6 Supersymmetry algebras

Supersymmetric theories avoid the restriction of the Coleman-Mandula theorem by relaxing one condition: they generalize the notion of a Lie-algebra and include in the defining relation anticommutators in addition to the usual commutators. We shall develop the form of the susy algebra from first principles, following Haag, Lopuszanski and Sohnius [32]. Under the conditions imposed in the Coleman-Mandula theorem the supersymmetric structure is almost uniquely fixed by the requirement of Lorentz invariance. The supermultiplett structure of one-particle states will later be deduced from the susy algebra.
In 4 dimensions the generalization of the Poincaré algebra to a superalgebra is obtained in its simplest version by the following procedure: one adds to the Poincaré-algebra a Majorana-spinor charge, with components $Q_\alpha, \alpha = 1, \dots, 4$, with the properties

$$\{Q_\alpha, \bar{Q}^\beta\} = 2(\gamma^\mu)_\alpha^{\ \beta} P_\mu, \qquad [Q_\alpha, P_\mu] = 0 \quad \text{and} \quad [M_{\mu\nu}, Q_\alpha] = (\Sigma_{\mu\nu} Q)_\alpha. \tag{6.1}$$

Here P_μ and $M_{\mu\nu}$ are the generators of displacement and homogeneous Lorentz-transformations of space-time. Since P_μ has the dimensions L^{-1}, the supersymmetry generators Q_α must have dimension $L^{-1/2}$. The $M_{\mu\nu}$ are dimensionless. We have seen that the Q_α map fermions into bosons and bosons into fermions. The relation $[M_{\mu\nu}, Q_\alpha] = (\Sigma_{\mu\nu} Q)_\alpha$ expresses the fact, that the Q_α transform as a spinor under Lorentz transformations. In the last section we have explicitly realized this algebraic structure on a Majorana-spinor field and spin-0 fields. Now we shall investigate, how unique this algebraic structure is.

6.1 Graded Lie algebras

Supersymmetry is expressed in terms of symmetry generators t_a that form a graded Lie algebra or Lie superalgebra. We have to deal with two types of elements: Bose and Fermi. Thus we use a graded vector space such that each of its vectors has a grade defined mod 2. Hence a $\mathbb{Z}_2$-graded algebra consists of a vector space $\mathcal{S}$ which is the set-theoretic union[1] of two subspaces,

$$\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1,$$

and is equipped with a binary operation which respects this grading. The dimension of $\mathcal{S}_0$ is N_0 and that of $\mathcal{S}_1$ is N_1. We assign to an element $t_a \in \mathcal{S}$ a grade η_a. The elements in $\mathcal{S}_0$ have grade

[1] Superselection rules forbid the linear combination of Bose with Fermi elements [36], and hence $\mathcal{S}$ is the set-theoretic union and not the direct sum of $\mathcal{S}_0$ and $\mathcal{S}_1$.

0 and those in $\mathcal{S}_1$ have grade 1. $\mathcal{S}_0$ consists of even elements and $\mathcal{S}_1$ of odd ones. The bilinear product

$$[.,.\} : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}, \qquad [t_a, t_b\} = iC_{ab}{}^c t_c, \tag{6.2}$$

has the following properties

$$\begin{aligned} \text{grading:} \quad & [\mathcal{S}_0, \mathcal{S}_0\} \subset \mathcal{S}_0, \quad [\mathcal{S}_0, \mathcal{S}_1\} \subset \mathcal{S}_1, \quad [\mathcal{S}_1, \mathcal{S}_1\} \subset \mathcal{S}_0 \\ \text{super-ACR:} \quad & [t_a, t_b\} = -(-)^{\eta_a \eta_b}[t_b, t_a\} \\ \text{super-Jacobi:} \quad & (-)^{\eta_a \eta_c}[t_a, [t_b, t_c\}\} + (-)^{\eta_c \eta_b}[t_c, [t_a, t_b\}\} + (-)^{\eta_b \eta_a}[t_b, [t_c, t_a\}\} = 0. \end{aligned} \tag{6.3}$$

Introducing the adjoint action

$$\mathrm{ad}_{t_a} : t_b \longrightarrow \mathrm{ad}_{t_a}(t_b) = [t_a, t_b\}$$

the super-Jacobi identity can be rewritten as

$$(\mathrm{ad}_{t_a} \circ \mathrm{ad}_{t_b})(t_c) - (-)^{\eta_a \eta_b}(\mathrm{ad}_{t_b} \circ \mathrm{ad}_{t_a})(t_c) = \mathrm{ad}_{[t_a, t_b\}}(t_c).$$

Since this holds for any generator t_c we conclude, that

$$[\mathrm{ad}_{t_a}, \mathrm{ad}_{t_b}\} \equiv \mathrm{ad}_{t_a} \circ \mathrm{ad}_{t_b} - (-)^{\eta_a \eta_b}\mathrm{ad}_{t_b} \circ \mathrm{ad}_{t_a} = \mathrm{ad}_{[t_a, t_b\}} = iC_{ab}{}^c \mathrm{ad}_{t_c}. \tag{6.4}$$

The even elements in $\mathcal{S}_0$ correspond to bosonic generators, the odd elements in $\mathcal{S}_1$ correspond to fermionic generators. Hence, the products of two bosonic or two fermionic operators are bosonic, and the products of a fermionic with a bosonic operator are fermionic, so that

$$C_{ab}{}^c = 0 \quad \text{unless} \quad \eta_c = \eta_a + \eta_b \bmod 2. \tag{6.5}$$

According to the symmetry properties of the product in (6.3) the structure constants in (6.2) must satisfy the conditions

$$C_{ba}{}^c = -(-1)^{\eta_a \eta_b} C_{ab}{}^c. \tag{6.6}$$

In applications the t_a are realized as linear operators on a Hilbert space and the product of operators $t_a t_b$ is defined. In this case

$$[t_a, t_b\} = t_a t_b - (-)^{\eta_a \eta_b} t_b t_a. \tag{6.7}$$

The super-Jacobi identity is then automatically fulfilled. If one operator is bosonic we get the commutator, if both operators are fermionic we get the anticommutator. If the t_a are hermitian, then the structure constants satisfy a reality condition

$$-i\bar{C}_{ab}{}^c t_c = [t_a, t_b\}^\dagger = t_b t_a - (-)^{\eta_a \eta_b} t_a t_b = iC_{ba}{}^c t_c \quad \text{or} \quad \bar{C}_{ab}{}^c = -C_{ba}{}^c. \tag{6.8}$$

Note that $\mathcal{S}$ is not a Lie algebra, since the product of two elements in $\mathcal{S}_1$ is symmetric and not antisymmetric,

$$[t_a, t_b\} = [t_b, t_a\} \quad \text{if} \quad t_a, t_b \in \mathcal{S}_1.$$

In the literature one writes $[.,.]$ instead of $[.,.\}$ if the product is antisymmetric, i.e. if at least one argument is even and $\{.,,\}$ instead of $[,.,\}$ if the product is symmetric, i.e. if both arguments are odd. The subspace $\mathcal{S}_0$ spans an ordinary Lie-algebra. The subspace $\mathcal{S}_1$ is not even a subalgebra, since $\mathcal{S}_1$ is not closed under $[.,.\}$.

6.1.1 Conditions for the structure constants

We construct a $\mathbb{Z}_2$ grading of a Lie algebra which is generated by N_0 generators T_i with commutation relations

$$[T_i, T_j] = iC_{ij}{}^k T_k, \tag{6.9}$$

with Lie-algebra structure constants $C_{ij}{}^k$. Let us see, what are the consequences of the super Jacobi-identity for the structure constants $C_{ab}{}^c$:

$$\begin{aligned} 0 &= (-)^{\eta_a\eta_c}[t_a,[t_b,t_c\}\} + (-)^{\eta_c\eta_b}[t_c,[t_a,t_b\}\} + (-)^{\eta_b\eta_a}[t_b,[t_c,t_a\}\} \\ &= i(-)^{\eta_a\eta_c}C_{bc}{}^d[t_a,t_d\} + i(-)^{\eta_c\eta_b}C_{ab}{}^d[t_c,t_d\} + i(-)^{\eta_b\eta_a}C_{ca}{}^d[t_b,t_d\} \\ &= -\big((-)^{\eta_a\eta_c}C_{bc}{}^dC_{ad}{}^e + (-)^{\eta_c\eta_b}C_{ab}{}^dC_{cd}{}^e + (-)^{\eta_b\eta_a}C_{ca}{}^dC_{bd}{}^e\big)\, t_e \end{aligned} \tag{6.10}$$

We denote the generators of $\mathcal{S}_1$ by Q_α. Since $[\mathcal{S}_0,\mathcal{S}_1] \subset \mathcal{S}_1$ and $\{\mathcal{S}_1,\mathcal{S}_1\} \subset \mathcal{S}_0$ we have, in addition to (6.9), the relations

$$[T_i, Q_\alpha] = iC_{i\alpha}{}^\beta Q_\beta \quad \text{and} \quad \{Q_\alpha, Q_\beta\} = iC_{\alpha\beta}^{\;i} T_i.$$

Because of (6.6) the non-vanishing structure constants obey

$$C_{ij}{}^k = -C_{ji}{}^k, \qquad C_{i\alpha}{}^\beta = -C_{\alpha i}{}^\beta \quad \text{and} \quad C_{\alpha\beta}{}^i = C_{\beta\alpha}{}^i. \tag{6.11}$$

If the generators T_i and Q_α are hermitian, then (6.8) implies

$$\bar{C}_{ij}{}^k = C_{ij}{}^k, \qquad \bar{C}_{i\alpha}{}^\beta = C_{i\alpha}{}^\beta \quad \text{and} \quad \bar{C}_{\alpha\beta}{}^i = -C_{\alpha\beta}{}^i.$$

Let us study the consequences of the super Jacobi identities for two bosonic generators and one fermionic generator. We just must set $a = i, b = j$ and $c = \alpha$ in (6.10) in which case all $(-)^{\eta_a\eta_b} = 1$ and find

$$0 = C_{j\alpha}{}^\beta C_{i\beta}{}^\gamma + C_{ij}{}^k C_{\alpha k}{}^\gamma + C_{\alpha i}{}^\beta C_{j\beta}{}^\gamma.$$

Now we introduce the N_1-dimensional matrices $C_i, i = 1, \ldots N_0$, with matrix elements $(C_i)_\alpha^{\;\beta}$ and using $C_{\alpha i}{}^\beta = -C_{i\alpha}{}^\beta$ we end up with

$$[C_i, C_j] = -C_{ij}{}^k C_k. \tag{6.12}$$

This proves the

Lemma 4 *The N_0 matrices C_i, whose matrix elements are the structure constants $C_{i\alpha}{}^\beta$, form a N_1-dimensional representation of the bosonic Lie-algebra $\mathcal{S}_0$.*

Now we investigate the super Jacobi identity for one bosonic and two fermionic generators. Hence we set $a=i$ and $b=\alpha, c=\beta$ in (6.10), so that $\eta_a=0$ and $\eta_b=\eta_c=1$ and find

$$C_{ij}{}^k C_{\alpha\beta}^{\ j} = C_{i\alpha}{}^\gamma C_{\beta\gamma}^{\ k} + C_{i\beta}{}^\gamma C_{\alpha\gamma}^{\ k} = C_{i\alpha}{}^\gamma C_{\gamma\beta}^{\ k} + C_{i\beta}{}^\gamma C_{\gamma\alpha}^{\ k}.$$

