5 The Birth of Supersymmetry

40 years ago the idea came up that perhaps the approximate $SU(3)$ symmetry of strong interaction is part of a larger $SU(6)$ symmetry. Under this bigger symmetry mesons (or baryons) with different spins belong to one multiplet. There were various attempts to generalize the $SU(6)$ symmetry of the non-relativistic quark model to a fully relativistic quantum field theory. These attempts failed, and several authors proved no-go theorems showing that this in in fact impossible. Well-known is the Coleman-Mandula theorem which we shall sketch now.

5.1 Coleman Mandula Theorem

Consider the theory for one free real scalar field and one free vector field of the same mass m,

$$\mathcal{L} = \tfrac{1}{2}\big(\partial_\mu\phi\partial^\mu\phi + \partial_\mu A^\rho\partial^\mu A_\rho - m^2\phi^2 - m^2 A^\rho A_\rho\big), \quad \Box\phi + m^2\phi = \Box A^\rho + m^2 A^\rho = 0.$$

There are an infinite number of conserved currents. For example, the tensor current

$$J_{\mu\nu} = \phi\overset{\leftrightarrow}{\partial}_\mu A_\nu = \phi\partial_\mu A_\nu - \partial_\mu\phi A_\nu$$

is conserved on shell,

$$\partial^\mu J_{\mu\nu} = \phi\Box A_\mu - \Box\phi A_\mu = -(m^2 - m^2)\phi A_\nu = 0.$$

This tensor is one out of an infinity of conserved currents of the form

$$J_{\mu_1\dots\mu_n} = \phi\overset{\leftrightarrow}{\partial}_{\mu_1}\dots\overset{\leftrightarrow}{\partial}_{\mu_n}\phi \quad (n\in 2\mathbb{N}) \quad \text{and} \quad J_{\mu_1\dots\mu_n\nu} = \phi\overset{\leftrightarrow}{\partial}_{\mu_1}\dots\overset{\leftrightarrow}{\partial}_{\mu_n}A_\nu \quad (n\in\mathbb{N}).$$

The corresponding charges obtained from these conserved currents are tensorial charges of higher rank. One may ask whether these conservation laws can be extended to the interacting case.
In $d>2$ dimensions the answer to this question is no [15, 16]! In a relativistic quantum theory with a discrete spectrum of massive (but no massless) one-particle states, and with some nontrivial scattering amplitudes, the only conserved tensorial charges that are not Lorentz scalars, are the energy-momentum vector P_μ and the angular momentum tensor $M_{\mu\nu}$ which span the Poincaré algebra. All other conserved charges must be Lorentz scalars. The total algebra is thus *always* of the form

$$iL + \mathcal{G}, \qquad [iL,\mathcal{G}] = 0.$$

In the total massless case iL can be extended to the conformal algebra $so(2,d)$.
Let us formulate the theorem more precisely.

Lemma 3 *The Coleman-Mandula-Theorem*
Let G be a connected symmetry-group of the S-matrix, i.e. a group whose generators commute with the S-matrix, and let the following five assumptions hold:

- *G contains a subgroup locally isomorphic to the Poincaré-group.*
- *All particle types correspond to positive-energy representations of the Poincaré–group. For any finite mass m there is only a finite number of particles with mass less than m.*
- *Elastic-scattering amplitudes are analytic functions of center-of-mass energy squared s and invariant momentum transfer squared t, in some neighborhood of the physical region, except at normal thresholds.*
- *Let $|p_i\rangle$, $i=1,2$ be any two 1-particle states and let $|p_1p_2\rangle$ be the 2-particle state constructed from these. Then $S|p_1p_2\rangle \neq |p_1p_2\rangle$ except, perhaps, for certain isolated values of s.*
- *The generators of G, considered as integral operators in momentum space, have distributions for their kernels.*

Then G is locally isomorphic to the direct product of an internal symmetry group and the Poincaré-group.

A unitary operator U on the Hilbert-space is said to be a symmetry transformation of the S-matrix if:

- U transforms 1-particle states into 1-particle states.
- U acts on many-particle states as if they were tensor products of one-particle states.
- U commutes with the scattering matrix S.

