha\\ \bar\theta^{\dot\alpha}\end{pmatrix} \qquad \bar\alpha = (\,\theta^\alpha\,,\bar\theta_{\dot\alpha}\,)$$

such that

$$\begin{aligned}
\bar\alpha\psi - \bar\alpha_c\psi_c &= 2(\theta\lambda + \bar\theta\bar\lambda)\\
\bar\alpha\gamma_5\psi - \bar\alpha_c\gamma_5\psi_c &= 2(-\theta\bar\lambda + \bar\theta\lambda)\\
\bar\alpha\gamma_\mu\psi + \bar\alpha_c\gamma_\mu\psi_c &= 2(\theta\sigma_\mu\bar\lambda + \bar\theta\tilde\sigma_\mu\lambda)
\end{aligned}$$

Inserting these relations into (8.40) yields

$$\begin{aligned}
\delta A_\mu &= 2i(\theta\sigma_\mu\bar\lambda + \bar\theta\tilde\sigma_\mu\lambda), \quad \delta A = 2i(\theta\lambda + \bar\theta\bar\lambda), \quad \delta B = 2(-\theta\lambda + \bar\theta\bar\lambda)\\
\delta\lambda &= iF^{\mu\nu}\sigma_{\mu\nu}\theta - i\sqrt{2}\,(\sigma_\mu D^\mu\phi)\,\bar\theta\, - ig[\phi,\phi^\dagger]\theta\\
\delta\bar\lambda &= iF^{\mu\nu}\tilde\sigma_{\mu\nu}\bar\theta - i\sqrt{2}\,(\tilde\sigma_\mu D^\mu\phi^\dagger)\theta + ig[\phi,\phi^\dagger]\bar\theta
\end{aligned} \tag{8.42}$$

8.3.4 Bogomol'nyi bound, Monopoles and Jackiw-Rebbi modes

Let us come back to the monopole and instanton solutions in this model. Assume that we have constructed a solution with $B = 0$ and $\psi = 0$. Such a solution is left invariant by the supersymmetry transformations if

$$\delta\psi = 0 \Longleftrightarrow iF^{\mu\nu}\Sigma_{\mu\nu}\alpha = \not{D}A\alpha \quad \text{or} \quad \tfrac{1}{2}F^{\mu\nu}\gamma_{\mu\nu}\alpha = \not{D}A\alpha \tag{8.43}$$

which in a chiral basis reads

$$\tfrac{1}{2}F^{\mu\nu}(\sigma_{\mu\nu})_\alpha{}^\beta\,\theta_\beta = \partial_\mu A(\sigma^\mu)_{\alpha\dot\beta}\,\bar\theta^{\dot\beta} \quad \text{and} \quad \tfrac{1}{2}F^{\mu\nu}(\tilde\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta} = \partial_\mu A(\tilde\sigma^\mu)^{\dot\alpha\beta}\,\theta_\beta, \quad \alpha = \begin{pmatrix}\theta_\alpha\\ \bar\theta^{\dot\alpha}\end{pmatrix}.$$

Let us see, what second order equation is implied by the condition (8.43). For that we act with $\not{D}$ on this equation which leads to

$$\tfrac{1}{2}\gamma_\rho\gamma_{\mu\nu}D^\rho F^{\mu\nu}\alpha = D^2A\alpha - \frac{ig}{2}\gamma^{\rho\mu}[F_{\rho\mu}, A]\alpha.$$

Now we use (8.6) and the Bianchi identity to simplify the left hand side

$$\gamma_\nu D_\mu F^{\mu\nu}\alpha = D^2A\alpha - \frac{ig}{2}\gamma^{\rho\mu}[F_{\rho\mu}, A]\alpha = D^2A\alpha - ig\gamma^\mu[D_\mu A, A]\alpha,$$

where in the last step we have inserted (8.43). This finally yields the second order equation

$$0 = \big(D_\mu F^{\mu\nu} - ig[A, D^\nu A]\big)\gamma^\nu\alpha = D^2A\alpha. \tag{8.44}$$

Next we argue that both sides must vanish separately and that these conditions are just the field equations for the scalar field and the gauge potential in cases where the pseudoscalar and Dirac field vanish. To see this we note that $\bar\alpha\gamma^\mu\alpha$ is imaginary and $\bar\alpha\alpha$ is real for a Majorana parameter α. Hence if we multiply (8.44) with $\bar\alpha$ from the left and assume that α is Majorana, then the left hand side becomes antihermitean whereas the right hand side becomes hermitean. It follows that both sides must vanish. This then proves that the first order equations (8.43) imply the second order field equations (8.33).
Let us now assume, that (A_μ, A) is a magnetic monopole. Then the first order equation (8.43) reduces to

$$0 = \delta\psi = i\begin{pmatrix}B_i\tau_i & 0\\ 0 & B_i\tau_i\end{pmatrix}\alpha - \begin{pmatrix}0 & \tau_i D_i A\\ -\tau_i D_i A & 0\end{pmatrix}\alpha.$$

This system of equations is satisfied if

$$i\theta_\alpha = (\sigma_0)_{\alpha\dot\beta}\bar\theta^{\dot\beta} \quad \text{and} \quad B_i = D_iA \tag{8.45}$$

hold true. The first condition means that only a $N = 1$ supersymmetry is left and the second condition is the Bogomol'nyi bound. To make this more clear, consider the energy or mass (8.35) for a static background (A_i, A)

$$M = \tfrac{1}{2}\int d^3x\,\mathrm{Tr}\big(\vec B^2 + (\vec DA)^2\big) = \tfrac{1}{2}\int d^3x\,\mathrm{Tr}\big(\vec B \pm \vec DA\big)^2 \mp \int \mathrm{Tr}\big(\vec B\cdot\vec DA\big).$$

Clearly we have the bound

$$M \geq \left| \int \mathrm{Tr}\,(\vec{B} \cdot \vec{D}A) \right|$$

which is just the celebrated Bogomol'nyi bound. The Bianchi identity for the 'chromo-magnetic' field reads

$$\vec{D} \cdot \vec{B} = 0 \Longrightarrow \nabla \mathrm{Tr}\,(\vec{B}\,A) = \mathrm{Tr}\,(\vec{B} \cdot \vec{D}A).$$

Inserting this into the Bogomol'nyi bound and using the Gauss theorem yields

$$M \geq \left| \oint \mathrm{Tr}\,(\vec{B}\,A) d\vec{S} \right|. \tag{8.46}$$

To continue we assume that the gauge group is $SU(2)$ with generators

$$T_a = \frac{1}{\sqrt{2}} \tau_a, \quad \text{such that} \quad f_{abc} = \epsilon_{abc}.$$

