7 Representations of supersymmetric algebras

We want to study the possible representations of some of the superalgebras we have introduced in the last chapter. Let us start with the algebra in 4 dimensions in the Weyl basis. I recall our conventions: We take the chiral representation

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad \sigma_\mu = (\sigma_0, -\sigma_i), \quad \tilde{\sigma}_\mu = (\sigma_0, \sigma_i), \tag{7.1}$$

for which the infinitesimal spinor-rotations have the form

$$\gamma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde{\sigma}_{\mu\nu} \end{pmatrix} \quad \text{with} \quad \sigma_{\mu\nu} = \tfrac{1}{2}(\sigma_\mu\tilde{\sigma}_\nu - \sigma_\nu\tilde{\sigma}_\mu), \quad \tilde{\sigma}_{\mu\nu} = \tfrac{1}{2}(\tilde{\sigma}_\mu\sigma_\nu - \tilde{\sigma}_\nu\sigma_\mu). \tag{7.2}$$

Note that the σ_{oi} are hermitean, whereas the σ_{ij} are antihermitian. A Dirac spinor consists of a lefthanded and a righthanded part,

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}, \quad \bar{Q} = (Q^\alpha, \bar{Q}_{\dot\alpha}) = Q^\dagger\gamma^0 \Longrightarrow \bar{Q}_{\dot\alpha} = Q^\dagger_\alpha, \quad \bar{Q}^{\dot\alpha} = (Q^\alpha)^\dagger. \tag{7.3}$$

The raising and lowering of the indices are done with ε:

$$Q^\alpha = \varepsilon^{\alpha\beta} Q_\beta, \quad Q_\alpha = \varepsilon_{\alpha\beta}Q^\beta \quad \text{and} \quad \bar{Q}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{Q}_{\dot\beta}, \quad \bar{Q}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{Q}^{\dot\beta}$$

where we introduced

$$(\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon^{\alpha\beta}) = (\varepsilon^{\dot\alpha\dot\beta}) \quad = \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.4}$$

The index structure of the relevant generators are

$$(\sigma_\mu)_{\alpha\dot\alpha}, \quad (\tilde{\sigma}_\mu)^{\dot\alpha\alpha}, \quad (\sigma_{\mu\nu})_\alpha{}^\beta, \quad (\tilde{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}. \tag{7.5}$$

We have discussed that the supercharges should have spin $\frac{1}{2}$ which fixes their commutation relations with the Lorentz boosts. They must commute with the translations such that the extended superalgebra contains the Poincaré algebra plus

$$
\begin{aligned}
&[J_{\mu\nu}, Q^i_\alpha] = \tfrac{1}{2i}(\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta \qquad && [J_{\mu\nu}, \bar{Q}^{\dot\alpha}_i] = \tfrac{1}{2i}(\tilde\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}_i\,, \\
&\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2\delta^{ij}(\sigma^\mu)_{\alpha\dot\beta}P_\mu \qquad && [Q^i_\alpha, P_\mu] = 0.
\end{aligned}
\tag{7.6}
$$

The missing supercommutators will be discussed soon.

7.1 Hamiltonian and Central charges

The sign in the anticommutator of Q and $\bar{Q}$ is determined by the requirement that the energy should be a positive definite operator: we get for each value of the index i

$$
\sum_{\alpha=1}^{2}\{Q^i_\alpha, Q^{i\dagger}_\alpha\} = 2\mathrm{Tr}\,\sigma^\mu P_\mu = 4P_0 \quad \text{no sum over i.} \tag{7.7}
$$

Since the left hand side is manifestly positive, since each term is positive,

$$
\langle\psi|\{Q^i_\alpha, Q^{i\dagger}_\alpha\}\psi\rangle = ||Q^i_\alpha\psi||^2 + ||Q^{i\dagger}_\alpha\psi||^2 \geq 0,
$$

it follows that the

- *spectrum of $H = P_0$ in a theory with supersymmetry contains no negative eigenvalues.*