If we introduce the N_0 symmetric N_1-dimensional matrices

$$C^i \quad \text{with matrix elements} \quad (C^i)_{\alpha\beta} = C_{\alpha\beta}^{\ i}$$

then these relations can be written as

$$C_{ij}{}^k C^j = C_i C^k + (C_i C^k)^T. \tag{6.13}$$

Finally we study the consequences of the super Jacobi identity for three fermionic operators. Hence we set $a=\alpha, b=\beta$ and $c=\gamma$ in (6.10). This results in

$$0 = C_{\beta\gamma}{}^i C_{\alpha i}^{\ \delta} + C_{\alpha\beta}^{\ i} C_{\gamma i}^{\ \delta} + C_{\gamma\alpha}{}^i C_{\beta i}^{\ \delta} = (C^i)_{\alpha\beta}(C_i)_\gamma^{\ \delta} + \text{cycl.}(\alpha,\beta,\gamma). \tag{6.14}$$

6.1.2 The grading of $SU(2)$

In this case $\mathcal{S}_0$ is generated by the 3 hermitian matrices

$$\tfrac{1}{2}\sigma_i, \quad \text{such that} \quad C_{ij}{}^k = \epsilon_{ijk}.$$

The condition (6.12) reads

$$[C_i, C_j] = -\epsilon_{ijk} C_k,$$

and hence the real matrices C_i must form a N_1 dimensional representation of $su(2)$. For the 2-dimensional representations we have

$$(C_i)_\alpha^{\ \beta} = \tfrac{i}{2}(\sigma_i)_\alpha^{\ \beta}, \quad \text{so that} \quad [T_i, Q_\alpha] = -\tfrac{1}{2}(\sigma_i Q)_\alpha.$$

Now we proceed to the constraints (6.13) for the symmetric matrices C^k. For the two-dimensional representation we set

$$C^k = \sigma_k\sigma_2 \Longrightarrow C^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad C^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Now one can prove, that (6.13) holds:

$$C_i C^k + (C_i C^k)^T = \tfrac{i}{2}\left(\sigma_i\sigma_k\sigma_2 - \sigma_2\sigma_k^T\sigma_i^T\right) = \tfrac{i}{2}[\sigma_i, \sigma_k]\sigma_2 = -\epsilon_{ikj}C^j = C_{ij}{}^k C^j,$$

where we used the identity $\sigma_2\sigma_i^T = -\sigma_i\sigma_2$. With this solution we find

$$\{Q_\alpha, Q_\beta\} = i(\sigma_i\sigma_2)_{\alpha\beta} T_i.$$

Now one can show, that for our choice for C_i and C^k the third set of constraints in (6.14) are automatically fulfilled. Hence we have obtained the following grading of $su(2)$:

$$[T_i,T_j]=i\epsilon_{ijk}T_k,\quad [T_i,Q_\alpha]=-\tfrac{1}{2}(\sigma_i Q)_\alpha \quad\text{and}\quad \{Q_\alpha,Q_\beta\}=i(\sigma_i\sigma_2)_{\alpha\beta}T_i. \tag{6.15}$$

One can write down an explicit matrix representation of the graded $SU(2)$ on a 3-dimensional graded vector space $V(1|2)$ over $\mathbb{C}$. We represent a vector in V as a column vector with $1+2$ entries. The Bose element is in the top entry whereas the Fermi elements are in the bottom 2 entries. Then the matrix representation of the generators $\{T_i, Q_\alpha\}$ takes the form

$$T_i=\begin{pmatrix}0&0\\0&\frac{1}{2}\sigma_i\end{pmatrix},\quad Q_1=\begin{pmatrix}0&1&0\\0&0&0\\-1&0&0\end{pmatrix}\quad\text{and}\quad Q_2=\begin{pmatrix}0&0&1\\1&0&0\\0&0&0\end{pmatrix}.$$

The superalgebra is then represented by the following matrices

$$M=\begin{pmatrix}0&B\\\varepsilon B^T&D\end{pmatrix},\qquad \varepsilon=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{6.16}$$

where $D=D^\dagger$ a 2-dimensional matrix which maps fermions into fermions and B is a 2-dimensional row vector which maps fermions into bosons, so that

$$\begin{pmatrix}0&0\\0&D\end{pmatrix}\in\mathcal{S}_0\quad\text{and}\quad\begin{pmatrix}0&B\\\varepsilon B^T&0\end{pmatrix}\in\mathcal{S}_1.$$

Using $\varepsilon D=-D^T\epsilon$ the super-anticommutator of 2 elements can be written as

$$[M_1,M_2\}=\begin{pmatrix}0&B_3\\\varepsilon B_3^T&D_3\end{pmatrix},\quad B_3=B_1D_2-B_2D_1,\quad D_3=[D_1,D_2]+\varepsilon(B_1^TB_2+B_2^TB_1).$$

It turns out, that for the 3-dimensional representation

$$(C_i)_\alpha^{\ \beta}=\epsilon_{i\alpha\beta}$$

we cannot satisfy the constraints (6.13,6.14).

6.1.3 Supertrace and super-Killingform

Guided by the grading of $SU(2)$ we consider a mod 2 graded vector space $V(N|M)$ over $\mathbb{C}$ with N Bose and M Fermi dimensions. A vector in V can be represented as a column vector with $M+N$ entries. For Bose (Fermi) elements the top N (bottom M) entries will be non-zero. Now consider the complex linear transformations on $V(N|M)$. The grading of V induces an obvious grading of these linear transformations,

$$M=\begin{pmatrix}A&B\\C&D\end{pmatrix}. \tag{6.17}$$

In the matrix representation a Bose linear transformation is block diagonal

$$M\in\mathcal{S}_0:\quad M=\begin{pmatrix}A&0\\0&D\end{pmatrix}$$

whereas a Fermi transformation is block off-diagonal

$$M \in \mathcal{S}_1 : \quad M = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

The matrices A, D are square matrices, whereas B, C are in general rectangular matrices. In the matrix representation the bracket is the usual commutator in all cases but one, namely when both bracketed transformations are Fermi in which case it is an anticommutator. The so obtained Lie superalgebra is $gl(N|M)$ and it is not simple. Simplicity of a superalgebra is defined as for the ordinary Lie algebra: a Lie superalgebra $\mathcal{S}$ is simple if any sub-superalgebra $\mathcal{A}$ of $\mathcal{S}$, such that $[\mathcal{A}, \mathcal{S}\} \subset \mathcal{A}$, is trivial, i.e. either $\mathcal{A} = 0$ or $\mathcal{A} = \mathcal{S}$. In ordinary Lie algebras one achieves simplicity by imposing the traceless condition. However, if M_1 and M_2 are traceless, then the bracket

$$[M_1, M_2\} = \begin{pmatrix} [A_1, A_2] + B_1C_2 + B_2C_1 & A_1B_2 - A_2B_1 + B_1D_2 - B_2D_1 \\ C_1A_2 - C_2A_1 + D_1C_2 - D_2C_1 & [D_1, D_2] + C_1B_2 + C_2B_1 \end{pmatrix} \tag{6.18}$$

need not be traceless since, it involves anti-commutators. We therefore need a new concept of trace, called *supertrace*, such that it vanishes for the bracketing of two matrices. It is easily checked that the supertrace, given by

$$\mathrm{S\,Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, D, \tag{6.19}$$

has the required property,

$$\mathrm{S\,Tr}\, [M_1, M_2\} = 0. \tag{6.20}$$

Recall, that the invariant Killing form of an ordinary Lie algebra is defined as

$$K(T_i, T_j) = \mathrm{Tr}\left(\mathrm{ad}_{T_i} \circ \mathrm{ad}_{T_j}\right) = -\sum_{k,l} C_{ik}{}^{l} C_{jl}{}^{k}, \quad \text{where} \quad [T_i, T_j] = iC_{ij}{}^{k} T_k.$$

This bilinear-form is non-degenerate for semi-simple Lie groups and positive definite for compact Lie groups. The generalized Killing form on $\mathcal{S}$ is analogously defined as

$$K(t_a, t_b) = \mathrm{S\,Tr}\left(\mathrm{ad}_{t_a} \circ \mathrm{ad}_{t_b}\right) = -\sum_{c,d} (-)^{\eta_d} C_{ac}{}^{d} C_{bd}{}^{c} \equiv K_{ab}. \tag{6.21}$$

To prove the second equality we introduce an orthogonal basis $\{t_a\}$ in $\mathcal{S}$ for which

$$(t_a, t_b) = \mathrm{S\,Tr}\,(t_a, t_b) = (-)^{\eta_a} \delta_{ab} \tag{6.22}$$

such that indeed

$$\mathrm{S\,Tr}\left(\mathrm{ad}_{t_a} \circ \mathrm{ad}_{t_b}\right) = \sum_c \left(t_c, [t_a, [t_b, t_c\}\}\right) = -C_{bc}{}^{e} C_{ae}{}^{d} (t_c, t_d) = -K_{ab}.$$

It reduces to a multiple of the usual Killing form on the bosonic subspace $\mathcal{S}_0$ and hence is a symmetric bilinear form when acting on the subspace $\mathcal{S}_0$. However, it is antisymmetric when acting on the fermionic subspace $\mathcal{S}_1$,

$$K(Q_\alpha, Q_\beta) = K_{\alpha\beta} = -K_{\beta\alpha}. \tag{6.23}$$

This follows from the following symmetry property of the super-Killingform

$$K_{ab} = (-)^{\eta_a \eta_b} K_{ba}, \tag{6.24}$$

which is easily gotten from (6.4), applied to 2 fermionic generators. Finally, we have

$$K(T_i, Q_\alpha) = 0, \tag{6.25}$$

because the structure constants respect the grading. The generalized Killingform is invariant under bracketing,

$$K\big([t_c, t_a\}, t_b\big) + (-)^{\eta_a \eta_b} K\big(t_a, [t_c, t_b\}\big) = 0. \tag{6.26}$$

The proof I leave as exercise. Let us finally calculate the super Killingform for the graded $SU(2)$ algebra with structure constants

$$C_{ij}{}^k = \epsilon_{ijk}, \quad C_{i\alpha}{}^\beta = -C_{\alpha i}{}^\beta = \tfrac{i}{2}(\sigma_i)^\beta_\alpha, \quad C_{\alpha\beta}{}^i = C_{\beta\alpha}{}^i = (\sigma_i \sigma_2)_{\alpha\beta}.$$

Inserting into (6.21) yields the following super Killing-matrix

$$\begin{aligned} K_{ij} &= -\sum_{kl} C_{ik}{}^l C_{jl}{}^k + \sum_{\alpha\beta} C_{i\alpha}{}^\beta C_{j\beta}{}^\alpha = -\sum_{kl} \epsilon_{ikl}\epsilon_{jlk} - \tfrac{1}{4}\sum_{\alpha\beta}(\sigma_i)^\beta_\alpha (\sigma_j)^\alpha_\beta = \tfrac{3}{2}\delta_{ij} \\ K_{\alpha\beta} &= -\sum_{\gamma i} C_{\alpha\gamma}{}^i C_{\beta i}{}^\gamma + (\alpha \leftrightarrow \beta) = \tfrac{i}{2}\sum_{\gamma i}(\sigma_i\sigma_2)_{\alpha\gamma}(\sigma_i)^\gamma_\beta - (\alpha \leftrightarrow \beta) = -3i(\sigma_2)_{\alpha\beta}. \end{aligned}$$

Writing $(a) = (i, \alpha)$ the matrix (K_{ab}) takes the form

$$(K_{ab}) = \tfrac{3}{2}\begin{pmatrix} \mathbb{1}_3 & 0 \\ 0 & -2\varepsilon \end{pmatrix},$$

where ε has been introduced in (6.16). K_{ab} is non-degenerated as expected for the simple graded $su(2)$ algebra.