If v denotes the length of A at infinity, then the last surface integral is just the magnetic flux of the $U(1)$-magnetic field in the direction of A times v so that

$$M \geq v\Phi.$$

Actually, using that $|\vec{D}A|$ tends to zero at spatial infinity it follows that

$$A_i^a = -\frac{1}{gv^2} \epsilon_{abc} A^b \partial_i A^c + A^a C_i$$

with C_i arbitrary. If we now compute the leading order behaviour of the non-Abelian gauge fields we find

$$F^{a\,ij} \sim \frac{1}{v} A^a f^{ij}$$

with

$$f_{ij} = -\frac{1}{gv^3} \epsilon_{abc} A^a \partial_i A^b \partial_j A^c + \partial_i C_j - \partial_j C_i$$

and the equation of motion imply $\partial_i f^{ij} = \partial_i {}^* f^{ij} \sim 0$. Outside the core of the monopole the non-Abelian field is purely in the direction of the Higgs field, that is, in the direction of the unbroken $U(1)$. The magnetic charge of this field configuration is

$$\Phi = \oint \vec{b} d\vec{S} = \frac{1}{2gv^3} \oint \epsilon_{ijk} \epsilon_{abc} A^a \partial^j A^b \partial^k A^c = \frac{4\pi N}{g} \tag{8.47}$$

with N the winding number of the Higgs field configuration, i.e. the winding of the map

$$S^2 \ni \hat{x} \longrightarrow A(\hat{x}) = \lim_{r \to \infty} A(r\hat{x}) \in S^2,$$

Thus we find the celebrated Dirac quantitation condition

$$g\Phi = 4\pi N \tag{8.48}$$

which relates the gauge coupling to the magnetic charge of the monopoles. The corresponding Bogomol'nyi bound

$$M \geq \frac{4\pi N}{g} v \tag{8.49}$$

saturates for $B_i = \pm D_i A$, that is for BPS-monopoles. The solutions to (8.49) are well-known. In the BPS limit monopoles neither attract nor repel each other. This must be the case, since the lower bound is attained independent of the distance between the monopoles. When one changes the collective coordinates the energy does not change and hence the monopole cannot increase or decrease the energy if they move relative to each other. This behaviour is typical for BPS states which saturate a Bogomol'nyi type bound.
Now we shall see that supersymmetry automatically generates the so-called Jackiw-Rebbi modes. Note that the Dirac equation in the gauge $A_0 = 0$ reads

$$i\dot{\psi} = -i\gamma^0\gamma^i D_i\psi + g\gamma^0[A, \psi].$$

For the static monopole background (A_i, A) we may separate off the energy dependence and arrive at the time-independent Dirac equation

$$E\psi = -i\gamma^0\gamma^i D_i\psi + g\gamma^0[A, \psi] = H\psi.$$

Now we shall prove that $\delta\psi$ is a zero-mode of H if (A_i, A) satisfy the Bogomol'nyi equation. For $B_i = D_i A$ the supersymmetry variation of the Dirac spinor simplifies to

$$\delta\psi = D_i A \begin{pmatrix} i\tau_i\,\kappa \\ \tau_i\,\kappa \end{pmatrix}, \quad \text{where} \quad \kappa = \theta + i\bar{\theta}.$$

It follows that

$$\gamma^j D_j \delta\psi = D_j D_i A \begin{pmatrix} \tau_j\tau_i\kappa \\ -i\tau_j\tau_i\kappa \end{pmatrix} = \tfrac{1}{2}[D_j, D_i] A \begin{pmatrix} i\epsilon_{jik}\,\kappa \\ \epsilon_{jik}\,\kappa \end{pmatrix} = i[B_k, A] \begin{pmatrix} i\tau_k\,\kappa \\ \tau_k\,\kappa \end{pmatrix} = -i[A, \delta\psi]$$

which is just the equation

$$H\delta\psi = 0$$

for a zero mode. Thus we have found an explicit expression for the Jackiw-Rebbi zero-modes about a magnetic monopoles. Supersymmetry opens up an elegant way to characterize and construct BPS-monopoles, it also gives the associated Jackiw-Rebbi modes. For the physical consequences of these modes I refer to [40].

8.3.5 $N = 2$-SYM in Euclidean spacetime

In Euclidean spacetime the gamma matrices must be hermitean so that we choose

$$(\gamma_0,\gamma_i)_M=(\gamma_0,-i\gamma_i)_E. \tag{8.50}$$

From

$$\mathcal{L}_M\sim\tfrac{1}{2}\mathrm{Tr}\,(F_{0i})^2-\tfrac{1}{4}\mathrm{Tr}\,(F_{ij})^2+\dots$$

we see, that the Lagrangian becomes negative definite, irrespective whether we multiply the time coordinate or the space coordinates with i. Because

$$\mathcal{L}_M=\tfrac{1}{2}\mathrm{Tr}\,(D_0A)^2-\tfrac{1}{2}\mathrm{Tr}\,(D_iA)^2$$

we conclude that we should continue the time coordinate for these terms to become negative definite. Hence we should choose

$$(\partial_0,\partial_i)_M=(i\partial_0,\partial_i)_E,\quad\text{such that}\quad(A_0,A_i,F_{0i},F_{ij})_M=(iA_0,A_i,iF_{0i},F_{ij})_E. \tag{8.51}$$

With this choices we have

$$\not{D}_M=i\not{D}_E.$$

In Euclidean spacetime the Dirac term must have the form

$$\pm i\psi^\dagger\not{D}\psi$$

in order to be hermitean and $SO(4)$ invariant. This tells us, that

$$(\psi,\bar\psi)_M=(\psi,i\psi^\dagger)_E\quad\text{such that}\quad(i\bar\psi\not{D}\psi)_M=-i(\psi^\dagger\not{D}\psi)_E. \tag{8.52}$$

For

$$A_M=A_E\quad\text{and}\quad B_M=B_E \tag{8.53}$$

the Yukawa interaction term

$$-g\mathrm{Tr}\,(\bar\psi[A,\psi])=-ig\mathrm{Tr}\,(\psi^\dagger[A,\psi])$$

becomes antihermitean in Euclidean spacetime,

$$\big\{-ig\mathrm{Tr}\,(\psi^\dagger[A,\psi])\big\}^\dagger=\big(gf_{abc}\psi^{a\dagger}A^b\psi^c\big)^\dagger=\big(gf_{abc}\psi^{c\dagger}A^b\psi^a\big)=ig\mathrm{Tr}\,(\psi^\dagger[A,\psi]).$$