We denote the state (or family of states) with the lowest energy by $|0\rangle$ and call it vacuum. The vacuum will have zero energy

$$
H|0\rangle = 0 \tag{7.8}
$$

if and only if

$$
Q^i_\alpha|0\rangle = 0 \quad \text{and} \quad Q^{i\dagger}_\alpha|0\rangle = 0 \qquad \forall\, \alpha, i. \tag{7.9}
$$

Any state with positive energy cannot be invariant under supersymmetry transformations. It follows in particular that every one-particle state $|1\rangle$ must have super partner states $Q^i_\alpha|1\rangle$ or $Q^{i\dagger}_\alpha|1\rangle$. The spin of these partners will differ by $\frac{1}{2}$ from that of $|1\rangle$. Thus

- *each supermultiplett must contain at least one boson and one fermion whose spins differ by $\frac{1}{2}$.*

The translation invariance of Q implies that Q does not change energy and momentum

$$
P_\mu|p\rangle = p_\mu|p\rangle \Longrightarrow P_\mu Q^i_\alpha|p\rangle = p_\mu Q^i_\alpha|p\rangle, \qquad P_\mu Q^{i\dagger}_\alpha|p\rangle = p_\mu Q^{i\dagger}_\alpha|p\rangle,
$$

and therefore

- *all states in a multiplet of unbroken supersymmetry have the same mass.*

Supersymmetry is spontaneously broken if the ground state will not be invariant under all supersymmetry transformations,

$$
Q^i_\alpha|0\rangle \neq 0 \quad \text{or} \quad Q^{i\dagger}_\alpha|0\rangle \neq 0
$$

for same α and i. We conclude that

• *supersymmetry is spontaneously broken if and only if the energy of the lowest lying state is not exactly zero.*

We come back to the breaking of supersymmetric theories in the next chapter. Q_i is a spinor operator. It could be a Dirac, Majorana or Weyl spinor. In certain dimensions (e.g. 2 and 10) it could be Majorana-Weyl.
As we have seen, in extended supersymmetry the Q generally carry some representation of the internal symmetry,

$$[T_r, Q^i_\alpha] = (t_r)^i{}_j Q^j_\alpha.$$

Since we assume this so-called R-symmetry to be compact, the representation matrices t can be chosen Hermitian, $t_r = t_r^\dagger$. Then

$$[T_r, \bar{Q}^i_{\dot\alpha}] = -\bar{Q}^j_{\dot\alpha}(t_r)_j{}^i. \tag{7.10}$$

Now let us consider the anticommutator $\{Q, Q\}$. It must be a linear combination of the bosonic operators in the representation $(0,0)$ and $(1,0)$ of the Lorentz group. The only three-dimensional $(1,0)$ representation in the bosonic sector is the (anti)selfdual part of $J_{\mu\nu}$. Such a term in $\{Q,Q\}$ would not commute with the 4-momentum, whereas $\{Q,Q\}$ does. Thus we are left with

$$\{Q^i_\alpha, Q^j_\beta\} = 2\varepsilon_{\alpha\beta}Z^{ij}, \tag{7.11}$$

were Z^{ij} commutes with the space-time symmetries and hence is some linear combination of the internal symmetry generators,

$$Z^{ij} = \alpha^{rij}T_r. \tag{7.12}$$

We show that the Z^{ij} commute with the superalgebra. For this reason they are called *central charges*. First we show that the commutator of any T_r with the central charges is a central charge: First we have

$$\{Q^i_\alpha, [T_r, Q^j_\beta]\} = (t_r)^j{}_k\{Q^i_\alpha, Q^k_\beta\} = 2\varepsilon_{\alpha\beta}(t_r)^j{}_k Z^{ik}.$$