6.1.4 Supergroups

An endomorphism $M : V(N|M) \to V(N|M)$ can be represented by a graded $(N+M)\times(N+M)$-matrix with the following structure

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (M_a{}^b) = \begin{pmatrix} A_i^{\ j} & B_i{}^\alpha \\ C_\alpha{}^i & D_\alpha{}^\beta \end{pmatrix}, \tag{6.27}$$

where $A : V_0 \to V_0$ maps bosons into bosons and $D : V_1 \to V_1$ maps fermions into fermions. A and D are square matrices. The matrices $B : V_1 \to V_0$ maps fermions into bosons and $C : V_0 \to V_1$ maps bosons into fermions. If $N \neq M$ then B and C are not square matrices:

$$B : N \times M \quad \text{and} \quad C : M \times N.$$

Since A, D do not change the grade they are even submatrices with degree $\eta = 0$. On the other hand, B, C change the grade and hence they are odd submatrices. In this subsection (6.1.4) we only use *commutators (no anticommutators)* and then we must assume, that the elements of B and C are anticommuting variables and therefore behave as Grassmann parameters:

$$B_{i\alpha} C_{\beta j} = -C_{\beta j} B_{i\alpha}.$$

From this we obtain the rules

$$(BC)^T = -C^T B^T \quad \text{and} \quad \mathrm{Tr}\,(BC) = -\mathrm{Tr}\,(CB). \tag{6.28}$$

Next we return to the *supertrace* which has been defined by

$$\mathrm{S\,Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, D = \mathrm{Tr}\,(-)^{N_F} M. \tag{6.29}$$

Here N_F is the fermionic number and it is zero on the bosonic subspace and 1 on the fermionic subspace,

$$N_F = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \Longrightarrow (-)^{N_F} = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_M \end{pmatrix}.$$

On the bosonic subspace the supertrace becomes the ordinary trace.

Lemma 5 *The supertrace is a symmetric bilinear-form on $L(V)$ which is cyclic, i.e.*

$$\mathrm{S\,Tr}\,(M_1 M_2) = \mathrm{S\,Tr}\,(M_2 M_1) \quad \textit{and} \quad \mathrm{S\,Tr}\,(M_1 M_2 M_3) = \mathrm{S\,Tr}\,(M_3 M_1 M_2). \tag{6.30}$$

The symmetry is shown by explicit calculation:

$$\begin{aligned}
\mathrm{S\,Tr}\,(M_1 M_2) &= \mathrm{Tr}\,(A_1 A_2 + B_1 C_2) - \mathrm{Tr}\,(C_1 B_2 + D_1 D_2) \\
&= \mathrm{Tr}\,(A_1 A_2) + \mathrm{Tr}\,(B_1 C_2) - \mathrm{Tr}\,(C_1 B_2) - \mathrm{Tr}\,(D_1 D_2) \\
&= \mathrm{Tr}\,(A_2 A_1) - \mathrm{Tr}\,(C_2 B_1) + \mathrm{Tr}\,(B_2 C_1) - \mathrm{Tr}\,(D_2 D_1) \\
&= \mathrm{Tr}\,(A_2 A_1 + B_2 C_1) - \mathrm{Tr}\,(C_2 B_1 + D_2 D_1) = \mathrm{S\,Tr}\,(M_2 M_1).
\end{aligned}$$

The cyclic property follows then at once. Having defined the supertrace we introduce the superdeterminant:
Definition: The determinant of a supermatrix M is the superdeterminant defined by

$$\mathrm{S}\det M = \exp\big\{\mathrm{S\,Tr}\,\log M\big\}. \tag{6.31}$$

On the bosonic subspace the superdeterminant is just the ordinary determinant. The superdeterminant shares many properties with the ordinary determinant. In particular we have the

Lemma 6 *Let M_1, M_2 be two graded matrices. Then we have the product rule*

$$\mathrm{S}\det(M_1 M_2) = \mathrm{S}\det(M_1) \cdot \mathrm{S}\det(M_2). \tag{6.32}$$

Proof: We set $M_i = \exp(P_i)$ and use the Baker-Campbell-Hausdorff formula

$$M_1 M_2 = e^{P_1} e^{P_2} = e^{P_1 + P_2 + \frac{1}{2}[P_1, P_2] + \dots}$$

and take the logarithm of both sides

$$\log(M_1 M_2) = P_1 + P_2 + \tfrac{1}{2}[P_1, P_2] + \dots .$$

Since the supertrace of the commutator of two supermatrices vanish, we immediately find

$$\mathrm{S\,Tr}\,\log(M_1 M_2) = \mathrm{S\,Tr}\,P_1 + \mathrm{S\,Tr}\,P_2 = \mathrm{S\,Tr}\,\log(M_1) + \mathrm{S\,Tr}\,\log(M_2).$$

After exponentiation we find the product rule for the determinant. Now one has the following

Lemma 7 *The superdeterminant can be expressed in terms of ordinary determinants by the following formulae*

$$\mathrm{S\,det}\,M = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}. \tag{6.33}$$

A corresponding formula holds for $\det M$ if all entries of M are c-numbers. Then the determinants in the denominators of these formulae are in the numerator. To prove this lemma we first calculate the logarithm of particular triangular matrices. When exponentiating both sides, it is straightforward to prove, that

$$\begin{aligned} R &= \begin{pmatrix} \mathbb{1} & B \\ 0 & D \end{pmatrix} \quad \Rightarrow \quad \log R = \begin{pmatrix} 0 & B(D-1)^{-1}\log D \\ 0 & \log D \end{pmatrix} \\ L &= \begin{pmatrix} A & 0 \\ C & \mathbb{1} \end{pmatrix} \quad \Rightarrow \quad \log L = \begin{pmatrix} \log A & 0 \\ C(A-1)^{-1}\log A & 0 \end{pmatrix}. \end{aligned} \tag{6.34}$$

It follows at once, that

$$\mathrm{S\,Tr}\,R = -\mathrm{Tr}\,\log D,\ \ \mathrm{S\,det}\,R = \frac{1}{\det D} \quad \text{and} \quad \mathrm{S\,Tr}\,L = \mathrm{Tr}\,\log A,\ \ \mathrm{S\,det}\,L = \det A.$$

Now we decompose the supermatrix M as

$$M = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ CA^{-1} & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} A & B \\ 0 & \mathbb{1} \end{pmatrix}.$$

Applying the product rule for superdeterminants to these two decomposition immediately leads to the formulae (6.33). Finally we need the generalization of transposition for supermatrices.
Definition: The supertransposition of a supermatrix is defined by

$$M^{ST} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{ST} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}. \tag{6.35}$$

This definition lead to the ordinary transposition law

$$(M_1M_2)^{ST} = M_2^{ST}M_1^{ST}. \tag{6.36}$$

This can be checked by explicitly calculating both sides and using (6.28). Also, one has the

Lemma 8 *The superdeterminant is invariant under supertransposition,*

$$\mathrm{S}\det(M^{ST}) = \mathrm{S}\det(M). \tag{6.37}$$

Proof: We use the formula (6.33) for the superdeterminants and find

$$\mathrm{S}\det(M^{ST}) = \frac{\det\left(A^T + C^T(D^T)^{-1}B^T\right)}{\det D^T} = \frac{\det\left(A - BD^{-1}C\right)^T}{\det D} = \mathrm{S}\det(M).$$

After we have introduced the supertrace, superdeterminant and super-transposition of supermatrices we can now proceed and define the supergroups relevant in physics. For the infinitesimal super-group transformation generated by elements of a super-Liealgebra one prefers to work with bosonic variables only. But then one needs to introduce anti-commutators in addition to commutators. The transition is similar as that from the Noether charge $\bar{\alpha}Q$ to the supercharges Q_α. Whereas $\bar{\alpha}Q$ obeys commutation relations the supercharges Q_α satisfy anti-commutation relations.

6.2 Superalgebras

The most important Lie superalgebras are the ones, which have no nontrivial invariant subalgebra. These simple finite-dimensional superalgebras are fully classified [34]. There are eight infinite families

$$s\ell(N|M),\quad osp(N|M),\quad P(N),\quad Q(N),\quad W(N),\quad S(N),\quad \tilde{S}(N),\quad H(N),$$

a continuum of 17-dimensional exceptional superalgebras and one exceptional superalgebra each in 31 and 40 dimensions,

$$D(2|1;\alpha),\quad G(3)\quad \text{and}\quad F(3).$$

In passing we note, that a semi-simple superalgebra is in general not the direct sum of simple superalgebras. A good review on superalgebras is [35].
The superalgebras of main interest in physics are the special linear, orthosymplectic and unitary ones. We shall now describe the relevant superalgebras in detail.

6.2.1 The special linear superalgebras $s\ell(N|M)$.

The supergroup $GL(N|M)$ consists of matrices of the form (6.27) with non-zero superdeterminant. This supergroup is not simple. But the sub-supergroup $S\ell(N|M)$ of matrices with superdeterminant one,

$$S\ell(N|M) = \left\{U \in GL(N|M)|\ \mathrm{S}\det U = 1\right\} = \left\{e^{iM} \in GL(N|M)|\ \mathrm{S}\,\mathrm{Tr}\, M = 0\right\}.$$

is simple. For $N \neq M$ the super-traceless $(N+M)\times(N+M)$ matrices form a simple $\big((N+M)^2-1\big)$-dimensional subalgebra $s\ell(N|M)$. For $m = n$ the unit matrix is super-traceless and generates a one-dimensional center of $s\ell(N|N)$. To achieve simplicity we must divide out this center and this leads to the superalgebras $ps\ell(N|N)$ with has dimension $4N^2 - 2$.
Instead of using anticommuting as well as commuting entries, it is possible to use only commuting entries. One defines the supertrace of M as in (6.19) or (6.29) but now A, B, C and D contain only commuting c-numbers. The generators of $s\ell(N|M)$ are still defined by $\mathrm{S\,Tr}\,(M) = 0$, but the composition rule is modified. Instead of a commutator one has an anticommutator for the bracket of B and C matrices with themselves and with each other. As earlier we denote this composition rule by $[M_1, M_2\}$.
The dimensions of the bosonic and fermionic subspaces for gl, sl and psl superalgebras are listed in the following table

superalgebra	$\dim(\mathcal{S})$	N_0=$\dim(\mathcal{S}_0)$	N_1=$\dim(\mathcal{S}_1)$
$g\ell(N\|M)$	$(N+M)^2$	N^2+M^2	$2NM$
$s\ell(N\|M)$	$(N+M)^2-1$	N^2+M^2-1	$2NM$
$ps\ell(N)$	$(2N)^2-2$	$2N^2-2$	$2N^2$