More generally, one can show that there are no consistent transformation rules of the coordinates, γ-matrices and fields such the resulting Euclidean action is hermitean and bounded from below. This is not as bad as it seems, since in Euclidean spacetime the mass term in $\psi^\dagger(i\not{D}+im)\psi$ is not hermitean either. But although the eigenvalues of $i\not{D}+im$ are not real, they come in pairs $\pm\lambda+im$ and the partition function stays real, since $(\lambda+im)(-\lambda+im)$ is real. Since after spontaneous symmetry breaking the Yukawa terms may lead to mass terms for the fermions we even expect an antihermitean Yukawa interaction. Hence we may accept the transformation rules (8.50-8.53) in which case

$$\begin{aligned}\mathcal{L}_E &= \tfrac14 \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \tfrac12 \mathrm{Tr}\,(D_\mu A)^2 + \tfrac12 \mathrm{Tr}\,(D_\mu B)^2 - \tfrac12 g^2\, \mathrm{Tr}\left([A,B]^2\right)\\ &\quad + i\mathrm{Tr}\,\psi^\dagger \not{D}\psi + ig\mathrm{Tr}\left(\psi^\dagger[A,\psi]\right) + g\mathrm{Tr}\left(\psi^\dagger\gamma_5[B,\psi]\right),\end{aligned} \tag{8.54}$$

where the indices are raised and lowered with the Euclidean metric $\delta_{\mu\nu}$. This seems to be a completely consistent Euclidean model. In the chiral representation the hermitean matrices γ-matrices and the hermitean γ_5 take the form

$$\gamma_0 = \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & i\tau_i \\ -i\tau_i & 0 \end{pmatrix} \quad \text{and} \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix}.$$

Also note that

$$\gamma_{0i} = \begin{pmatrix} -i\tau_i & 0 \\ 0 & i\tau_i \end{pmatrix} \quad \text{and} \quad \gamma_{ij} = i\epsilon_{ijk}\begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix},$$

so that $\sigma_{\mu\nu}$ in

$$\gamma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde\sigma_{\mu\nu} \end{pmatrix}$$

is anti-selfdual and $\tilde\sigma_{\mu\nu}$ selfdual.
The gauge- and general covariant derivative of the fields are

$$D_\mu \Phi = \partial_\mu \Phi - i[A_\mu, \Phi].$$

If there is a supersymmetry, then the symmetry transformations should have the form

$$\delta_\alpha A_\mu = i(\alpha^\dagger\gamma_\mu\psi - \psi^\dagger\gamma_\mu\alpha), \quad \delta_\alpha A = -\alpha^\dagger\psi + \psi^\dagger\alpha, \quad \delta_\alpha B = i(\alpha^\dagger\gamma_5\psi - \psi^\dagger\gamma_5\alpha). \tag{8.55}$$

Now we have got a serious problem. The second formula shows that A should be antihermitean. But with an antihermitean A the terms $\mathrm{Tr}\,(DA)^2$ and the potential term quartic in the scalar fields become unbounded from below. The corresponding model in unstable and probably not consistent. Note however, that for an antihermitean A also $D_\mu A$ is antihermitean.
The remaining transformation of the Dirac spinor field reads

$$\begin{aligned}\delta_\alpha\psi &= -iF^{\mu\nu}\Sigma_{\mu\nu}\alpha - i\not{D}(A - i\gamma_5 B)\alpha + g\,[A,B]\gamma_5\alpha\\ \delta_\alpha\psi^\dagger &= i\alpha^\dagger F^{\mu\nu}\Sigma_{\mu\nu} - i\alpha^\dagger\not{D}(A + i\gamma_5 B) + g\alpha^\dagger\,[A,B]\gamma_5.\end{aligned} \tag{8.56}$$

The invariance of S is proven as follows. The terms in $\delta\mathcal{L}$ containing the scalar fields is easy to calculate and leads to $\partial_\mu \mathrm{Tr}\,(K^\mu)$ with

$$K^\mu = -ig[A,B](\alpha^\dagger\gamma^\mu\gamma_5\psi) + \alpha^\dagger D^\mu(A - i\gamma_5 B)\psi + \psi^\dagger D^\mu(A - i\gamma_5 B)\alpha - \alpha^\dagger\gamma^\mu\not{D}(A - i\gamma_5 B)\psi.$$

As always, the term containing only the fermions is the hard one.
To continue our calculation we note, that (3.66) holds with our conventions in Euclidean spacetime, i.e.

$$4\psi\chi^\dagger = -(\chi^\dagger\psi) - \gamma_\mu(\chi^\dagger\gamma^\mu\psi) + \tfrac12\gamma_{\mu\nu}(\chi^\dagger\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\chi^\dagger\gamma_5\gamma^\mu\psi) - \gamma_5(\chi^\dagger\gamma_5\psi). \tag{8.57}$$

But the formulae (8.5) and (8.6) now read

$$2\Sigma_{\mu\nu}\gamma_\rho = i\delta_{\mu\rho}\gamma_\nu - i\delta_{\nu\rho}\gamma_\mu + i\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 \ \ , \ \ 2\gamma_\rho\Sigma_{\mu\nu} = -i\delta_{\mu\rho}\gamma_\nu + i\delta_{\nu\rho}\gamma_\mu + i\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 \tag{8.58}$$

where $\epsilon_{0123} = 1$. Using the last set of relations together with the Bianchi identity one arrives at the intermediate result

$$\begin{aligned}
&-i\partial_\mu \mathrm{Tr}\,(F^{\mu\nu}\psi^\dagger\gamma_\nu\alpha) - i\partial_\mu \mathrm{Tr}\,({}^*F^{\mu\nu}\alpha^\dagger\gamma_\nu\gamma_5\psi) \\
&+gf_{abc}\{(\psi_a^\dagger\gamma^\mu\psi_b)(\alpha^\dagger\gamma_\mu\psi^c) - (\psi_a^\dagger\psi_b)(\alpha^\dagger\psi_c) + (\psi_a^\dagger\gamma_5\psi_b)(\alpha^\dagger\gamma_5\psi_c\} \\
&-gf_{abc}\{(\psi_a^\dagger\gamma^\mu\psi_b)(\psi_c^\dagger\gamma_\mu\alpha) - (\psi_a^\dagger\psi_b)(\psi_c^\dagger\alpha) + (\psi_a^\dagger\gamma_5\psi_b)(\psi_c^\dagger\gamma_5\alpha)\}
\end{aligned} \tag{8.59}$$