By using the super-Jacobi identity the left hand side can be rewritten as follows,

$$\{Q^i_\alpha, [T_r, Q^j_\beta]\} = [T_r, \{Q^j_\beta, Q^i_\alpha\}] - \{Q^j_\beta, [T_r, Q^i_\alpha]\} = 2\varepsilon_{\beta\alpha}[T_r, Z^{ji}] - (t_r)^i{}_k\{Q^j_\beta, Q^k_\alpha].$$

If not all $\epsilon_{\alpha\beta}$ vanish then

$$[T_r, Z^{ji}] = (t_r)^j{}_k Z^{ki} + (t_r)^i{}_k Z^{jk}. \tag{7.13}$$

With (7.12) we conclude that

$$[Z^{ij}, Z^{kl}] = a^{rij}[T_r, Z^{kl}] = a^{rij}\big((t_r)^k{}_p Z^{pl} + (t_r)^l{}_p Z^{kp}\big). \tag{7.14}$$

The last two equations imply that the Z^{ij} span an *invariant subalgebra* of the internal symmetry algebra. We use once more the super Jacobi identity for $\{\bar{Q},[Q,T]\}$ and with (7.10) conclude

$$\{\bar{Q}^j_{\dot\beta},[Q^i_\alpha,T_r]\}+[T_r,\{\bar{Q}^j_{\dot\beta},Q^i_\alpha\}]=\{Q^i_\alpha,[T_r,\bar{Q}^j_{\dot\beta}]\}=-\{Q^i_\alpha,\bar{Q}^k_{\dot\beta}\}(t_r)_k{}^j.$$

Now we use (7.11) in the commutator of the supercharges with the central charges and find

$$2\varepsilon_{\beta\gamma}[\bar{Q}^i_{\dot\alpha},Z^{jk}]=[\bar{Q}^i_{\dot\alpha},\{Q^j_\beta,Q^k_\gamma\}]=-[Q^k_\gamma,\{\bar{Q}^i_{\dot\alpha},Q^j_\beta\}]-[Q^j_\beta,\{Q^k_\gamma,\bar{Q}^i_{\dot\alpha}\}]=0,$$

where we have used that $\{Q,\bar{Q}\}\sim P$ commutes with the supercharges. Hence, the central charges commute with the supercharges $\bar{Q}$. It follows then, that

$$0=[\bar{Q}^i_{\dot A},Z^{jk}]=-a^{rjk}(t_r)_l{}^i\bar{Q}^l_{\dot A}\quad\text{or}\quad a^{rjk}(t_r)_l{}^i=0$$

so that the central charges also commute with the remaining supercharges Q

$$[Z^{jk},Q^i_\alpha]=a^{rjk}(t_r)^i{}_lQ^l_\alpha=0,$$

and hence among themselves, $[Z^{ij},Z^{kl}]=0$. We see that the invariant subalgebra spanned by the Z^{ij} is Abelian. Since the internal symmetry group is assumed to be compact it follows that the Z^{ij} commute with the T_r. Thus

$$[T_r,Z^{ij}]=0. \tag{7.15}$$

This proves that the Z^{ij} commute with all elements of the superalgebra and this explains their name.

7.2 Representations

Is maybe useful to collect the (anti)commutation relations of the N-extended superalgebra in 4 dimensions. It contains the Poincaré algebra

$$\begin{aligned}
[J_{\mu\nu},J_{\rho\sigma}] &= i\big(\eta_{\mu\rho}J_{\nu\sigma}+\eta_{\nu\sigma}J_{\mu\rho}-\eta_{\mu\sigma}J_{\nu\rho}-\eta_{\nu\rho}J_{\mu\sigma}\big)\\
[P_\rho,J_{\mu\nu}] &= -i\big(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu\big),\qquad [P_\mu,P_\nu]=0
\end{aligned}$$