The theorem of Coleman and Mandula implies that the most general symmetry algebra of the S-matrix contains the energy momentum operator P_μ, the Lorentz-generators $M_{\mu\nu}$ and a finite number of Lorentz-scalar operators B_l, that is:

$$[P_\mu, B_l] = 0 \quad , \quad [M_{\mu\nu}, B_l] = 0,$$

where the B_l constitute a Lie-algebra with structure constants ${c_{lm}}^k$:

$$[B_l, B_m] = i{c_{lm}}^k B_k.$$

It follows at once, that the Casimir operators of the Poincaré algebra, P^2 and W^2 commute not only with the Poincaré group but also with all internal symmetry generators,

$$[B_l, P^2] = 0 \quad \text{and} \quad [B_l, W^2] = 0.$$

The first of equation says that all members of an irreducible multiplet of the internal symmetry group must have the same mass. This is known as O'RAIFEARTAIGH'S THEOREM [31]. The second equation says that they must also have the same spin. For massless states with discrete helicities we have

$$W_\mu = \lambda P_\mu$$

and no B_l can change the helicity since $[B_l, W_\mu] = \lambda[B_l, P_\mu] = 0$, and the statement that internal symmetries cannot change the spin holds true.
To see the arguments leading to the Coleman-Mandula theorem consider a forbidden tensorial charge $\mathbf{Q}_{\mu\nu}$ which for simplicity we shall assume traceless, $\mathbf{Q}^\mu_\mu = 0$. Assume a scalar particle of mass m, carrying the charge $\mathbf{Q}_{\mu\nu}$ appears in the theory and let $|p\rangle$ be a corresponding one-particle state with mass $m^2 = p^2$. Then

$$\langle p|\mathbf{Q}_{\mu\nu}|p\rangle = \Big(p_\mu p_\nu - \frac{p^2}{d}\eta_{\mu\nu}\Big)C, \quad C \neq 0.$$

We consider a $2 \to 2$ scattering process. The incoming particles with momenta p_1, p_2 scatter and then go out with final momenta p_1' and p_2'. The conservation law of $\mathbf{Q}$ applied between asymptotic incoming and outgoing states requires

$$C\Big(p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu} + \frac{1}{d}\eta_{\mu\nu}(m^2 + m^2)\Big) = C\Big(p'_{1\mu}p'_{1\nu} + p'_{2\mu}p'_{2\nu} + \frac{1}{d}\eta_{\mu\nu}(m^2 + m^2)\Big).$$

If $C \neq 0$, these equations imply that the scattering must proceed either in the forward or backward direction whereas in all other directions there is no scattering. This conflicts the analycity properties of scattering amplitudes in more then 2 dimensions and thus $C = 0$. No interacting theory can carry the charge $\mathbf{Q}_{\mu\nu}$. Similar arguments can be used for amplitudes of non-identical particles to prove this 'no-go' theorem.
The Coleman-Mandula theorem shows the impossibility of nontrivial symmetries that connect particles of different spins, if all the particles have integer spin or if all the particles have half-odd-integer spins.

5.2 The Wess-Zumino-Model

We study the simplest $N = 1$ supersymmetric model in four spacetime dimensions. It was constructed by Wess and Zumino [26] when they extended the 2-dimensional supersymmetric string-model of Gervais and Sakita [27] (which was based on earlier work of Ramond [28] and Neveu and Schwartz [29]) to four dimensions. The model contains a spin-0-spin-$\frac{1}{2}$ multiplet, i.e. a single Majorana field ψ, a pair of real scalar and pseudo-real bosonic fields A and B and a pair of real scalar and pseudoscalar bosonic auxiliary fields $\mathcal{F}$ and $\mathcal{G}$.
With (3.61) we have

$$\bar\alpha\psi,\ \bar\alpha\gamma^{\mu\nu}\psi,\ \bar\alpha\gamma_5\gamma^\mu\psi: \text{ hermitian}, \qquad \bar\alpha\gamma_5\psi,\ \bar\alpha\gamma^\mu\psi: \text{ antihermitian}.$$

Furthermore,

$\bar{\alpha}\psi$: scalar, $\bar{\alpha}\gamma_5\psi$ pseudoscalar, $\bar{\alpha}\gamma^\mu\psi$ vector and $\bar{\alpha}\gamma_5\gamma^\mu\psi$ pseudovector.