Now we apply the Fierz-identity (8.57) and find

$$\begin{aligned}
\gamma^\mu\chi\psi^\dagger\gamma_\mu &= -(\psi^\dagger\chi) + \tfrac{1}{2}\gamma_\mu(\psi^\dagger\gamma^\mu\chi) + \tfrac{1}{2}\gamma_5\gamma_\mu(\psi^\dagger\gamma_5\gamma^\mu\chi) + \gamma_5(\psi^\dagger\gamma_5\chi) \\
\chi\psi^\dagger - \gamma_5\chi\psi^\dagger\gamma_5 &= -\tfrac{1}{2}\gamma_\mu(\psi^\dagger\gamma^\mu\chi) + \tfrac{1}{2}\gamma_5\gamma_\mu(\psi^\dagger\gamma_5\gamma^\mu\chi)
\end{aligned}$$

which yield

$$\gamma^\mu\chi\psi^\dagger\gamma_\mu - \chi\psi^\dagger + \gamma_5\chi\psi^\dagger\gamma_5 = \gamma_\mu(\psi^\dagger\gamma^\mu\chi) - (\psi^\dagger\chi) + \gamma_5(\psi^\dagger\gamma_5\chi). \tag{8.60}$$

With this result the terms in (8.59) containing the ψ^3-terms can be rewritten as

$$\begin{aligned}
&+gf_{abc}\{(\psi_a^\dagger\gamma_\mu\psi_c)(\alpha^\dagger\gamma^\mu\psi_b) - (\psi_a^\dagger\psi_c)(\alpha^\dagger\psi_b) + (\psi_a^\dagger\gamma_5\psi_c)(\alpha^\dagger\gamma_5\psi_b)\} \\
&-gf_{abc}\{(\psi_c^\dagger\gamma^\mu\psi_b)(\psi_a^\dagger\gamma_\mu\alpha) - (\psi_c^\dagger\psi_b)(\psi_a^\dagger\alpha) + (\psi_c^\dagger\gamma_5\psi_b)(\psi_a^\dagger\gamma_5\alpha)\}
\end{aligned}$$

It follows that the terms in curly brackets are symmetric in (b,c) and (a,c), respectively. Because of the antisymmetry of the structure constants the contribution of the 3-fermion terms to the variation of the Lagrangian vanishes. Thus we remain with the following result for the variation of $\mathcal{L}$ under supersymmetry transformations:

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\mu \mathrm{Tr}\, V^\mu \quad \text{where} \\
V^\mu &= -ig[A,B](\alpha^\dagger\gamma^\mu\gamma_5\psi) + \alpha^\dagger D^\mu(A - i\gamma_5 B)\psi + \psi^\dagger D^\mu(A - i\gamma_5 B)\alpha \\
&\quad -\alpha^\dagger\gamma^\mu \not{D}(A - i\gamma_5 B)\psi - iF^{\mu\nu}\psi^\dagger\gamma_\nu\alpha - i\,{}^*F^{\mu\nu}\alpha^\dagger\gamma_\nu\gamma_5\psi
\end{aligned} \tag{8.61}$$

Now it is easy to construct the Noether current. One finds

$$\begin{aligned}
J^\mu = &\quad \alpha^\dagger\{i\gamma_\nu({}^*F^{\mu\nu}\gamma_5 + F^{\mu\nu}) + ig[A,B]\gamma^\mu\gamma_5 - \not{D}(A + iB\gamma_5)\gamma^\mu\}\psi \\
&+ \ \psi^\dagger\{i\gamma_\nu({}^*F^{\mu\nu}\gamma_5 - F^{\mu\nu}) + ig[A,B]\gamma^\mu\gamma_5 + \gamma^\mu\not{D}(A - iB\gamma_5)\}\alpha
\end{aligned} \tag{8.62}$$

8.3.6 Selfdual instantons and small fluctuations

As previously we are interested in background configurations which preserve part of the supersymmetry. We assume that all fields with the exception of the gauge potential vanish. This condition is preserved by a supersymmetry transformations if

$$\delta\psi = -iF^{\mu\nu}\Sigma_{\mu\nu}\alpha = 0, \quad \alpha = \begin{pmatrix}\theta\\ \bar\theta\end{pmatrix} \quad \Longleftrightarrow \quad F^{\mu\nu}\sigma_{\mu\nu}\theta = F^{\mu\nu}\tilde\sigma_{\mu\nu}\bar\theta = 0 \tag{8.63}$$

There are two ways to fulfill these conditions:

$$\theta = 0,\ F^{\mu\nu}\tilde\sigma_{\mu\nu} = 0 \quad \text{or} \quad \bar\theta = 0,\ F^{\mu\nu}\sigma_{\mu\nu} = 0. \tag{8.64}$$

In the first case only the righthanded supersymmetry survives and $F^{\mu\nu}$ is anti-selfdual, in the second case the lefthanded supersymmetry survives and $F^{\mu\nu}$ is selfdual. As expected these selfdual and anti-selfdual gauge fields are solutions of the classical field equations which are the Yang-Mills, Klein-Gordon and Dirac equations

$$\begin{aligned} D_\nu F^{\mu\nu} &= ig[A, D^\mu A] + ig[B, D^\mu B] + g[\psi^\dagger, \gamma^\mu\psi] \\ D^2 A &= ig[\psi^\dagger, \psi] - g^2[[A,B],B]] \\ D^2 B &= g[\bar\psi, \gamma_5\psi] - g^2[A,[A,B]] \\ i\not{D}\psi &= -ig[A - i\gamma_5 B, \psi]. \end{aligned} \tag{8.65}$$

Similarly to the previously considered monopoles we can interprete the (anti)selfdual instantons as BPS-states which preserve only half of the supersymmetry.
Now we pick a selfdual instanton configuration $\bar A_\mu$ and consider its supersymmetry variation

$$\delta_\alpha A_\mu = 0, \quad \delta_\alpha A = 0, \quad \delta_\alpha B = 0, \quad \delta_\alpha\psi = -iF^{\mu\nu}\Sigma_{\mu\nu}\alpha.$$

Since $\bar A_\mu$ is a classical solution we have

$$S[\bar A_\mu] = S[\bar A_\mu, \delta_\alpha\psi] \sim S[\bar A_\mu] + \big(\delta_\alpha\psi^\dagger, i\not{D}\,\delta_\alpha\psi\big)$$

which should imply that $\delta_\alpha\psi$ is a zero-mode of the Dirac operator. This is easily proven:

$$\begin{aligned} \gamma^\mu\delta_\alpha\psi &= \gamma_\nu({}^*F^{\mu\nu}\gamma_5 - F_{\mu\nu})\alpha \Longrightarrow \\ i\not{D}\delta_\alpha\psi &= i\gamma_\nu D_\mu({}^*F^{\mu\nu}\gamma_5 - F^{\mu\nu})\alpha = -i\gamma_\nu D_\mu F^{\mu\nu} = 0, \end{aligned}$$

where we have used the Bianchi identity and the Yang-Mills equation. This proves that $\delta_\alpha\psi$ is a zeromode of the Dirac operator. Since

$$\delta_\alpha\psi = -\tfrac{1}{2}\begin{pmatrix}0 & 0\\ 0 & \tilde\sigma F\end{pmatrix}\alpha,$$

this zero mode is right-handed. Actually, from the index theorem we infer that in an $SU(2)$-instanton background with charge q the operator $i\not{D}$ possesses

$$\tfrac{2}{3}(2j+1)(j+1)j\cdot q \tag{8.66}$$

righthanded zero-modes in the spin-j representation. In particular for adjoint fermions there are $4q$ zero modes.
Now we wish to relate the various fluctuation fields about selfdual instantons [43]. We start with perturbing an arbitrary background field by small fluctuations

$$A_\mu = \bar{A}_\mu + a_\mu, \quad A = \bar{A} + a, \quad B = \bar{B} + b. \tag{8.67}$$