as subalgebra. The supercharges commute with the translations,

$$[Q^i_\alpha,P_\mu]=0,$$

transform under the spin-$\frac{1}{2}$ representation of the Lorentz group

$$[J_{\mu\nu},Q^i_\alpha]=\tfrac{1}{2i}(\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta\quad,\quad[J_{\mu\nu},\bar{Q}^{i\dot\alpha}]=\tfrac{1}{2i}(\tilde\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{i\dot\beta}$$

and under the R-symmetry group as

$$[T_r,Q^i_\alpha]=(t_r)^i{}_jQ^j_\alpha\quad,\quad[T_r,\bar{Q}^i_{\dot\alpha}]=-\bar{Q}^j_{\dot\alpha}(t_r)_j{}^i.$$

The generators of the R-symmetry fulfill

$$[T_r,T_s]=iC_{rs}{}^tT_t$$

and commute with the bosonic generators of the Poincaré algebra,

$$[T_r, J_{\mu\nu}] = [T_r, P_\mu] = 0.$$

The supercharges fulfill the following anticommutation relations

$$\{Q^i_\alpha, Q^j_\beta\} = 2\varepsilon_{\alpha\beta} Z^{ij} \quad , \quad \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 2\varepsilon_{\dot\alpha\dot\beta} \bar{Z}^{ij} \quad , \quad \{Q^i_\alpha, \bar{Q}^j_{\dot\alpha}\} = 2\delta^{ij}\sigma^\mu_{\alpha\dot\alpha} P_\mu \,, \tag{7.16}$$

where Z^{ij} is the antisymmetric central charge matrix. It commutes with all other generators of the super-Poincaré algebra.
Let us now discuss the representation theory of N-extended supersymmetry in four dimensions. In general we can have more than one supersymmetry (extended supersymmetry). For each conserved Weyl spinor charge we have one supersymmetry. We will first assume that the *central charges are zero*.

7.2.1 Massive representations

For massive particles we may go to the rest frame in which $P \sim (M, \vec{0})$. In this inertial systems the relations (7.16) simplify as follows:

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\alpha}\} = 2M\delta^{ij}(\sigma_0)_{\alpha\dot\alpha} = 2M\delta_{\alpha\dot\alpha}\delta^{ij} \quad , \quad \{Q^i_\alpha, Q^j_\beta\} = \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 0\,. \tag{7.17}$$

Q is a tensor operator of spin $\frac{1}{2}$, as follows from

$$[Q, \vec{J}] = \tfrac{1}{2}\vec{\sigma} Q.$$

Therefore, the result of the action of Q on a state with spin s will be a linear combination of states with spin $s + \frac{1}{2}$ and $s - \frac{1}{2}$,

$$Q|m\, s\, s_3\rangle = \sum_{\tilde{s}_3} c^+_{s_3\tilde{s}_3} |m\, s + \tfrac{1}{2}\, \tilde{s}_3\rangle + \sum_{\tilde{s}_3} c^-_{s_3\tilde{s}_3} |m\, s - \tfrac{1}{2}\, \tilde{s}_3\rangle.$$

Now we define the following 2N fermionic creation and annihilation operators

$$A^i_\alpha = \frac{1}{\sqrt{2M}}\, Q^i_\alpha \qquad \bar{A}^i_{\dot\alpha} = \frac{1}{\sqrt{2M}}\, \bar{Q}^i_{\dot\alpha}\,. \tag{7.18}$$

which satisfy the simple algebra

$$\{A^i_\alpha, \bar{A}^j_{\dot\alpha}\} = \delta^{ij}\delta_{\alpha\dot\alpha}.$$

Building the representation is now easy. We start with the *Clifford vacuum* $|\Omega\rangle$, which is annihilated by the A^i_α and we generate the representation by acting with the creation operators. A typical state would be

$$\bar{A}^{i_1}_{\dot\alpha_1} \cdots \bar{A}^{i_n}_{\dot\alpha_n}\, |\Omega\rangle.$$