5.2.1 The off-shell formulation

The Lagrangian density is

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{1}{2}\partial_\mu B\partial^\mu B + \tfrac{i}{2}\bar{\psi}\not{\partial}\psi + \tfrac{1}{2}(\mathcal{F}^2+\mathcal{G}^2) \\
&+ m\Big[\mathcal{F}A + \mathcal{G}B - \tfrac{1}{2}\bar{\psi}\psi\Big] + g\big[\mathcal{F}(A^2-B^2) + 2\mathcal{G}AB - \bar{\psi}(A-i\gamma_5 B)\psi\big] \\
&= \mathcal{L}_0 + \mathcal{L}_m + \mathcal{L}_g.
\end{aligned} \tag{5.1}$$

The unfamiliar factor of $\frac{1}{2}$ in the fermion term arises because ψ is a Majorana spinor; a Dirac spinor is a linear combination of two Majorana spinors. The fields $\mathcal{F}$ and $\mathcal{G}$ enter only algebraically. Parity invariance demands, that the scalar $\bar{\psi}\psi$ couples to a scalar field and the pseudo-scalar $\bar{\psi}\gamma_5\psi$ to a pseudo-scalar field. The dimensions of the fields are

$$[A] = [B] = L^{-1}, \quad [\psi] = L^{-3/2} \quad \text{and} \quad [\mathcal{F}] = [\mathcal{G}] = L^{-2}.$$

The field equations are

$$\begin{aligned}
0 &= -\Box A + m\mathcal{F} + 2g\mathcal{F}A + 2g\mathcal{G}B - g\bar{\psi}\psi \\
0 &= -\Box B - m\mathcal{G} - 2g\mathcal{F}B + 2g\mathcal{G}A + ig\bar{\psi}\gamma_5\psi \\
0 &= \mathcal{F} + mA + g(A^2-B^2) \\
0 &= \mathcal{G} + mB + 2gAB \\
0 &= i\not{\partial}\psi - m\psi - 2g(A-i\gamma_5 B)\psi
\end{aligned} \tag{5.2}$$

To discover the Fermi-Bose symmetries of $\mathcal{L}$ we make the following infinitesimal supersymmetry transformations:

$$\begin{aligned}
\delta_\alpha A &= \bar{\alpha}\psi, \qquad \delta_\alpha B = i\bar{\alpha}\gamma_5\psi \\
\delta_\alpha\psi &= -i\not{\partial}\big(A + i\gamma_5 B\big)\alpha + (\mathcal{F} + i\gamma_5\mathcal{G})\alpha \\
\delta_\alpha\mathcal{F} &= -i\bar{\alpha}\not{\partial}\psi, \qquad \delta_\alpha\mathcal{G} = \bar{\alpha}\gamma_5\not{\partial}\psi,
\end{aligned} \tag{5.3}$$

where α is an arbitrary constant infinitesimal Majorana fermion c-number parameter with dimensions $[\alpha] = L^{1/2}$. These transformations transform bosons into fermions and vice versa,

$$\delta(\text{Boson}) = \text{Fermion} \quad \text{and} \quad \delta(\text{Fermion}) = \text{Boson}.$$

Note that for example $\bar{\alpha}\psi$ is a hermitean scalar and $i\bar{\alpha}\gamma_5\psi$ a hermitean pseudo-scalar, as required, since A and B are hermitean (pseudo)-scalars. With $\gamma^0\gamma_\mu = \gamma_\mu^\dagger\gamma^0$ and $\gamma_5^\dagger = \gamma_5$ the variation of the Dirac conjugate spinor is

$$\delta_\alpha\bar{\psi} = \delta_\alpha\psi^\dagger\gamma^0 = i\bar{\alpha}\not{\partial}\big(A - i\gamma_5 B\big) + \bar{\alpha}\big(\mathcal{F} + i\gamma_5\mathcal{G}\big). \tag{5.4}$$

The corresponding action is invariant under the supersymmetry transformation. We shall now

prove in several steps, that the Lagrangian density is invariant up to a total divergence. This requires some lengthy calculations. But for newcomers to supersymmetry this is a very useful exercise.
To prove the invariance of the first line in (5.1) we need

$$\begin{aligned} \tfrac{1}{2}\delta_\alpha\big(\partial_\mu A\partial^\mu A \;+\; \partial_\mu B\partial^\mu B\big) &= \bar\alpha\partial^\mu\big(A+i\gamma_5 B\big)\partial_\mu\psi \\ \tfrac{1}{2}\delta_\alpha\big(\mathcal{F}^2+\mathcal{G}^2\big) &= -i\bar\alpha\big(\mathcal{F}+i\gamma_5\mathcal{G}\big)\partial\!\!\!/\,\psi \\ \delta_\alpha(\bar\psi\partial\!\!\!/\,\psi) &= i\bar\alpha\partial\!\!\!/\,(A-i\gamma_5 B)\partial\!\!\!/\,\psi+\bar\alpha(\mathcal{F}+i\gamma_5\mathcal{G})\partial\!\!\!/\,\psi \\ &\quad -i\bar\alpha\Box(A+i\gamma_5 B)\psi-\bar\alpha\partial\!\!\!/\,(\mathcal{F}-i\gamma_5\mathcal{G})\psi, \end{aligned}$$

where we used

$$\bar\alpha\psi=\bar\psi\alpha,\quad \bar\alpha\gamma_5\psi=\bar\psi\gamma_5\alpha,\quad \bar\alpha\gamma^\mu\psi=-\bar\psi\gamma^\mu\alpha\quad\text{and}\quad \bar\alpha\gamma_5\gamma^\mu\psi=\bar\psi\gamma_5\gamma^\mu\alpha. \tag{5.5}$$