6.2.2 The orthosymplectic superalgebras $osp(N|M)$

The orthosymplectic supergroup $OSp(N|M)$ can be defined by considering the bilinear form

$$x^i \eta_{ij} y^j + \theta^\alpha \Omega_{\alpha\beta} \zeta^\beta = x^T \eta\, y + \theta^T \Omega\, \zeta \tag{6.38}$$

with commuting x^i $(i = 1, \dots, N)$ and y^j and anticommuting θ^α $(\alpha = 1, \dots, M)$ and ζ^β. The η_{ij} and $\Omega_{\alpha\beta}$ are symmetric real and antisymmetric real metrics, respectively. Let (x, θ) and (y, ζ) transform linearly under a supermatrix U. Thus

$$\begin{pmatrix} x \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = U \begin{pmatrix} x \\ \theta \end{pmatrix}, \qquad \begin{pmatrix} x \\ \theta \end{pmatrix}^T = (x^T, \theta^T) \longrightarrow (x^T, \theta^T) U^{ST}.$$

The supergroups $OSp(N|M)$ leave the line 'line element' (6.38) invariant which means

$$U^{ST} \begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix} U = \begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix}. \tag{6.39}$$

Writing $U = \mathbb{1} + M$ this implies the following condition for the infinitesimal transformations

$$M^{ST} \begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix} + \begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix} M = 0. \tag{6.40}$$

One may verify that the set of matrices satisfying the condition (6.40) generate a closed algebraic system: if M_1 and M_2 satisfy this condition, so does the ordinary commutator $[M_1, M_2]$. Recall, that the sub-blocks B and C are anticommuting, while A and D are commuting. Again we make the transition to commuting entries. One defines the transposition on U as in (6.35) but now A, B, C

and D contain only commuting c-numbers. The generators of $OSp(N|M)$ are still defined by (6.40), but the composition rule is modified. Instead of a commutator one has an anticommutator for the bracket of B and C matrices with themselves and with each other. As earlier we denote this composition rule by $[M_1, M_2\}$. The explicit form of (6.40) reads

$$A^T\eta + \eta A = 0, \quad D^T\Omega + \Omega D = 0, \quad C^T\Omega + \eta B = 0 \quad \text{and} \quad B^T\eta - \Omega C = 0. \tag{6.41}$$

The solution of the last two relations are

$$B = -\eta^{-1}C^T\Omega.$$

Using these relations for the submatrices A_i, B_i, C_i and D_i of $M_i, i = 1, 2$, one can explicitly check that $[M_1, M_2\}$ in (6.18) satisfies the condition (6.40). Let us recall that $Sp(M, \mathbb{R})$ is generated by real matrices D with $D^T\Omega + \Omega D = 0$ with antisymmetric and real Ω. Thus, D generates $Sp(M, \mathbb{R})$, A generates $SO(N, \mathbb{R})$. The dimension of the bosonic sector is thus $\frac{1}{2}N(N-1) + \frac{1}{2}M(M+1)$, and that of the fermionic sector NM. The total dimension of the superalgebra is therefore

$$\dim\bigl(osp(N|M)\bigr) = \frac{1}{2}\bigl[(N+M)^2 + M - N\bigr].$$

6.2.3 The unitary superalgebras $su(N|M)$

We consider the line element in superspace

$$(ds)^2 = z^\dagger\eta z + \theta^\dagger\Omega\theta. \tag{6.42}$$

and linear transformations

$$\begin{pmatrix} z \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} z \\ \theta \end{pmatrix} = U\begin{pmatrix} z \\ \theta \end{pmatrix}, \qquad (z^\dagger, \theta^\dagger) \longrightarrow (z^\dagger, \theta^\dagger)\begin{pmatrix} A^\dagger & C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix} = (z^\dagger, \theta^\dagger)U^\dagger$$

which leave the line element invariant. Writing $U = \mathbb{1} + M$ we see, that the infinitesimal symmetry transformation obey

$$M^\dagger\begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix} + \begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix}M = 0, \quad \text{where} \quad M^\dagger = \begin{pmatrix} A^\dagger & C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}. \tag{6.43}$$

Again we take commuting entries, but now complex, and define $u(N|M)$ by those ordinary complex matrices M which satisfy (6.43). The explicit form of this condition reads

$$A^\dagger\eta + \eta A = 0, \quad D^\dagger\Omega + \Omega D = 0, \quad C^\dagger\Omega + \eta B = 0 \quad \text{and} \quad B^\dagger\eta - \Omega C = 0. \tag{6.44}$$

The solution of the last two relations are

$$B = -\eta^{-1}C^\dagger\Omega, \qquad \eta^\dagger = \eta, \quad \Omega^\dagger = -\Omega.$$

The superalgebra $su(N|M)$ is obtained by retaining only the generators with vanishing supertrace. Since $\mathrm{S\,Tr}[M_1, M_2\} = 0$ always, these matrix form also a closed algebraic system called $su(N|M)$.

6.2.4 Further superalgebras

The elements of $s\ell(N+1|N+1)$ defined by matrices of the form (now we work with commuting entries)

$$M = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \quad \text{with} \quad \operatorname{Tr} A = 0,\ B = B^T,\ C = -C^T$$

form a sub-superalgebra denoted by $P(N)$. The dimension of the superalgebra $P(N)$ is $2(N+1)^2 - 1$.
$Q(N)$ is the $2(N+1)^2 - 1$-dimensional sub-superalgebra of $s\ell(N+1|N+1)$ defined by

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad \text{with} \quad \operatorname{S}\operatorname{Tr} M = 0,\ \operatorname{Tr} B = 0.$$

Just as for ordinary Lie algebras, there exist superalgebras in some particular dimensions. They are called *exceptional*. I refer to the literature for the definition and discussion of the exceptional superalgebras $D(2|1;\alpha), G(3)$ and $F(4)$. The dimensions of these superalgebras are

$$\dim\big(D(2|1;\alpha)\big) = 9+8 = 17, \quad \dim\big(G(3)\big) = 17+14 = 31, \quad \dim\big(F(4)\big) = 30+10 = 40.$$

Besides there are the superalgebras of *Cartan type*. These are $W(N), S(N), \tilde{S}(N)$ and $H(N)$.
Note, that in any spacetime dimensions the ordinary Poincaré algebra is not even semisimple. So one may ask why one puts so much attention on simple Lie superalgebras. But similarly as for the bosonic Poincaré algebra one can obtain the super Poincaré algebra as Wigner-Inönü contraction from the simple anti de-Sitter algebra.

6.3 Superalgebras containing spacetime algebras

The superalgebras of interest contain spacetime algebras. There are four spacetime algebras of interest for supergravity and supersymmetry. These are:

- The *Poincaré group* which leave the line element $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ invariant. The Poincaré algebra has been discussed in section (2.1).
- The *conformal group*, the group of transformations of x^μ which leave $ds^2 = 0$ invariant. In contains, in addition to the Poincaré transformations the dilatations and special conformal transformations. The corresponding algebra has been investigated in section (4.2).
- The *anti-de Sitter group*, $SO(2, d-1)$. This is the group of symmetries of a maximally symmetric spacetime with constant negative curvature. It admits $\frac{1}{2}d(d+1)$ Killing fields which generate a $so(2, d-1)$ algebra. This group and its algebra has been discussed in section (2.4).
- The *de-Sitter group*, $SO(1, d)$. This is the group of a maximally symmetric spacetime, whose constant curvature is positive. The $\frac{1}{2}d(d+1)$ Killing fields generate a $so(1, d)$ algebra.

Compact	Real Form	Maximal compact subalg.
$SU(N)$	$SU(p, N-p)$	$SU(p) \times SU(N-p) \times U(1)$
$SU(N)$	$S\ell(N)$	$SO(N)$
$SU(2N)$	$SU^*(2N)$	$USp(2N)$
$SO(N)$	$SO(p, N-p)$	$SO(p) \times SO(q)$
$SO(2N)$	$SO^*(2N)$	$U(N)$
$USp(2N)$	$Sp(2N)$	$U(N)$
$USp(2N)$	$USp(2p, 2N-2p)$	$USp(2p) \times USp(2N-2p)$
$G_{2,-14}$	$G_{2,2}$	$SU(2) \times SU(2)$
$F_{4,-52}$	$F_{4,-20}$	$SO(9)$
$F_{4,-52}$	$F_{4,4}$	$USp(6) \times SU(2)$
$E_{6,-78}$	$E_{6,-26}$	$F_{4,-52}$
$E_{6,-78}$	$E_{6,-14}$	$SO(10) \times SO(2)$
$E_{6,-78}$	$E_{6,2}$	$SU(6) \times SU(2)$
$E_{6,-78}$	$E_{6,6}$	$USp(8)$
$E_{7,-133}$	$E_{7,-25}$	$E_{6,-78} \times SO(2)$
$E_{7,-133}$	$E_{7,-5}$	$SO(12) \times SU(2)$
$E_{7,-133}$	$E_{7,7}$	$SU(8)$
$E_{8,-248}$	$E_{8,-24}$	$E_{7,-133} \times SU(2)$
$E_{8,-248}$	$E_{8,8}$	$SO(16)$

Table 6.1: Real forms of simple bosonic Lie algebras.

The de Sitter and Anti-de Sitter spacetimes also admit conformal Killing vector fields, and in both cases the conformal group is $SO(2, d)$, as it is in Minkowski spacetime. At this point it might be useful to recapitulate the real forms of bosonic algebras of 'classical type'. That is done in table (6.1) (taken from [11]) The conventions which has been used for groups is that $Sp(2N) = Sp(2N, \mathbb{R})$ (always even entry), and $USp(2N, 2M) = U(N, M, \mathbb{H})$. $S\ell(N)$ is $S\ell(N, \mathbb{R})$. Further, $SU^*(2N) = S\ell(N, \mathbb{H})$ and $SO^*(2N) = O(N, \mathbb{H})$. Here it maybe useful to recall the Cartan classification of the classical groups:

groups	A_N	B_N	C_N	D_N
complexified group	$S\ell(N+1, \mathbb{C})$	$Spin(2N+1, \mathbb{C})$	$Sp(2N, \mathbb{C})$	$Spin(2N, \mathbb{C})$
compact form	$SU(N+1, \mathbb{C})$	$Spin(2N+1, \mathbb{R})$	$U(N, \mathbb{H})$	$Spin(2N, \mathbb{R})$

In these bosonic algebras there are isomorphisms, some of which will be important later. To discuss these, it is of help to recall the Dynkin diagrams of the classical groups. These are depicted in figure (6.1).