It is totally antisymmetric under interchange of index-pairs $(\dot\alpha, i) \leftrightarrow (\dot\beta, j)$. Since there are $2N$ creation operators $\bar A^i$ there are $\binom{2N}{n}$ states at the n-th oscillator level. The total number of states built on *one* vacuum state is

$$\sum_{n=0}^{2N} \binom{2N}{n} = (1+1)^{2N} = 4^N,$$

half of them being bosonic and half of them fermionic. If the vacuum sector is degenerate which happens if the Clifford vacuum $|\Omega\rangle$ is a member of a spin multiplet, then the number of states is

$$\text{NUMBER OF STATES} = 4^N \cdot \text{DIMENSION OF VACUUM SECTOR.}$$

The *maximal spin* s_{max} is carried by states like

$$\bar A^{i_1}_{\dot 1} \cdots \bar A^{i_n}_{\dot 1} \; |\Omega\rangle.$$

and their spins is equal to the spin s_0 of the ground-state plus $N/2$,

$$s_{max} = s_0 + N/2.$$

The *minimal spin* is 0 if $N/2 \geq s_0$ or $s_0 - N/2$ otherwise.
Since renormalisability requires *massive matter* to have spin $\leq \frac{1}{2}$. We conclude from the above expression for s_{max} that we must have

• $N = 1$ *for renormalizable coupling of massive matter.*

We see that in the absence of central charges, the only relevant massive multiplet is that of the massive Wess-Zumino model which has $N = 1$ and $s_0 = 0$. It contains a scalar, a pseudo-scalar and the two spin states of a massive Majorana spinor

N=1	s^P:	0^+	$\frac{1}{2}$	0^-
	states:	1	2	1

The two spin-zero states correspond to the Clifford vacuum $|\Omega\rangle$ and to $\bar A_{\dot 1} \bar A_{\dot 2} |\Omega\rangle$.

7.2.2 Massless representations

In this case we can go to the frame $P_\mu = (E, 0, 0, E)$. The anticommutation relations now become

$$\{Q^i_\alpha, \bar Q^j_{\dot\alpha}\} = 2E\delta^{ij}(\sigma_0 + \sigma_3)_{\alpha\dot\alpha} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \delta^{ij}\,. \tag{7.19}$$

the rest being zero. Since

$$\{Q^i_2, \bar Q^i_{\dot 2}\} = Q^i_2 Q^{i\dagger}_2 + Q^{i\dagger}_2 Q^i_2 = 0$$

they are represented by zero in a unitary theory. Thus we have N non-trivial creation and annihilation operators

$$A^i = \frac{1}{2\sqrt{E}}\, Q^i_1 \quad \text{and} \quad \bar A^i = \frac{1}{2\sqrt{E}}\, \bar Q^i_{\dot 1}.$$

and the representation is 2^N-dimensional. It is much shorter than the massive one which contains 4^N states.
The following Lorentz generators commute with $P_\mu = (E, 0, 0, E)$:

$$J_1 = J_{10} + J_{13}, \quad J_2 = J_{20} + J_{23} \quad \text{and} \quad J_3 = -J_{12}.$$

Note that J_3 is just the helicity operator λ defined in (2.16) for a massless particle moving in the 3-direction. The J_i generate the little group E_2 of translations and rotations in the 2-plane,

$$[J_1, J_3] = iJ_2, \quad [J_2, J_3] = -iJ_1 \quad \text{and} \quad [J_1, J_2] = 0.$$

In any finite dimensional representation J_1 and J_2 are trivially represented. Note that A^i increases the helicity by $\frac{1}{2}$ and $\bar{A}^i$ decreases it by $\frac{1}{2}$,

$$[\lambda, A^i] = \tfrac{1}{2}A^i \quad \text{and} \quad [\lambda, \bar{A}^i] = -\tfrac{1}{2}\bar{A}^i.$$