We perform the corresponding Taylor expansion of the (unstable) Euclidean action

$$S = S_0 + S_1 + S_2 + \dots,$$

where S_i is of order i in the fluctuation fields. In what follows we only work with the background fields and the perturbation and hence may safely omit the bar for the background fields. The first term S_1 leads to the field equations (8.65). The expression for S_2 for general background fields is rather complicated. But for backgrounds with vanishing A, B and ψ it becomes rather simple

$$S_2 = \int a_\nu \{ - D^2 \delta_{\nu\mu} + D_\nu D_\mu - 2ig[F^{\mu\nu}, .] \} a_\mu - aD^2 a - bD^2 b + i\psi^\dagger \not{D}\psi. \tag{8.68}$$

We read off the following fluctuation operators

$$M_{\nu\mu} = -D^2 \delta_{\nu\mu} + D_\nu D_\mu + 2ig\,\mathrm{ad}(F_{\nu\mu}), \quad M_A = -D^2, \quad M_B = -D^2, \quad M_\psi = i\not{D}, \tag{8.69}$$

where all covariant derivatives are to be taken in the adjoint representation.
In a one-loop approximation to the partition function one needs the eigenvalues of these fluctuation operators,

$$M_{\nu\mu} a_\mu = \lambda^2 a_\mu, \quad M_A a = \lambda^2 a, \quad M_B b = \lambda^2 b \quad \text{and} \quad M_\psi \psi = \lambda\psi. \tag{8.70}$$

Let us assume that ψ is an eigenmode of the Dirac operator,

$$i\not{D}\psi = \lambda\psi \Longrightarrow iD_\mu \psi^\dagger \gamma^\mu = -\lambda\psi^\dagger. \tag{8.71}$$

The squared Dirac operator,

$$\not{D}^2 = \gamma_\mu \gamma_\nu D_\mu D_\nu = (\delta_{\mu\nu} + \gamma_{\mu\nu}) D_\mu D_\nu = D^2 + \tfrac{1}{2}\gamma_{\mu\nu}[D_\mu, D_\nu] = D^2 + \Sigma_{\mu\nu}\mathrm{ad}(F_{\mu\nu}), \tag{8.72}$$

simplifies for selfdual gauge fields as follows:

$$(i\not{D})^2 = -D^2 + \tfrac{i}{2}\begin{pmatrix} \sigma\mathrm{ad}(F) & 0 \\ 0 & \tilde{\sigma}\mathrm{ad}(F) \end{pmatrix} \longrightarrow -D^2 + \tfrac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \tilde{\sigma}\mathrm{ad}(F) \end{pmatrix}. \tag{8.73}$$

We see that the 'left-handed' fermions

$$\psi_L = P_L \psi, \qquad P_L = \tfrac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8.74}$$

fulfill the equation

$$-D^2 \psi_L = \lambda^2 \psi_L$$

which is identical to the eigenvalue equation for the scalar and pseudoscalar field. Therefore any eigenmode ψ_L of $-\not{D}^2$ with eigenvalue λ^2 transforms under susy transformations into eigenmodes

$$a = -\alpha_L^\dagger \psi_L + \psi_L^\dagger \alpha_L \quad \text{and} \quad b = i(\alpha_L^\dagger \psi_L - \psi_L^\dagger \alpha_L) \tag{8.75}$$

of $-D^2$ with the same eigenvalue.
The same procedure applies to the eigenvalue equation of the vector fluctuations. We start with a righthanded eigenmode of $-\not{D}^2$ which fulfills

$$\big(- D^2 + \tfrac{i}{2}\tilde{\sigma}\text{ad}\,(F)\big)\psi_R = \lambda^2\psi_R.$$

Now we multiply this equation with σ_ν from the left and use the first of the following two identities

$$\sigma_\rho\tilde{\sigma}_{\mu\nu} = \delta_{\rho\mu}\sigma_\nu - \delta_{\rho\nu}\sigma_\mu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma \quad , \quad \tilde{\sigma}_\rho\sigma_{\mu\nu} = \delta_{\rho\mu}\tilde{\sigma}_\nu - \delta_{\rho\nu}\tilde{\sigma}_\mu - \epsilon_{\mu\nu\rho\sigma}\tilde{\sigma}_\sigma,$$

which follow from (8.58). This leads to

$$\big\{ - D^2\delta_{\nu\mu} + i\text{ad}\,(F_{\nu\mu} + {}^*F_{\nu\mu})\big\}(\sigma_\mu\psi_R) = \lambda^2(\sigma_\nu\psi_R). \tag{8.76}$$

For a selfdual gauge field this equation becomes

$$\begin{aligned} \big\{ - D^2\delta_{\nu\mu} + 2i\,\text{ad}(F_{\nu\mu})\big\}\,\sigma_\mu\psi_R &= \lambda^2\gamma_\nu\psi_R \\ \big\{ - D^2\delta_{\nu\mu} + 2i\,\text{ad}(F_{\nu\mu})\big\}\,\psi_R^\dagger\tilde{\sigma}_\mu &= \lambda^2\psi_R^\dagger\tilde{\sigma}_\nu \end{aligned} \tag{8.77}$$

and it implies

$$-D^2(\sigma D)\psi_R = \lambda^2(\sigma D)\psi_R. \tag{8.78}$$

Now we take the vector field which is gotten as supersymmetry transformation of a righthanded eigenmode,

$$a_\mu = i(\alpha^\dagger\sigma_\mu\psi_R - \psi_R^\dagger\tilde{\sigma}_\mu\alpha). \tag{8.79}$$

For $\lambda \neq 0$ this vector field has a non-vanishing divergence,

$$D_\mu a_\mu = \lambda(\alpha_L^\dagger\psi_L + \psi_L^\dagger\alpha_L). \tag{8.80}$$