The isomorphisms one needs in supersymmetry and supergravity are:

$$B_1 \sim A_1 : \qquad SO(3) \simeq SU(2), \quad SO(2,1) \simeq SU(1,1) \simeq Sp(2, \mathbb{R}) \simeq Sl(2)$$
$$D_2 \sim A_1 \times A_1 : \qquad SO(4) \simeq SU(2) \times SU(2), \quad SO(2,2) \simeq SU(1,1) \times SU(1,1)$$

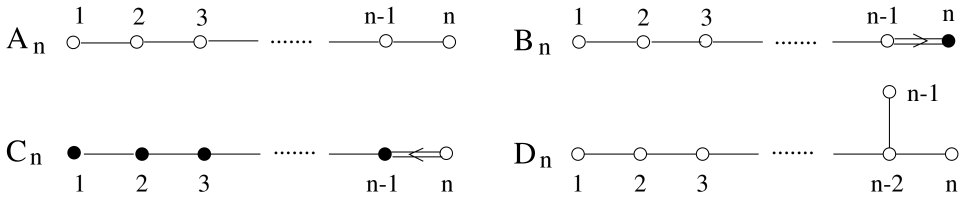

Figure 6.1: *Dynkin diagrams of the A_N, B_N, C_N and D_N series. $\bullet$ are short roots.*

$$
\begin{array}{ll}
B_2 \sim C_2 : & SO(5) \simeq USp(4), \quad SO(4,1) \simeq USp(2,2), \quad SO(2,3) \simeq Sp(4,\mathbb{R}) \\
D_3 \sim A_3 : & SO(6) \simeq SU(4), \quad SO(1,5) \simeq SU^*(4) \\
 & SO(2,4) \simeq SU(2,2), \quad SO(3,3) \simeq S\ell(4)
\end{array}
$$

Note that the equality sign is not correct for the groups. For the algebras there are these isomorphisms, but it is rather the covering group of the orthogonal groups which are listed at the right hand sides.
It is remarkable that for the Poincaré, conformal, de Sitter and anti-de Sitter algebras super-extensions exist. I will not discuss the de Sitter superalgebras, because the corresponding supergravity theories (which do exist) contain ghosts [33]. The anti-de Sitter algebras are free from ghosts. The superalgebras which correspond to supersymmetric theories can now be found by looking up which superalgebra contains a given spacetime group. According to the spin-statistic theorem the odd generators (the columns of C and rows of B) must transform under the spacetime group as spinors. This means that the spacetime generators must be in the spinor representations of the $SO(p,q)$ groups. The supergroups containing the AdS and conformal algebras as spacetime symmetries in $2, 4$ and 5 dimensions are

dimensions	spacetime-group	supergroup	remarks
$d=2$	adS	$OSp(N\|2)$	
	conformal	$OSp(N\|2) \times OSp(N\|2)$	
$d=4$	adS	$OSp(N\|4)$	$Sp(4) \simeq SO(3,2)$ is adS
	conformal	$SU(N\|2,2)$	$SU(2,2) \simeq SO(4,2)$
$d=5$	adS	$SU(N\|2,2)$	complex spinors
	conformal	F_4?	

In the appendix to this chapter I give a list of superalgebras of 'classical type' (taken from [11]). A superalgebra is of classical type if the representation of $\mathcal{S}_0$, according to which the fermionic generators transform, is completely reducible. The non-classical superalgebras are the Cartan-type superalgebras $W(N), S(N), \tilde{S}(N)$ and $H(N)$.

6.4 Physically relevant examples of superalgebras

Let us see in more detail, how to construct supergroups and superalgebras explicitly. We shall investigate the important anti-de Sitter and Poincaré superalgebras in detail.

6.4.1 Anti-de Sitter algebras in $2, 4$ and 5 dimensions

The Poincaré algebra is not a simple algebra, but a Inönü-Wigner contraction of the simple anti-de Sitter algebra $SO(2, d-1)$. The anti-de Sitter algebra is

$$[M_{mn}, M_{pq}] = i\big(\eta_{mp}M_{nq} + \eta_{nq}M_{mp} - \eta_{mq}M_{np} - \eta_{np}M_{mq}\big), \qquad m, n = 0, \ldots d, \tag{6.45}$$

where $\eta_{mn} = \mathrm{diag}(+ - \ldots - +)$. Setting

$$M_{d\mu} = -M_{\mu d} \equiv RP_{\mu}, \quad \text{where} \quad \mu = 0, \ldots d-1 \tag{6.46}$$

and denoting the remaining components of M_{mn} by $M_{\mu\nu}$, this algebra takes the form

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i\big(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\big) \\ [P_{\rho}, M_{\mu\nu}] &= -i\big(\eta_{\rho\mu}P_{\nu} - \eta_{\rho\nu}P_{\mu}\big) \\ [P_{\mu}, P_{\nu}] &= \frac{i}{R^2}M_{\mu\nu}. \end{aligned} \tag{6.47}$$

The Inönü-Wigner contraction is the statement that the $SO(2, d-1)$ algebra reduces to the Poincaré algebra when we take the limit $R \to \infty$ with constant P_{μ} in the limit. The same applies to the corresponding superalgebras as we shall see later.
For the following calculations it is useful to recall the formulae we have derived in (3.26)

$$\Big(\mathcal{C}^{-1}\gamma^{(n)}\Big)^T = -\varepsilon(-)^{[n/2]}(-\eta)^n\mathcal{C}^{-1}\gamma^{(n)}. \tag{6.48}$$

We also need the formulae (3.53) and (3.64). But now ψ and χ are commuting objects, so that

$$\bar{\psi}\gamma^{(n)}\chi = -\epsilon(-)^{n(n+1)/2}\eta^t\ \bar{\chi}\gamma^{(n)}\psi \tag{6.49}$$

and

$$\Delta\psi\bar{\chi} = \sum_{p=0}^{D}\frac{(-)^{p(p-1)/2}}{p!}\gamma_{\mu_1\ldots\mu_p}\ \big(\bar{\chi}\gamma^{\mu_1\cdots\mu_p}\psi\big). \tag{6.50}$$

Remember, that $\Delta = 2^{[d/2]}$ and $D = d$ for even d and $D = (d-1)/2$ for odd dimensions. If we choose Majorana spinors, then we take $B = \mathcal{C}A^T = \mathbb{1}$ such that Majorana spinors are real. For an antisymmetric charge conjugation matrix this implies $\mathcal{C}^{-1} = -A$ and $\bar{\psi} = -\psi^T\mathcal{C}^{-1}$.

Two spacetime dimensions

First we consider the anti-de Sitter spacetime in 2 spacetime dimensions. The corresponding supergroup is $OSp(N, 2)$. According to (6.41) the generators of $sp(2, \mathbb{R})$ satisfy

$$D^T\Omega + \Omega D = -(\Omega D)^T + \Omega D = 0, \qquad \Omega^T = -\Omega, \tag{6.51}$$

i.e. the real matrices ΩD are symmetric. Let us relate $sp(2,\mathbb{R})$ to the isomorphic $so(2,1)$. Consulting our tables in section 3 we see that there exists a Majorana representation of $SO(2,1)$. We take $\mathcal{C} = \gamma_0\gamma_2$ such that Majorana spinors are real. Since $\epsilon = \eta = 1$ we have

$$\mathcal{C}^T = -\mathcal{C}, \quad (\mathcal{C}^{-1}\gamma_m)^T = \mathcal{C}^{-1}\gamma_m, \quad (\mathcal{C}^{-1}\gamma_{mn})^T = \mathcal{C}^{-1}\gamma_{mn}, \quad \gamma_{mn} = \tfrac{1}{2}[\gamma_m, \gamma_n], \tag{6.52}$$

where the indices m, n take the values $0, 1$ and 2. It follows in particular, that

$$\gamma_0^T = \gamma_0, \quad \gamma_1^T = -\gamma_1 \quad \text{and} \quad \gamma_2^T = \gamma_2.$$

From the known hermiticity properties of the gamma-matrices we conclude, that the γ_m and hence the charge conjugation matrix $\mathcal{C}$ are all real. The real $\frac{1}{2}\gamma_{mn}$ satisfy the $SO(2,1)$ algebra (6.45) with i on the right hand side replaced by -1,

$$[\tfrac{1}{2}\gamma_{mn}, \tfrac{1}{2}\gamma_{pq}] = -\tfrac{1}{2}\big(\eta_{mp}\gamma_{nq} + \eta_{nq}\gamma_{mp} - \eta_{mq}\gamma_{np} - \eta_{np}\gamma_{mq}\big), \quad \eta = \mathrm{diag}(1,-1,1). \tag{6.53}$$

The following matrices form a Majorana representation of $so(2,1)$:

$$\gamma_0 = \sigma_1, \quad \gamma_1 = -i\sigma_2, \quad \gamma_2 = \sigma_3 \Longrightarrow \{\gamma_m, \gamma_n\} = 2\eta_{mn}\mathbb{1}_2, \quad \eta_{mn} = \mathrm{diag}(1,-1,1)$$

In this representation the generators of $SO(2,1)$ have the form

$$\{\gamma_{01}, \gamma_{02}, \gamma_{12}\} = \{\sigma_3, -i\sigma_2, \sigma_1\},$$

and the charge conjugation matrix reads

$$\mathcal{C} = \gamma_0\gamma_2 = -i\sigma_2.$$

Now we compare (6.51) with (6.52) we conclude that we may choose $\Omega = \mathcal{C}^{-1}$ in which case D is a linear combination of the $SO(2,1)$ generators with real coefficients,

$$D = \tfrac{1}{2}\omega^{mn}\gamma_{mn}, \qquad \Omega = \mathcal{C}^{-1} \tag{6.54}$$

Now we choose $\eta = \mathbb{1}$ in (6.41). Then A is real and antisymmetric, that is in the Lie algebra of $SO(N)$. Hence

$$A = \alpha^{ij}\Omega_{ij}, \quad (\Omega_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \quad \text{e.g.} \quad \Omega_{12} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.55}$$

The generators Ω_{ij} fulfill the $so(N)$ algebra, that is the algebra (6.45) with i on the right hand side replaced by -1 and η replaced by δ. Now we take as generators of the bosonic sector of the super Lie algebra

$$M_{mn} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\gamma_{mn} \end{pmatrix} \quad \text{and} \quad T_{ij} = \begin{pmatrix} \Omega_{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.56}$$

Clearly, the M_{mn} generate the $so(2,1)$ subalgebra and the T_{ij} the $so(N)$ subalgebra of $\mathcal{S}_0$. The generators of $so(2,1)$ commute with those of $so(N)$ such that $\mathcal{S}_0 = so(2,1) \oplus so(N)$.