The (weighted) sum of these three terms yields the variation of $\mathcal{L}_0$:

$$\delta_\alpha\mathcal{L}_0=\partial_\mu\big(\bar\alpha V_0^\mu\big),\qquad V_0^\mu=\tfrac{1}{2}\gamma^\mu\big\{\partial\!\!\!/\,(A+i\gamma_5 B)-i(\mathcal{F}-i\gamma_5\mathcal{G})\big\}\psi. \tag{5.6}$$

This already shows, that the massless non-interacting model is invariant under susy-transformations. Actually this simple subsector is invariant not only under susy but under superconformal transformations. The superconformal group is the generalization of the conformal group.
For the variation of the mass-term we need

$$\begin{aligned} \delta_\alpha(\mathcal{F}A+\mathcal{G}B) &= \bar\alpha(\mathcal{F}+i\gamma_5\mathcal{G})\psi-i\bar\alpha(A+i\gamma_5 B)\partial\!\!\!/\,\psi \\ \delta_\alpha(\bar\psi\psi) &= 2\bar\alpha(\mathcal{F}+i\gamma_5\mathcal{G})\psi+2i\bar\alpha\partial\!\!\!/\,(A-i\gamma_5 B)\psi \end{aligned}$$

from which follows that

$$\delta_\alpha\mathcal{L}_m=\partial_\mu\big(\bar\alpha V_m^\mu\big),\qquad V_m^\mu=-im(A+i\gamma_5 B)\gamma^\mu\psi. \tag{5.7}$$

This shows, that the massive non-interacting model is invariant under susy-transformations. Finally, for the variation of the interaction term we need

$$\begin{aligned} &\delta_\alpha\big(\mathcal{F}A^2-\mathcal{F}B^2\big)=\bar\alpha\big\{2\mathcal{F}(A-i\gamma_5 B)-i(A^2-B^2)\partial\!\!\!/\,\big\}\psi \\ &\delta_\alpha(2AB\mathcal{G})=2\bar\alpha\big(AB\gamma_5\partial\!\!\!/\,+B\mathcal{G}+iA\mathcal{G}\gamma_5\big)\psi \\ &\delta_\alpha(A\bar\psi\psi)=(\bar\alpha\psi)(\bar\psi\psi)+2A\bar\alpha\big\{\mathcal{F}+i\gamma_5\mathcal{G}+i\partial\!\!\!/\,(A-i\gamma_5 B)\big\}\psi \\ &\delta_\alpha(\bar\psi\gamma_5\psi)=2\bar\alpha\big\{(\mathcal{F}+i\gamma_5\mathcal{G})+i\partial\!\!\!/\,(A-i\gamma_5 B)\big\}\gamma_5\psi \\ &\delta_\alpha\big(B\bar\psi\gamma_5\psi\big)=i(\bar\alpha\gamma_5\psi)(\bar\psi\gamma_5\psi)+2B\bar\alpha\big\{(\mathcal{F}+i\gamma_5\mathcal{G})+i\partial\!\!\!/\,(A-i\gamma_5 B)\big\}\gamma_5\psi. \end{aligned}$$

Using $(\bar\psi\psi)(\bar\alpha\psi)+(\bar\psi\gamma_5\psi)(\bar\alpha\gamma_5\psi)=0$ which follows from (3.67), we see that the variation of the interaction term is also a total divergence,

$$\delta_\alpha\mathcal{L}_g=\partial_\mu(\bar\alpha V_g^\mu),\qquad V_g^\mu=-ig\big(A+i\gamma_5 B\big)^2\gamma^\mu\psi. \tag{5.8}$$

Hence, the Lagrangian density transforms under supersymmetry transformations into a total divergence,

$$\begin{aligned} \delta_\alpha \mathcal{L} &= \partial_\mu(\bar{\alpha} V^\mu) \\ V^\mu &= \gamma^\mu\big\{\tfrac{1}{2}\not{\partial}(A+i\gamma_5 B) - \tfrac{i}{2}(\mathcal{F}-i\gamma_5\mathcal{G}) - im(A-i\gamma_5 B) - ig(A-i\gamma_5 B)^2\big\}\psi. \end{aligned} \tag{5.9}$$