However, out of a_μ we can construct the following source free vector field

$$b_\mu = a_\mu + \frac{1}{\lambda^2}D_\mu D_\nu a_\nu$$

which satisfies the so-called *background gauge condition*

$$D_\mu b_\mu = 0. \tag{8.81}$$

The difference between b_μ and a_μ is an infinitesimal gauge transformation. The proof is simple and makes use of (8.78):

$$\begin{aligned} D_\mu b_\mu &= D_\mu a_\mu + \frac{1}{\lambda^2}D^2\big\{\alpha_L^\dagger i(\sigma D)\psi_R - iD_\nu\psi_R^\dagger\,\tilde{\sigma}_\nu\alpha_L\big\} \\ &= D_\mu a_\mu - \big\{\alpha_L^\dagger i(\sigma D)\psi_R - iD_\nu\psi_R^\dagger\tilde{\sigma}_\nu\alpha_L\big\} = D_\mu a_\mu - D_\nu a_\nu = 0. \end{aligned}$$

After this preparations we come back to the fluctuation operator $M_{\nu\mu}$ for the gauge bosons in (8.70). It is just the sum of the the operator on the left in (8.77) plus $D_\nu D_\mu$ which annihilates b_μ. Hence b_μ is eigenmode of the fluctuation operator with eigenvalues λ^2,

$$M_{\nu\mu}b_\mu = \lambda^2 b_\nu, \qquad D_\mu b_\mu = 0.$$

Hence we have already diagonalized the fluctuation operator on fluctuations fulfilling the background gauge condition.
We summarize the main results of this section: the bosonic fluctuation operators and $-\not{D}^2$ have all the same spectrum and the eigenmodes are related by supersymmetry. Now we turn to calculating the partition function in the one-loop approximation.

8.3.7 One-loop partition function

After our previous investigations it seems proper to use background gauge fixing. Hence we add a gauge fixing term $\frac{1}{2}(D_\mu a_\mu)^2$ to the Lagrangian, and the first eigenvalue equation in (8.70) is then modifies to

$$(M_{\nu\mu} - D_\nu D_\mu)a_\mu = \lambda^2 a_\nu.$$

We use the zeta-function regularization to 'calculate' the determinant of an selfadjoint and non-negative operator A:

$$\log\det A = -\frac{d}{ds}\zeta_A(s)|_{s=0}, \quad \zeta_A(s) = \sum_{\lambda_n>0}\lambda_n^{-s}.$$

The so defined determinant has a simple scaling property,

$$\log\det\frac{1}{\lambda}A = \log\det A - \log\lambda\cdot\zeta_A(0).$$

Using the fact that the fluctuation operators have the same non-zero eigenvalues

$$\log\det{}'(-\not{D}^2) = 4\log\det{}'(-D^2) = \log\det{}'(M_{\nu\mu}),$$

where in the last equations we assumed the background gauge condition $D_\mu b_\mu = 0$. Now we insert our result in the general formula for the one-loop partition function in the q-instanton sector[44]

$$\begin{aligned} Z_q(V,g) &= e^{W_q(V,g)} = \frac{1}{\mathcal{N}}\int\mathcal{D}(a_\mu,\psi^\dagger,\psi,A,B)e^{-S+\int(\psi^\dagger\eta+\eta^\dagger\psi)} \\ &= e^{-S(\bar{A}_\mu)}\left[g\sqrt{\frac{2\pi}{V}}\right]^{d_H}\frac{1}{V_H}\prod_1^p d\gamma_r(\det J)^{1/2}\frac{\det{}'^{1/2}(-\not{D}^2)\det{}'M_{gh}}{\left\{\det{}'(-D^2)\det{}'(-D^2)\det{}'(-M_{\nu\mu}\right\}^{1/2}} \\ &\qquad\times\prod_n(\eta^\dagger,\psi_n)(\psi_n^\dagger,\eta)\exp\big(-\int\eta^\dagger G'\eta\big). \end{aligned}$$

For the background gauge fixing $M_{gh} = -D^2$. Here d_H and V_H are the dimensions and volume of the stability group H which commutes with the $su(2)$-algebra defined by the instanton [45]. For $G = SU(2)$ we have $d_H = 0$. The fluctuation operator $M_{\nu\mu}$ may possess p additional zero-modes arising from the variation of the collective parameters $\{\gamma_r\}$. J denotes the Jacobian when one converts p expansion parameters (in the expansion of the gauge field) into the collective parameters. Now we consider the rescaled theory with

$$\bar{A}(\lambda x) = \bar{A}(x)$$

to the original one. Again we refer to [44] and take the general result

$$\begin{aligned} W_q[\lambda V, g] &= W_q(V,g) + \frac{\log\lambda}{16\pi^2}\int \mathrm{Tr}\left(a_2^A(x) + a_2^B(x) - 2a_2^{gh}(x) - a_2^{\psi}(x) + a_2^{A_\mu}(x)\right) \\ &= W_q(V,g) + \frac{\log\lambda}{16\pi^2}\int \mathrm{Tr}\left(- a_2^{\psi}(x) + a_2^{A_\mu}(x)\right), \end{aligned}$$

where the second Seeley-deWitt coefficients of the various fluctuation operators appear. These coefficients are

$$a_2^{A_\mu}(x) = -\tfrac{5}{3}g^2\mathrm{Tr}_A F^{\mu\nu}F_{\mu\nu}, \quad a_2^A(x) = \tfrac{1}{12}g^2\mathrm{Tr}_A F^{\mu\nu}F_{\mu\nu} \quad \text{and} \quad a_2^{\psi}(x) = -\tfrac{2}{3}\mathrm{Tr}_A g^2 F^{\mu\nu}F_{\mu\nu} \tag{8.82}$$

where Tr_A is the trace in the adjoint representation, we obtain

$$W_q[\lambda V, g] = W_q(V,\eta,g) - \frac{\log\lambda g^2}{16\pi^2}\int \mathrm{Tr}_A\left(F^{\mu\nu}F_{\mu\nu}\right) = W_q(V,g) - T_A\frac{\log\lambda g^2}{16\pi^2}\int \left(F^a_{\mu\nu}F^{\mu\nu}_a\right),$$

where T_A is the second-order Casimir of the adjoint representation. This then implies the following scaling law for the effective action,

$$\Gamma_q(\lambda V, g) = \frac{1}{4g^2}\left(1 - T_A\frac{\log\lambda g^2}{4\pi^2}\right)\int F^{\mu\nu}_a F^a_{\mu\nu}.$$

The effective action is invariant, if we rescale the field and coupling constant according to

$$A_\mu \to \sqrt{Z_3}A_\mu, \quad g^2(\lambda) = \frac{g^2}{Z_3} \quad \text{with} \quad Z_3 = 1 - \frac{\log\lambda}{4\pi^2}T_A g^2.$$

For example, for $G = SU(2)$ we have $T_A = 2$ and the β-function and anomalous dimension of the gauge fields are found to be

$$\beta(g) = \mu\frac{\partial}{\partial\mu}g(\mu) = -\frac{1}{4\pi^2}g^3(\mu) \quad \text{and} \quad \gamma_A(g) = \mu\frac{\partial}{\partial\mu}\log Z_3 = \frac{1}{2\pi^2}g^2(\mu), \qquad \mu = \frac{1}{\lambda}.$$

We see, that the theory is asymptotically free, similarly as QCD with a not too big number of fermions. This ends the first part of our discussion of the Seiberg-Witten model. To make further progress we need the superfield formulation which we shall introduce soon. The non-perturbative structure of the $N = 2$-SYM can much easier seen in this formalism. But before we turn to superfields we shall discuss a new technique to construct supersymmetric theories, namely the dimensional reduction.