Now we turn to the fermionic generators. According to (6.41) we have $B = -C^T\mathcal{C}^{-1}$ so that these generators have the form

$$M = \begin{pmatrix} 0 & -C^T\mathcal{C}^{-1} \\ C & 0 \end{pmatrix} \in \mathcal{S}_1.$$

Here C is a general rectangular $2 \times N$ matrix and therefore a linear combination of $2N$ basic generators with all but one entries equals zero. We choose as basis elements the matrices

$$e_\alpha \otimes e_i^T, \tag{6.57}$$

where the 2-dimensional column-vector e_α has entries $(e_\alpha)_\beta = \delta_{\alpha\beta}$ and the N-dimensional column vector e_i has entries $(e_i)_j = \delta_{ij}$. and correspondingly the following generators of the fermionic subspace

$$Q_\alpha^i \quad = \quad \begin{pmatrix} 0 & -e_\alpha^T\mathcal{C}^{-1} \otimes e_i \\ e_\alpha \otimes e_i^T & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \bar{e}_\alpha \otimes e_i \\ e_\alpha \otimes e_i^T & 0 \end{pmatrix}. \tag{6.58}$$

We used, that $-e_\alpha^T\mathcal{C}^{-1} = \bar{e}_\alpha$. The commutators of the bosonic with these fermionic generators are

$$[M_{mn}, Q_\alpha^i] = \tfrac{1}{2}(\gamma_{mn}Q^i)_\alpha, \quad [T_{ij}, Q_\alpha^i] = -(Q_\alpha\Omega)^i. \tag{6.59}$$

This shows, that the real $Q^1, \dots, Q^m$ transform according to to the spin $\frac{1}{2}$-representation of the $so(2,1)$. These Majorana charges do not mix under spin rotations. On the other hand, the $so(N)$ transforms the Q^i without affecting the spinor index α. They are rotated into each other with the N-dimensional defining representation. The anticommutator of two Q is easily found to be

$$\{Q_\alpha^i, Q_\beta^j\} = \begin{pmatrix} -\mathcal{C}_{\alpha\beta}^{-1}\left(e_ie_j^T - e_je_i^T\right) & 0 \\ 0 & \delta^{ij}\left(e_\alpha\bar{e}_\beta + e_\beta\bar{e}_\alpha\right) \end{pmatrix}. \tag{6.60}$$

Now it is easy to calculate the anticommutator of two fermionic generators for $i \neq j$. One finds

$$\{Q_\alpha^i, Q_\beta^j\} = -(\mathcal{C}^{-1})_{\alpha\beta}T_{ij}. \tag{6.61}$$

To calculate the anticommutator for $i = j$ we use (6.50) which implies for $so(1,2)$ the Fierz identity

$$2e_\alpha\bar{e}_\beta = \bar{e}_\alpha e_\beta \mathbb{1} + \gamma_m(\bar{e}_\alpha\gamma^m e_\beta) \Longrightarrow (e_\alpha\bar{e}_\beta + e_\beta\bar{e}_\alpha) = -(\mathcal{C}^{-1}\gamma^m)_{\alpha\beta}\gamma_m.$$

Now we use that $\gamma^0\gamma^1\gamma^2 = \pm\mathbb{1}$ which implies

$$\gamma_m = \pm\epsilon_{mpq}\gamma^{pq} \ (\epsilon_{012} = 1) \quad \text{so that} \quad (-\mathcal{C}^{-1}\gamma^m)_{\alpha\beta}\gamma_m = \tfrac{1}{2}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}\gamma_{mn}.$$

This implies the following anticommutator for $i = j$:

$$\{Q_\alpha^i, Q_\beta^i\} = (\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn}. \tag{6.62}$$

For contracting the super-AdS algebra to the super-Poincaré algebra, we set

$$M_{20} = RP_0 \quad \text{and} \quad M_{21} = RP_1$$

so that the anti-de Sitter algebra reads

$$[M_{01},P_0]=iP_1,\quad [M_{01},P_1]=-iP_0,\quad [P_0,P_1]=iR^{-2}M_{01}. \tag{6.63}$$

For $R\to\infty$ it contracts to the Poincaré algebra, as discussed above. Now, in addition, we replace Q_α by $Q_\alpha/\sqrt{R}$ in which case

$$\begin{aligned} \{Q^i_\alpha,Q^j_\beta\} &= 2\delta^{ij}\big\{(\mathcal{C}^{-1}\gamma^2\gamma^\mu)_{\alpha\beta}P_\mu+\tfrac{1}{R}(\mathcal{C}^{-1}\gamma^{01})_{\alpha\beta}M_{01}\big\}-\tfrac{1}{R}(\mathcal{C}^{-1})_{\alpha\beta}T_{ij}\\ [P_\mu,Q^i_\alpha] &= \tfrac{1}{2R}(\gamma_{2\mu}Q^i)_\alpha,\quad [M_{01},Q^i_\alpha]=\tfrac12(\gamma_{01}Q^i)_\alpha,\quad [T_{ij},Q^k_\alpha]=-(Q_\alpha\Omega_{ij})^k. \end{aligned} \tag{6.64}$$

Now we perform the limit $R\to\infty$ in which case (6.63) contracts to the Poincaré algebra as expected. Using that $\mathcal{C}^{-1}\gamma^2=-\gamma^0=-\mathcal{C}_2$ is, up to a sign, the charge conjugation matrix in 2-dimensional Minkowski space (with $\eta=\epsilon=-1$), the anticommutation relations in (6.64) simplify to

$$\begin{aligned} \{Q^i_\alpha,Q^j_\beta\} &= -2\delta^{ij}(\mathcal{C}_2\gamma^\mu)_{\alpha\beta}P_\mu,\quad [P_\mu,Q^i_\alpha]=0\\ [M_{01},Q^i_\alpha] &= \tfrac12(\gamma_{01}Q^i)_\alpha,\quad [T_{ij},Q^k_\alpha]=-(Q_\alpha\Omega_{ij})^k. \end{aligned} \tag{6.65}$$

We have found a superalgebra extending the Poincaré algebra consisting of N Majorana charges Q^i. The Q^i are transformed into each other by the internal $SO(N)$ transformation. The $so(N)$ rotate the Majorana supercharges Q^i. This automorphism group is called *R-symmetry.* In accordance with the Coleman-Mandula theorem the R-symmetry commutes with the Poincaré transformations.

Four spacetime dimensions

Again we must solve $D^T\Omega+\Omega D=0$ with a real 4-dimensional, nondegenerate and antisymmetric matrix Ω. This condition is equivalent to

$$(\Omega D)^T=\Omega D\Longrightarrow D\in sp(4). \tag{6.66}$$

To relate $Sp(4)$ to $SO(2,3)$ we recall, that $SO(2,3)$ admits a Majorana representation with $\epsilon=1$ and $\eta=-1$. From (6.48) it follows, that

$$\mathcal{C}=-\mathcal{C}^T,\quad (\mathcal{C}^{-1}\gamma^m)^T=-\mathcal{C}^{-1}\gamma^m\quad\text{and}\quad(\mathcal{C}^{-1}\gamma_{mn})^T=\mathcal{C}^{-1}\gamma_{mn},\qquad m=0,\dots,4. \tag{6.67}$$

The γ_{mn} satisfy the $SO(2,4)$-algebra (6.53) with $\eta_{mn}=$diag$(1,-1,-1,-1,1)$. As charge conjugation matrix we take

$$\mathcal{C}=\gamma_0\gamma_4 \tag{6.68}$$

such that Majorana spinors are real. Then

$$\gamma_0^T=-\gamma_0,\quad \gamma_4^T=-\gamma_4$$

which implies, that the γ^m are imaginary. For example, we may choose the following explicit

Majorana representation

$$\gamma^0 = \sigma_0 \otimes \sigma_2, \quad \gamma^1 = i\sigma_0 \otimes \sigma_3, \quad \gamma^2 = i\sigma_1 \otimes \sigma_1, \quad \gamma^3 = i\sigma_3 \otimes \sigma_1 \quad \text{and} \quad \gamma^4 = \sigma_2 \otimes \sigma_1$$

in which case $\mathcal{C} = -i\sigma_2 \otimes \sigma_3$. Comparing (6.66) with (6.67) we see, that we can choose

$$\Omega = \mathcal{C}^{-1} \quad \text{and} \quad D = \tfrac{1}{2}\omega^{mn}\gamma_{mn}, \qquad \omega^{mn} \in \mathbb{R}.$$

This identification yields the isomorphism between $sp(4)$ and $so(2,3)$. Choosing $\eta = \mathbb{1}$ in the other constraint in (6.41) we conclude that $A \in so(N)$ and hence

$$A = \alpha^{ij}\Omega_{ij}, \quad (\Omega_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{6.69}$$

As in two dimensions we take the following basis of the bosonic subalgebra

$$M_{mn} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\gamma_{mn} \end{pmatrix} \quad \text{and} \quad T_{ij} = \begin{pmatrix} \Omega_{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.70}$$

The M_{mn} satisfy the $so(2,3)$ commutation relations whereas the T_{ij} generate $so(N)$. These two subalgebras commute, in accordance with the Coleman-Mandula theorem. As basis for the fermionic subsector we again choose

$$Q^i_\alpha = \begin{pmatrix} 0 & -e^T_\alpha \mathcal{C}^{-1} \otimes e_i \\ e_\alpha \otimes e^T_i & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \bar{e}_\alpha \otimes e_i \\ e_\alpha \otimes e^T_i & 0 \end{pmatrix}, \tag{6.71}$$

where now e_α is the 4-dimensional column vector with matrix elements $(e_\alpha)_\beta = \delta_{\alpha\beta}$ and we used $\bar{e}_\alpha = -e^T_\alpha \mathcal{C}^{-1}$. The commutator of the bosonic with the fermionic generators are

$$[M_{mn}, Q^i_\alpha] = \tfrac{1}{2}(\gamma_{mn}Q^i)_\alpha, \quad [T_{ij}, Q^k_\alpha] = -(Q_\alpha \Omega)^k. \tag{6.72}$$

As expected, the 4-dimensional real column vectors $Q^1, \dots, Q^N$ transform as Majorana spinors under the spacetime group $SO(2,3)$ and the N-dimensional real row vectors $Q_1, \dots Q_4$ transform as vectors under the group $SO(N)$. As in the 2-dimensional case the R-symmetry group is $SO(N)$. To calculate the anticommutator of two fermionic charges we use

$$4(e_\alpha \bar{e}_\beta + e_\beta \bar{e}_\alpha) = \gamma_{mn}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}$$

which follows from (6.50) and the (anti)symmetry properties of the matrices $\mathcal{C}^{-1}\gamma^{(n)}$. Therefore the anticommutator of two supercharges is given by

$$\{Q^i_\alpha, Q^j_\beta\} = \tfrac{1}{2}\delta^{ij}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn} - (\mathcal{C}^{-1})_{\alpha\beta}T_{ij}. \tag{6.73}$$

Now we perform the $4+1$-split and rescale the supercharges and $M_{d\mu}$ and end up with the algebra

$$\{Q^i_\alpha, Q^j_\beta\} = (\mathcal{C}^{-1}\gamma^4\gamma^\mu)_{\alpha\beta}\delta^{ij}P_\mu + \tfrac{1}{2R}(\mathcal{C}^{-1}\gamma^{\mu\nu})_{\alpha\beta}\delta^{ij}M_{\mu\nu} - \tfrac{1}{R}(\mathcal{C}^{-1})_{\alpha\beta}T_{ij}. \tag{6.74}$$

Since $\mathcal{C}^{-1}\gamma^4 = -\gamma^0 = \mathcal{C}_4$ is the antisymmetric charge conjugation matrix in 4-dimensional Minkowski spacetime we obtain in the limit $R \to \infty$ the Poincaré algebra plus the following (anti)commutators:

$$
\begin{aligned}
\{Q^i_\alpha, Q^j_\beta\} &= (\mathcal{C}_4\gamma^\mu)_{\alpha\beta}\delta^{ij}P_\mu, \qquad [P_\mu, Q^i_\alpha] = 0, \qquad [T_{ij}, Q^k_\alpha] = -(Q_\alpha\Omega_{ij})^k \\
[M_{\mu\nu}, Q^i_\alpha] &= \tfrac{1}{2}(\gamma_{\mu\nu}Q^i)_\alpha, \qquad [M_{\mu\nu}, T_{ij}] = [P_\mu, T_{ij}] = 0.
\end{aligned}
\tag{6.75}
$$

Together with the Poincaré algebra we have derived the (anti)commutation for supersymmetric theories with N-Majorana supercharges. Later we shall restrict the values of N by physical arguments. For the simplest $N = 1$ theories there are no additional generators besides $M_{\mu\nu}, P_\mu$ and Q_α and we rediscover the supersymmetry algebra of the corresponding Noether charges in the Wess-Zumino model.