Now we introduce the Clifford vacuum $|\Omega\rangle$ with helicity λ. It is annihilated by all A^i, $A^i|\Omega = 0\rangle$. The states in an irreducible representation are gotten by acting with the creation operators on this state. For example,

$$\bar{A}^{i_1} \cdots \bar{A}^{i_n} |\Omega\rangle$$

has helicity $\lambda - n/2$. This way we get the following states

helicity:	λ	$\lambda - 1/2$	$\dots$	$\lambda - N/2$
multiplicities:	1	$\binom{N}{1}$	$\dots$	$\binom{N}{N}$

The total number of states in an irreducible massless representation is

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N.$$

According to the CPT theorem a physical massless state contains always the two helicities λ and $-\lambda$. Hence a single supermultiplett can contain all massless states for $\lambda = N/4$. Otherwise the states must be doubled starting with a second Clifford vacuum with helicity $\lambda' = N/2 - \lambda$. Here we will describe some useful examples with helicities up to one.

N=1 supersymmetry:
The number of Clifford-states in an irreducible multiplet is just $1 + 1$ and we need at least two Clifford vacua to built a CPT invariant model.
For the *chiral multiplet* $\lambda = \frac{1}{2}$ and $\lambda' = 0$ and we have the following states

helicity	$1/2$	1 Majorana spinor
	0	1 complex scalar
	$-1/2$	1 Majorana spinor

These are the fields of the massless Wess-Zumino model.
The *vector multiplet* with $\lambda = 1$ and $\lambda' = -1/2$ consists of

helicity	1	1 gauge field
	1/2	1 Majorana spinor
	−1/2	1 Majorana spinor
	−1	1 gauge field

N=2 supersymmetry
A irreducible representation contains 3 helicities and $1+2+1$ 'states'.
The simplest multiplet is the *hyper-multiplet* $\lambda = 1/2$ with the following content:

helicity	1/2	1 Majorana spinor
	0	1 complex scalar
	−1/2	1 Majorana spinor

To built a *vector multiplet* we need two Clifford vacua with $\lambda = 1$ and $\lambda' = 0$ so that it contains

helicity	1	1 gauge field
	1/2	2 Majorana spinors
	0	1 complex scalar
	−1/2	2 Majorana spinors
	−1	1 gauge field

The two Majorana spinors can be combined to a Dirac spinor. This multiplet is considered in the Seiberg-Witten solution to $N = 2$ supersymmetric gauge theories.

N=4 supersymmetry
A irreducible supermultiplett consists of $1+4+6+4+1 = 16$ states. The unique multiplet giving rise to a renormalizable field theory in flat spacetime is the *vector multiplet*. It has $\lambda = 1$ and contains

helicity	1	1 gauge field
	1/2	4 Majorana spinors
	0	6 complex scalars
	−1/2	4 Majorana spinors
	−1	1 gauge field

This multiplet is relevant in the recently discovered AdS/CFT correspondence.

7.2.3 Non-zero central charges

In this case massive supermultipletts can be as short as massless ones. Under a $U(N)$ transformation of the super charges,

$$Q^i_\alpha \to U^i{}_j Q^j_\alpha \quad \text{and} \quad \bar{Q}^i_{\dot{\alpha}} \to \bar{Q}^j_{\dot{\alpha}} (U^\dagger)_j{}^i$$

the antisymmetric central charge matrix $Z = (Z^{ij})$ transforms as

$$Z \longrightarrow UZU^T, \quad \bar{Z} \longrightarrow \bar{U}\bar{Z}U^\dagger.$$

In [37] it was shown that there exists a unitary U such that Z becomes block diagonal[1],

$$\begin{pmatrix} 0 & Z_1 & 0 & 0 & & \dots & \\ -Z_1 & 0 & 0 & 0 & & \dots & \\ 0 & 0 & 0 & Z_2 & & \dots & \\ 0 & 0 & -Z_2 & 0 & & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & & & & 0 & Z_{N/2} \\ & \dots & & & & -Z_{N/2} & 0 \end{pmatrix}. \tag{7.20}$$