8.4 Appendix 1: useful formulae in the chiral representation

Notation, conventions and identities

The previously defined ε fulfill the relations:

$$\begin{aligned} \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} &= \delta_\alpha^\gamma \quad , \\ \varepsilon_{\dot\alpha\dot\eta}\varepsilon^{\dot\beta\dot\gamma} &= \delta_{\dot\alpha}^{\dot\gamma} \quad , \\ \varepsilon_{\alpha\beta}\varepsilon^{\delta\gamma} &= \delta_\alpha^\gamma\delta_\beta^\delta - \delta_\alpha^\delta\delta_\beta^\gamma \quad , \\ \varepsilon_{\dot\alpha\dot\eta}\varepsilon^{\dot\delta\dot\gamma} &= \delta_{\dot\alpha}^{\dot\gamma}\delta_{\dot\beta}^{\dot\delta} - \delta_{\dot\alpha}^{\dot\delta}\delta_{\dot\beta}^{\dot\gamma} \quad . \end{aligned} \tag{8.83}$$

We recall

$$\sigma^\mu = (\tau_0, \vec\tau) = \tilde\sigma_\mu \quad \text{and} \quad \tilde\sigma^\mu = (\tau_0, -\vec\tau) = \sigma_\mu \quad .$$

σ^μ has undotted-dotted indices: $\sigma^\mu_{\alpha\dot\beta}$

$\tilde\sigma^\mu$ has dotted-undotted indices: $\tilde\sigma^{\mu\dot\alpha\beta}$

We also have the completeness relations:

$$\mathrm{Tr}\,\sigma^\mu\tilde\sigma^\nu = 2\eta^{\mu\nu} \quad , \quad \sigma^\mu_{\alpha\dot\beta}\tilde\sigma_\mu^{\dot\gamma\delta} = 2\delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma}. \tag{8.84}$$

σ^μ and $\tilde\sigma^\mu$ are related by the Levi-Civita symbols:

$$\tilde\sigma^{\mu\dot\alpha\beta} = \varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\beta\delta}\sigma^\mu_{\delta\dot\gamma} \quad , \quad \sigma^\mu_{\alpha\dot\beta} = \varepsilon_{\dot\delta\dot\beta}\varepsilon_{\gamma\alpha}\tilde\sigma^{\mu\dot\delta\gamma}. \tag{8.85}$$

It is occasionally convenient to do a “fake” conversion of an undotted to a dotted index or vice versa using the fact that σ^0 and $\tilde\sigma^0$ are just the identity matrix:

$$\psi^\alpha = (\bar\psi_{\dot\beta})^*\tilde\sigma^{0\dot\beta\alpha} \quad ; \qquad \bar\psi^{\dot\alpha} = (\psi_\beta)^*\sigma^{0\beta\dot\alpha}.$$

Because the Pauli matrices anticommute, i.e.

$$\{\tau^i, \tau^j\} = 2\delta^{ij}\tau_0 \qquad i, j = 1, 2, 3,$$

we have the relations:

$$(\sigma^\mu\tilde\sigma^\nu + \sigma^\nu\tilde\sigma^\mu)_\alpha^\beta = 2\eta^{\mu\nu}\delta_\alpha^\beta \quad , \quad (\tilde\sigma^\mu\sigma^\nu + \tilde\sigma^\nu\sigma^\nu)_{\dot\alpha}^{\dot\beta} = 2\eta^{\mu\nu}\delta_{\dot\alpha}^{\dot\beta}\,. \tag{8.86}$$

Spinor Summation Convention:

$$\begin{aligned} \psi\chi &= \psi^\alpha\chi_\alpha &&= -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \\ \bar\psi\bar\chi &= \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} &&= -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi\,. \end{aligned} \tag{8.87}$$

Note that these quantities are Lorentz scalars. We also have:

$$(\chi\psi)^\dagger = (\chi^\alpha\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\chi\bar\psi \quad , \quad (\chi\sigma^\mu\bar\psi)^\dagger = \psi\sigma^\mu\bar\chi = \text{Lorentz vector}.$$

Other useful relations are:

$$\psi^\alpha\psi^\beta = -\tfrac{1}{2}\varepsilon^{\alpha\beta}\psi\psi, \quad \psi_\alpha\psi_\beta = \tfrac{1}{2}\varepsilon_{\alpha\beta}\psi\psi, \quad \bar\psi^{\dot\alpha}\bar\psi^{\dot\beta} = \tfrac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\psi\bar\psi, \quad \bar\psi_{\dot\alpha}\bar\psi_{\dot\beta} = -\tfrac{1}{2}\varepsilon_{\dot\alpha\dot\eta}\bar\psi\bar\psi. \tag{8.88}$$

The SL(2,C) generators are defined as

$$\sigma_\alpha^{\mu\nu\beta} = \frac{1}{4i}\left[\sigma^\mu_{\alpha\dot\gamma}\tilde\sigma^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\tilde\sigma^{\mu\dot\gamma\beta}\right] \quad , \quad \tilde\sigma^{\mu\nu\dot\alpha}_{\dot\beta} = \frac{1}{4i}\left[\tilde\sigma^{\mu\dot\alpha\gamma}\sigma^\nu_{\gamma\dot\beta} - \tilde\sigma^{\nu\dot\alpha\gamma}\sigma^\mu_{\gamma\dot\beta}\right].$$

We have

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} &= 2i\sigma^{\mu\nu} && \text{self dual } (1,0) \\ \epsilon^{\mu\nu\rho\sigma}\tilde\sigma_{\rho\sigma} &= -2i\tilde\sigma^{\mu\nu} && \text{anti self dual } (0,1), \end{aligned} \tag{8.89}$$

and thus the trace relation:

$$\mathrm{Tr}\left[\sigma^{\mu\nu}\sigma^{\rho\sigma}\right] = \tfrac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\sigma}\right) + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}.$$