Five spacetime dimensions: AdS_5

I discuss this case since it plays an important role in recent developments in field theory. The so-called AdS/CFT-correspondence is base on the relation between supersymmetric theories on AdS_5 and conformal field theories on the boundary of AdS_5 which can be identified with Minkowski spacetime. The symmetry algebra of AdS_5 is the same as the conformal algebra in Minkowski spacetime, namely $so(2,4)$.
There is a Majorana representation of $SO(2,4)$ and it has $(\epsilon,\eta) = (1,-1)$. Then the 8-dimensional γ-matrices and charge conjugation have the following symmetry properties,

$$
(\mathcal{C}^{-1}\gamma^{n)})^T = -(-)^{[n/2]}\mathcal{C}^{-1}\gamma^n, \qquad n = 0,\dots,6. \tag{6.76}
$$

It follows

$$
\mathcal{C}^{-1}\gamma_{mn}, \quad \mathcal{C}^{-1}\gamma_{mnp} \quad \text{and} \quad \mathcal{C}^{-1}\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 \equiv \mathcal{C}^{-1}\gamma_* \tag{6.77}
$$

are symmetric[2] and the others are antisymmetric. Note that $\gamma_*^2 = -\mathbb{1}$ and $\gamma_*^T = -\gamma_*$. Choosing $CA^T = \mathbb{1}$, such that Majorana spinors are real, we find

$$
C = \gamma_0\gamma_5 = -\mathcal{C}^T = -\mathcal{C}^{-1} = -\gamma_5\gamma_0, \quad \gamma_0^T = \mathcal{C}^{-1}\gamma_0\mathcal{C} = -\gamma_0. \tag{6.78}
$$

Hence, γ_0 and γ_5 are antisymmetric, whereas the others γ_i are symmetric. All γ_m are imaginary in this representation.
We want to use the isomorphism $SO(2,4) \sim SU(2,2)$ and construct the superalgebra $su(N|2,2)$ which contains the anti-de Sitter algebra as subalgebra of the bosonic algebra $su(N) \times su(2,2) \times u(1)$. If we want to use the 8-dimensional real Majorana representation then we should use the 8-dimensional real forms of $su(2,2)$ and $su(N)$. For that one notes, that a unitary matrix $U = \Re U + i\Im U \in U(N)$ can be represented as

$$
R = \begin{pmatrix} \Re U & -\Im U \\ \Im U & \Re U \end{pmatrix} \in SO(2N).
$$

To see that we note that $UU^\dagger = \mathbb{1}$ implies

$$
\Re U(\Re U)^T + \Im U(\Im U)^T = \mathbb{1}, \quad \Re U(\Im U)^T - \Im U(\Re U)^T = 0
$$

[2]For convenience we have chosen a choice for γ_* which differs from the choice we made in section 3.

so that R is orthogonal, $R^TR = \mathbb{1}$. Correspondingly the Lie algebra $u(N)$ can be represented by matrices

$$A = \begin{pmatrix} a & -s \\ s & a \end{pmatrix} = \sigma_0 \otimes a - i\sigma_2 \otimes s, \quad \text{where} \quad a^T = -a, \quad s^T = s.$$

An equivalent way to characterize the real form of the Lie algebra of $u(N)$ is

$$u(N) \sim \{A \in \mathrm{Mat}_{2N} \,|\, A^T = -A,\ A\Omega = \Omega A\}, \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix}. \tag{6.79}$$

Note that Ω itself is element of the subalgebra and commutes with all elements. Hence we may decompose $s(N) = su(N) \times U(1)$ as follows

$$su(N) = \{A \in u(N) \,|\, \mathrm{Tr}\,\Omega A = 0\}, \qquad u(1) = \{A = \alpha\Omega\}. \tag{6.80}$$

In other words, for the generators of $su(N)$ the symmetric matrix s is traceless. With similar arguments one shows that there exists a 8-dimensional real form $u(2,2)$. In this representation the elements have the form

$$u(2,2) = \{D \in Mat_8 | (\mathcal{C}^{-1}D)^T = \mathcal{C}^{-1}D,\ \gamma_* D = D\gamma_*\}. \tag{6.81}$$

Instead of repeating the arguments leading to the real form of $u(N)$ we show, that the infinitesimal generators satisfy the $so(2,4) \times so(2) \sim u(2,2)$ commutation relation. According to (6.77) the matrix D must be a linear combination of

$$\gamma^{mn}, \quad i\gamma^{mnp} \quad \text{and} \quad \gamma_*.$$

But the product of three gamma-matrices does not commute with γ_* and hence D must be a linear combination of the real matrices γ^{mn} and γ_*. Since γ_* commutes with the γ^{mn} and the latter generate $so(2,4)$ this proves (6.81). Now we are ready to define our bosonic generators. For that we define the following traceless matrices

$$a_{ij} = e_i e_j^T - e_j e_i^T \quad \text{and} \quad s_{ij} = e_i e_j^T + e_j e_i^T - \frac{2}{N}\delta_{ij}\,\mathbb{1}_N \tag{6.82}$$

and the associated generators of $su(N)$:

$$A_{ij} = \begin{pmatrix} \sigma_0 \otimes a_{ij} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S_{ij} = \begin{pmatrix} -i\sigma_2 \otimes s_{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.83}$$

As generators of $su(2,2)$ we take

$$M_{mn} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\gamma_{mn} \end{pmatrix}. \tag{6.84}$$

As further generator which commutes with $su(N)$ and with $su(2,2)$ we take

$$S=\frac{1}{N}\begin{pmatrix}\Omega & 0\\ 0 & 0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0 & 0\\ 0 & \gamma_*\end{pmatrix}. \tag{6.85}$$

In the complex form this would correspond to the $u(1)$ generator with vanishing supertrace. The real bosonic generators

$$\{A_{ij},S_{ij},S,M_{mn}\}$$

generate the sub Lie algebra $su(N)\times u(1)\times su(2,2)$. Now we come the the fermionic generators. Guided by our earlier experience or by rewriting the complex into the real form one is lead to the following form for these generators:

$$Q^i_\alpha=\begin{pmatrix}0 & 0 & -e_\alpha^T\mathcal{C}^{-1}\otimes e_i\\ 0 & 0 & -(\gamma_* e_\alpha)^T\mathcal{C}^{-1}\otimes e_i\\ e_\alpha\otimes e_i^T & \gamma_* e_\alpha\otimes e_i^T & 0\end{pmatrix}. \tag{6.86}$$

The e_i are the usual basis column vectors in $\mathbb{R}^N$ and the e_α the basis column vectors in $\mathbb{R}^8$. When calculating the anticommutator of Q^i_α with Q^j_β one encounters the following expressions: The $u(N)$ part in the anticommutator is

$$-(\mathcal{C}^{-1})_{\alpha\beta}\sigma_0\otimes a_{ij}-i(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}\sigma_2\otimes\big(s_{ij}+\tfrac{2}{N}\delta_{ij}\mathbb{1}\big),$$

whereas the $u(2,2)$ part reads

$$-\delta_{ij}(e_\alpha e_\beta^T+e_\beta e_\alpha^T)\mathcal{C}^{-1}+\delta_{ij}\gamma_*(e_\alpha e_\beta^T+e_\beta e_\alpha^T)\gamma_*\mathcal{C}^{-1}.$$

Now one uses the following Fierz identity for the $so(2,4)$ Clifford algebra:

$$(e_\alpha e_\beta^T+e_\beta e_\alpha^T)\mathcal{C}^{-1}=-\gamma_{mn}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}-\tfrac{1}{3}\gamma_{mnp}(\mathcal{C}^{-1}\gamma^{mnp})_{\alpha\beta}-2\gamma_*(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta} \tag{6.87}$$

to prove, that the $U(2,2)$ part becomes

$$\tfrac{1}{4}\delta_{ij}\big\{\gamma_{mn}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}+2\gamma_*(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}\big\}.$$

Collecting our results we end up with the following anticommutator

$$\{Q^i_\alpha,Q^j_\beta\}=-(\mathcal{C}^{-1})_{\alpha\beta}A_{ij}+(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}S_{ij}+2(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}\delta_{ij}S+\tfrac{1}{2}\delta_{ij}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn} \tag{6.88}$$

The commutator of the bosonic charges with the supercharges is easily computed to be

$$\begin{aligned}{}[A_{jk},Q^i_\alpha]&=(a_{jk})_{pi}Q^p_\alpha \quad , \quad [S_{jk},Q^i_\alpha]=-(s_{jk})_{pi}(\gamma_*)_{\sigma\alpha}Q^p_\sigma\\ [S,Q^i_\alpha]&=\big(\tfrac{1}{4}-\tfrac{1}{N}\big)(\gamma_*)_{\beta\alpha}Q^i_\beta \quad , \quad [M_{mn},Q^i_\alpha]=\tfrac{1}{2}(\gamma_{mn})_{\beta\alpha}Q^i_\beta.\end{aligned} \tag{6.89}$$

Note that for $N=4$ the $U(1)$ charge S commutes with all other generators. But the anticommutator of two supercharges still contains S. Hence, for $N=4$ the anti-de Sitter superalgebra becomes non-simple. The non-extended $N=1$-superalgebra simplifies considerably since in this case the R-symmetry group is just a $U(1)$. The super-CR read

$$\{Q_\alpha, Q_\beta\} = 2(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta} S + \tfrac{1}{2}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta} M_{mn} \quad \text{and} \quad [S, Q_\alpha] = \tfrac{3}{4}(\gamma_*)_{\alpha\beta} Q_\beta. \tag{6.90}$$

This makes clear, that the $U(1)$-transformation is just a chiral rotation of the supercharges. The lefthanded charges have the opposite transformation rule as the righthanded ones.