We have labeled the *real positive* eigenvalues by Z_m, $m = 1, 2, \dots, N/2$. We will split the index $i \to (a, m)$: $a = 1, 2$ labels positions inside the 2×2 blocks while m labels the blocks. Then

$$\{Q^{am}_\alpha, \bar{Q}^{bn}_{\dot\alpha}\} = 2M\delta_{\alpha\dot\alpha}\delta^{ab}\delta^{mn} \quad , \quad \{Q^{am}_\alpha, Q^{bn}_\beta\} = 2Z_n\varepsilon_{\alpha\beta}\varepsilon^{ab}\delta^{mn}\,. \tag{7.21}$$

Now we define the following fermionic oscillators

$$A^m_\alpha = \frac{1}{\sqrt{2}}[Q^{1m}_\alpha + \varepsilon_{\alpha\beta}Q^{\dagger 2m}_\beta] \quad , \quad B^m_\alpha = \frac{1}{\sqrt{2}}[Q^{1m}_\alpha - \epsilon_{\alpha\beta}Q^{\dagger 2m}_\beta]\,, \tag{7.22}$$

and similarly for the conjugate operators. The anticommutators become

$$\begin{aligned} \{A^m_\alpha, A^n_\beta\} &= \{A^m_\alpha, B^n_\beta\} = \{B^m_\alpha, B^n_\beta\} = 0\,, \\ \{A^m_\alpha, A^{\dagger n}_\beta\} &= 2\delta_{\alpha\beta}\delta^{mn}(M + Z_m) \\ \{B^m_\alpha, B^{\dagger n}_\beta\} &= 2\delta_{\alpha\beta}\delta^{mn}(M - Z_m)\,. \end{aligned} \tag{7.23}$$

Unitarity requires that the right-hand sides in (7.23) be non-negative. This in turn implies the celebrated BOGOMOL'NYI BOUND

$$M \geq \max[Z_n]\,. \tag{7.24}$$

It is essential for the unitarity of the underlying theory.
Consider for example the $N = 4$ theory and assume, that $M = Z_1 > Z_2$. Then

$$\{B^1_\alpha, B^{\dagger 1}_\beta\} = 0$$

implies that the B^1_α vanish identically and we are left with the following 6 creation and annihilation operators

$$A^m_\alpha,\, A^{\dagger m}_\alpha, \quad m = 1, 2, \quad \text{and} \quad B^2_\alpha,\, B^{\dagger 2}_\alpha.$$

[1] Consider even N.

They generate a representation with 2^6 states instead of 2^8 states. More generally, consider $0 \leq r \leq N/2$ of the Z_m's to be equal to M. Then $2r$ of the B-oscillators vanish identically and we are left with $2N - 2r$ creation and annihilation operators. The representation has 2^{2N-2r} states.
The maximal case $r = N/2$ gives rise to the *short BPS multiplet* whose number of states are the same as in the massless multiplet. The other multiplets with $0 < r < N/2$ are known as *intermediate BPS multiplets.*
BPS states are important probes of non-perturbative physics in theories with extended ($N \geq 2$) supersymmetry. The BPS states are special for the following reasons:

- Due to their relation with central charges, and although they are massive, they form multiplets under extended SUSY which are shorter than the generic massive multiplet. Their mass is given in terms of their charges and Higgs (moduli) expectation values.

- They are the only states that can become massless when we vary coupling constants and Higgs expectation values.

- When they are at rest they exert no force on each other.

- Their mass-formula is supposed to be exact if one uses renormalized values for the charges and moduli. The argument is that quantum corrections would spoil the relation of mass and charges, and if we assume unbroken SUSY at the quantum level there would be incompatibilities with the dimension of their representations.

We will come back to these remarks later during this lecture.