Fierz identities

$$\begin{aligned}
(\theta\phi)(\theta\psi) &= -\tfrac{1}{2}(\theta\theta)(\phi\psi) && (8.90)\\
(\bar\theta\bar\phi)(\bar\theta\bar\psi) &= -\tfrac{1}{2}(\bar\phi\bar\psi)(\bar\theta\bar\theta) && (8.91)\\
\phi\sigma_\mu\bar\chi &= -\bar\chi\tilde\sigma_\mu\phi && (8.92)\\
(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &= \tfrac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta) && (8.93)\\
(\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) &= \tfrac{1}{2}\eta^{\mu\nu}\theta_\alpha(\bar\theta\bar\theta) - i(\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta) && (8.94)\\
(\theta\phi)(\bar\theta\bar\psi) &= \tfrac{1}{2}(\theta\sigma^\mu\bar\theta)(\phi\sigma_\mu\bar\psi) && (8.95)\\
(\bar\theta\bar\psi)(\theta\phi) &= \tfrac{1}{2}(\theta\sigma^\mu\bar\theta)(\phi\sigma_\mu\bar\psi) = (\theta\phi)(\bar\theta\bar\psi) && (8.96)\\
(\theta\phi)(\bar\psi\tilde\sigma^\mu\theta) &= -\tfrac{1}{2}\theta^2(\bar\psi\tilde\sigma^\mu\phi) && (8.97)\\
(\bar\theta\bar\phi)(\psi\sigma^\mu\bar\theta) &= -\tfrac{1}{2}\bar\theta^2(\psi\sigma^\mu\bar\phi) && (8.98)\\
(\theta\phi)(\psi\sigma^\mu\bar\theta) &= -\tfrac{1}{2}(\theta\sigma^\nu\bar\theta)(\phi\sigma_\nu\tilde\sigma^\mu\psi) && (8.99)\\
(\bar\theta\bar\phi)(\bar\psi\tilde\sigma^\mu\theta) &= \tfrac{1}{2}(\theta\sigma^\nu\bar\theta)(\bar\phi\tilde\sigma_\nu\sigma^\mu\bar\psi) && (8.100)\\
(\bar\phi\tilde\sigma^\mu\theta)(\theta\sigma^\nu\bar\theta) &= -\tfrac{1}{2}\theta^2\bar\theta(\tilde\sigma^\nu\sigma^\mu)\bar\phi && (8.101)\\
(\phi\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &= \tfrac{1}{2}\bar\theta^2\theta(\sigma^\nu\tilde\sigma^\mu)\phi && (8.102)
\end{aligned}$$

For example, the identity (8.92) is proven as follows,

$$\phi\sigma_\mu\bar\chi = \phi^\alpha\sigma_{\mu\alpha\dot\alpha}\bar\chi^{\dot\alpha} = \varepsilon^{\alpha\beta}\phi_\beta\sigma_{\mu\alpha\dot\alpha}\,\varepsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta} = -\bar\chi_{\dot\beta}(\varepsilon^{\dot\beta\dot\alpha}\varepsilon^{\beta\alpha}\sigma_{\mu\alpha\dot\alpha})\phi_\beta = -\bar\chi_{\dot\beta}\tilde\sigma_\mu^{\dot\beta\beta}\phi_\beta = -\bar\chi\tilde\sigma_\mu\phi.$$

8.5 Appendix 2: Closing of the $N = 2$ algebra on the fermions.

We show explicitly that the superalgebra (8.55,8.56) in Euclidean spacetime close on the fermions. This calculation is rather tricky and we skipped it in the main body of the chapter. When calculating the commutator of two susy transformations on ψ one obtains three types of terms. Those which are linear in A, those which are linear in B and those containing neither the scalar nor the pseudo-scalar field,

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\psi = ig[A, X_A] + g[B, X_B] + R.$$

One finds

$$\begin{aligned} X_A &= \big\{(\alpha_1^\dagger\gamma_5\psi) - (\psi^\dagger\gamma_5\alpha_1)\big\}\gamma_5\alpha_2 + \big\{(\alpha_1^\dagger\gamma_\mu\psi) - (\psi^\dagger\gamma_\mu\alpha_1)\big\}\gamma^\mu\alpha_2 - \Big\{\alpha_1 \leftrightarrow \alpha_2\Big\} \\ X_B &= \big\{(\alpha_1^\dagger\psi) - (\psi^\dagger\alpha_1)\big\}\gamma_5\alpha_2 + \big\{(\alpha_1^\dagger\gamma_\mu\psi) - (\psi^\dagger\gamma_\mu\alpha_1)\big\}\gamma^\mu\gamma_5\alpha_2 - \Big\{\alpha_1 \leftrightarrow \alpha_2\Big\} \end{aligned}$$

and

$$\begin{aligned} R = \quad & 2\big\{(\alpha_1^\dagger\gamma_\nu D_\mu\psi) - (D_\mu\psi^\dagger\gamma_\nu\alpha_1)\big\}\Sigma_{\mu\nu}\alpha_2 \\ & +i\big\{(\alpha_1^\dagger D_\mu\psi) - (D_\mu\psi^\dagger\alpha_1)\big\}\gamma^\mu\alpha_2 \\ & -i\big\{(\alpha_1^\dagger\gamma_5 D_\mu\psi) - (D_\mu\psi^\dagger\gamma_5\alpha_1)\big\}\gamma^\mu\gamma_5\alpha_2 \\ & -\Big\{\alpha_1 \leftrightarrow \alpha_2\Big\}. \end{aligned}$$

Using the Fierz identity the he terms in R containing ψ (and not $\psi^\dagger$) can be rewritten as

$$\begin{aligned} & \tfrac{3i}{4}(\alpha_1^\dagger\alpha_2)\not{D}\psi + i(\alpha_1^\dagger\gamma^\mu\alpha_2)D_\mu\psi - \tfrac{3i}{4}(\alpha_1^\dagger\gamma^\rho\alpha_2)\gamma_\rho\not{D}\psi \\ & -\tfrac{i}{4}(\alpha_1^\dagger\gamma^{\rho\sigma}\alpha_2)\gamma_{\rho\sigma}\not{D}\psi - 3i(\alpha_1^\dagger\gamma_5\gamma^\mu\alpha_2)\gamma_5 D_\mu\psi \\ & +\tfrac{7i}{4}(\alpha_1^\dagger\gamma_5\gamma^\rho\alpha_2)\gamma_5\gamma_\rho\not{D}\psi + \tfrac{3i}{4}(\alpha_1^\dagger\gamma_5\alpha_2)\gamma_5\not{D}\psi. \end{aligned}$$

To be done!