6.4.2 Appendix: AdS_5 in chiral basis

The elements of the superalgebras $SU(N|2,2)$ have the form

$$M = \begin{pmatrix} A & -C^\dagger\Omega \\ C & D \end{pmatrix} \quad \text{with} \quad A^\dagger = -A, \quad (\Omega D)^\dagger = \Omega D, \quad \text{where} \quad \Omega^\dagger = -\Omega. \tag{6.91}$$

We see, that $A \in u(N)$. In addition, since the columns of C transform according to the 4-dimensional complex representation of $su(2,2) \sim so(2,4)$ this can only be a chiral spinor representation of $SO(2,4)$. To continue we note that

$$(\mathcal{A}\gamma^{(n)})^\dagger = -(-)^{n(n+1)/2}\mathcal{A}\gamma^{(n)}, \qquad \mathcal{A} = \gamma_0\gamma_5 = -\mathcal{A}^\dagger. \tag{6.92}$$

Since we want to a chiral representation we can only use those matrices which preserve chirality. Out of these we have the following hermitian and antihermitian matrices

$$\text{hermitian:} \quad \mathcal{A}\gamma^{mn},\ i\mathcal{A}\gamma_* \quad , \quad \text{antihermitian:} \quad \mathcal{A},\ \mathcal{A}\gamma^{mnpq}. \tag{6.93}$$

Thus we may take

$$\Omega = P_L\mathcal{A} \equiv \mathcal{A}, \quad \text{where} \quad P_L = \tfrac{1}{2}(1-\gamma_*), \quad \gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$$

and

$$D = \tfrac{1}{2}\omega^{mn}\sigma_{mn} + \tfrac{i}{4}\omega\,\mathbb{1}_4, \qquad \sigma_{mn} = P_L\gamma_{mn}$$

since $\mathcal{A}D$ is then a general hermitian matrix. Note that now we have chosen a hermitian γ_*, contrary to our choice in the Majorana basis. Also we have used that on the chiral sectors γ_* is $\pm\mathbb{1}$. Next we observe that σ_{mn} is traceless:

$$\mathrm{Tr}\,\gamma_m\gamma_n = 8\eta_{mn}, \quad \mathrm{Tr}\,\gamma_*\gamma_m\gamma_n = \mathrm{Tr}\,\gamma_*\gamma_m\gamma_n\gamma_0\gamma_0 = -\mathrm{Tr}\,\gamma_0\gamma_*\gamma_m\gamma_n\gamma_0 = -\mathrm{Tr}\,\gamma_*\gamma_m\gamma_n$$

from which immediately follows that $\mathrm{Tr}\,\sigma_{mn} = \mathrm{Tr}\,P_L\gamma_{mn} = 0$. As a consequence

$$\mathrm{Tr}\,D = i.$$

In passing we note, that

$$\mathcal{A}^2 = -\mathbb{1},\ \mathrm{Tr}\,\mathcal{A} = 0,\ \mathcal{A}^\dagger = -\mathcal{A}, \quad \text{so that} \quad \mathcal{A} \sim \mathrm{diag}\,(i, i, -i, -i)$$

which shows, that $D \in u(2,2)$. As bosonic generators of $su(N|2,2)$ we may take the following supertraceless supermatrices

$$M_{mn} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sigma_{mn} \end{pmatrix}, \quad X = iS, \quad X_{ij} = \begin{pmatrix} s_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_{ij} = \begin{pmatrix} a_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.94}$$

where the a_{ij}, s_{ij} have been introduce in the previous subsection. As earlier e_i denotes the N-dimensional column vector with $(e_i)_j = \delta_{ij}$. This shows that the bosonic subalgebra of $su(N|2,2)$ is $su(N) \times su(2|2) \times u(1)$. In particular we have

$$[X, M_{mn}] = [X, X_{ij}] = [X, Y_{ij}] = 0.$$

Note that for $N = 4$ the $U(1)$ factor is generated by the identity and hence the superalgebra is not simple in this case. To get a simple superalgebra for $N = 4$ we must not include X in the list of generators and then the bosonic subalgebra of $su(4|2,2)$ is $su(4) \times su(2,2)$.
Since $\psi^\dagger \mathcal{A} = \bar{\psi}$ we may take as fermionic generators

$$Q^i_\alpha = \begin{pmatrix} 0 & -\bar{e}_\alpha \otimes e_i \\ e_\alpha \otimes e_i^T & 0 \end{pmatrix} \quad \text{and} \quad \tilde{Q}^i_\alpha = \begin{pmatrix} 0 & i\bar{e}_\alpha \otimes e_i \\ ie_\alpha \otimes e_i^T & 0 \end{pmatrix}. \tag{6.95}$$

The 4-dimensional column vector e_α has entries $(e_\alpha)_\beta = \delta_{\alpha\beta}$. The bosonic generators M_{mn} satisfy the $so(2,4)$ commutation relations. In addition

$$\begin{aligned}
[M_{mn}, Q^i_\alpha] &= \tfrac{1}{2}(\Re\sigma_{mn})_{\beta\alpha} Q^i_\beta + \tfrac{1}{2}(\Im\sigma_{mn})_{\beta\alpha}\tilde{Q}^i_\beta \\
[M_{mn}, \tilde{Q}^i_\alpha] &= \tfrac{1}{2}(\Re\sigma_{mn})_{\beta\alpha} \tilde{Q}^i_\beta - \tfrac{1}{2}(\Im\sigma_{mn})_{\beta\alpha} Q^i_\beta \\
[X_{jk}, Q^i_\alpha] &= i(s_{jk})_{pi}\tilde{Q}^p_\alpha \quad , \quad [Y_{jk}, Q^i_\alpha] = Q^p_\alpha (a_{jk})_{pi} \\
[X_{jk}, \tilde{Q}^i_\alpha] &= -i(s_{jk})_{pi} Q^p_\alpha \quad , \quad [Y_{jk}, \tilde{Q}^i_\alpha] = \tilde{Q}^p_\alpha (y_{jk})_{pi}. \\
[X, Q^i_\alpha] &= \big(\tfrac{1}{4} - \tfrac{1}{N}\big)\tilde{Q}^i_\alpha \quad \text{and} \quad [X, \tilde{Q}^i_\alpha] = -\big(\tfrac{1}{4} - \tfrac{1}{N}\big) Q^i_\alpha.
\end{aligned}$$

The supercharges $Q^i, \quad i = 1, \ldots, N$ transform as left handed Weyl-spinors under the generators M_{mn}. The $Q_\alpha, \alpha = 1 \ldots, 4$ transform under the defining N-dimensional representation of $su(N)$. The anticommutator of the supercharges with $i \neq j$ yields

$$\begin{aligned}
\{Q^i_\alpha, Q^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta} Y_{ij} - (\Im\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) - \delta_{ij}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha\bar{e}_\beta + e_\beta\bar{e}_\alpha - \frac{i}{2}\mathbb{1}(\Im\mathcal{A})_{\alpha\beta} \end{pmatrix} \\
\{Q^i_\alpha, \tilde{Q}^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) + (\Im\mathcal{A})_{\alpha\beta} Y_{ij} + i\delta_{ij}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha\bar{e}_\beta - e_\beta\bar{e}_\alpha + \frac{1}{2}\mathbb{1}(\Re\mathcal{A})_{\alpha\beta} \end{pmatrix}.
\end{aligned}$$

where we have decomposed

$$\mathcal{A} = \Re\mathcal{A} + i\Im\mathcal{A}, \qquad (\Re\mathcal{A})_{\alpha\beta} = -(\Re\mathcal{A})_{\beta\alpha}, \quad (\Im\mathcal{A})_{\alpha\beta} = (\Im\mathcal{A})_{\beta\alpha}.$$

Now we use the Fierz identities

$$\begin{aligned}
4(e_\alpha\bar{e}_\beta + e_\beta\bar{e}_\alpha) &= -\tfrac{1}{2}\sigma_{mn}\big\{(\bar{e}_\beta\sigma^{mn}e_\alpha) + (\bar{e}_\alpha\sigma^{mn}e_\beta)\big\} + 2\,\mathbb{1}_4(\Im\mathcal{A})_{\alpha\beta} \\
4i(e_\alpha\bar{e}_\beta - e_\beta\bar{e}_\alpha) &= -\tfrac{i}{2}\sigma_{mn}\big\{(\bar{e}_\beta\sigma^{mn}e_\alpha) - (\bar{e}_\alpha\sigma^{mn}e_\beta)\big\} - 2i\,\mathbb{1}_4(\Re\mathcal{A})_{\alpha\beta}
\end{aligned}$$

and find

$$\begin{aligned}
\{Q^i_\alpha, Q^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta} Y_{ij} - (\Im\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) + \tfrac{1}{4}\delta_{ij}\big\{(\mathcal{A}\sigma^{mn})_{\alpha\beta} + (\mathcal{A}\sigma^{mn})_{\beta\alpha}\big\} M_{mn} \\
\{Q^i_\alpha, \tilde{Q}^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) + (\Im\mathcal{A})_{\alpha\beta} Y_{ij} + \tfrac{i}{4}\delta_{ij}\big\{(\mathcal{A}\sigma^{mn})_{\alpha\beta} - (\mathcal{A}\sigma^{mn})_{\beta\alpha}\big\} M_{mn}.
\end{aligned}$$

The $\tilde{Q}$ fulfill the same anticommutation rules as the Q. In addition, one has

$$\{Q^i_\alpha, \tilde{Q}^j_\beta\} = -\{\tilde{Q}^i_\alpha, Q^j_\beta\}.$$

Again we see, that for $N = 4$ the $U(1)$ generator commutes with all other generators and the superalgebra ceases to be simple.
I leave it to an exercise to prove the equivalence of the superalgebras in the Majorana and the Weyl basis. In any case, once again we see that it is advantageous to the real basis, if possible.

6.5 Appendix: superalgebras of classical type

Table 6.2: Lie superalgebras of classical type.

Name	Range	Bosonic algebra	Defining repres.	Number of generators
$SU(N\|M)$	$N \geq 2$ $N \neq M$ $N = M$	$SU(N) \oplus SU(M)$ $\oplus U(1)$ no $U(1)$	$(N, \bar{M}) \oplus$ $(\bar{N}, M)$	$N^2 + M^2 - 1$, $2NM$ $2(N^2-1), 2N^2$
$S\ell(N\|M)$ $SU(N-p, p\|M-q, q)$ $SU^*(2N\|2N)$		$S\ell(N) \oplus S\ell(M) \quad \oplus SO(1,1)$ $SU(N-p,p) \oplus SU(M-q,q) \oplus U(1)$ $SU^*(2N) \oplus SU^*(2M) \quad \oplus SO(1,1)$		if $N \neq M$
$OSp(N\|M)$	$N \geq 1$ $M = 2, 4, ..$	$SO(N) \oplus Sp(M)$	(N, M)	$\frac{1}{2}(N^2 - N +$ $M^2 + M), NM$
$OSp(N-p, p\|M)$ $OSp(N^*\|M-q, q)$		$SO(N-p, p) \oplus Sp(M)$ $SO^*(N) \oplus USp(M-q, q)$		M even N, M, q even
$D(2,1,\alpha)$	$0 < \alpha \leq 1$	$SO(4) \oplus S\ell(2)$	$(2,2,2)$	9, 8
$D^p(2,1,\alpha)$		$SO(4-p, p) \oplus S\ell(2)$		$p = 0, 1, 2$
$F(4)$		$\overline{SO(7)} \oplus S\ell(2)$	$(8, 2)$	21, 16
$F^p(4)$		$SO(7-p, p) \oplus S\ell(2)$		$p = 0, 1, 2, 3$
$G(3)$		$G_2 \oplus S\ell(2)$	$(7, 2)$	14, 14
$G_p(3)$		$G_{2,p} \oplus S\ell(2)$		$p = -14, 2$
$P(N-1)$	$N \geq 3$	$S\ell(N)$	$(N \otimes N)$	$N^2 - 1, N^2$
$Q(N-1)$	$N \geq 3$	$SU(N)$	Adjoint	$N^2 - 1, N^2 - 1$
$Q(N-1)$ $Q((N-1)^*)$ $UQ(p, N-1-p)$		$S\ell(N)$ $SU^*(N)$ $SU(p, N-p)$		