This shows, that action (5.1) is invariant under the supersymmetry transformations (5.3). Note that this transformations do not contain the masses or coupling constants of the model.
The commutator of two successive transformations of a symmetry group must itself be a symmetry transformation. This way we can identify the algebra of group generators. Let us see what the commutator of two susy transformations looks like. For example,

$$\begin{aligned} [\delta_{\alpha_1},\delta_{\alpha_2}]A &= \delta_{\alpha_1}(\bar{\alpha}_2\psi) - \delta_{\alpha_2}(\bar{\alpha}_1\psi) = -i\bar{\alpha}_2\not{\partial}(A+i\gamma_5 B)\alpha_1 + \bar{\alpha}_2(\mathcal{F}+i\gamma_5\mathcal{G})\alpha_1 - (1\leftrightarrow 2) \\ &= -2i\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu A. \end{aligned} \tag{5.10}$$

Similarly one finds

$$[\delta_{\alpha_1},\delta_{\alpha_2}]B = -2i\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu B, \quad [\delta_{\alpha_1},\delta_{\alpha_2}]\mathcal{F} = -2i\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu \mathcal{F}, \quad [\delta_{\alpha_1},\delta_{\alpha_2}]\mathcal{G} = -2i\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu \mathcal{G}.$$

Computing the commutator on the spin-$\frac{1}{2}$ field is more involved. First,

$$\begin{aligned} [\delta_{\alpha_1},\delta_{\alpha_2}]\psi &= \partial_\mu\big\{-i(\bar{\alpha}_1\psi)\gamma^\mu + i(\bar{\alpha}_1\gamma_5\psi)\gamma^\mu\gamma_5 - i(\bar{\alpha}_1\gamma^\mu\psi) + i(\bar{\alpha}_1\gamma_5\gamma^\mu\psi)\gamma_5\big\}\alpha_2 - \big\{1\leftrightarrow 2\big\} \\ &= \big\{-i\gamma^\mu M + i\gamma^\mu\gamma_5 M\gamma_5 - iM\gamma^\mu + i\gamma_5 M\gamma_5\gamma^\mu\big\}\partial_\mu\psi, \end{aligned} \tag{5.11}$$

where we have introduced

$$M = \alpha_2\bar{\alpha}_1 - \alpha_1\bar{\alpha}_2 = -\tfrac{1}{2}\gamma_\rho(\bar{\alpha}_1\gamma^\rho\alpha_2) + \gamma_{\rho\sigma}(\bar{\alpha}_1\gamma^{\rho\sigma}\alpha_2) \tag{5.12}$$

such that

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = -2i\big(\bar{\alpha}_2\gamma^\mu\alpha_1\big)\partial_\mu\psi. \tag{5.13}$$

We see that on all fields the commutator of the supersymmetry transformations yield a translation,

$$[\delta_{\alpha_1},\delta_{\alpha_2}] = -2i\big(\bar{\alpha}_2\gamma^\mu\alpha_1\big)\partial_\mu. \tag{5.14}$$

Is is not difficult to see, that the supersymmetry transformations commute with the translations,

$$[\delta_\alpha,\delta_a] = 0, \qquad \delta_a = a^\mu\partial_\mu.$$

Let us finally calculate the commutator of Lorentz transformations and supersymmetry transformations. For example, for the A field

$$[\delta_\alpha,\delta_\omega]A = \delta_\alpha\big\{\tfrac{i}{2}(\omega,L)A\big\} - \delta_\omega\bar{\alpha}\psi = \tfrac{i}{2}\big\{(\omega,L)\bar{\alpha}\psi - \bar{\alpha}(\omega,J)\psi\big\} = -\tfrac{i}{2}\omega_{\mu\nu}\bar{\alpha}\,\Sigma^{\mu\nu}\psi,$$

where $J = L + \Sigma$. Similarly one finds

$$\begin{aligned} [\delta_\alpha,\delta_\omega]B &= \tfrac{1}{2}\omega_{\mu\nu}\bar{\alpha}\gamma_5\Sigma^{\mu\nu}\psi & [\delta_\alpha,\delta_\omega]\mathcal{F} &= -\tfrac{1}{2}\omega_{\mu\nu}\bar{\alpha}\not{\partial}\,\Sigma^{\mu\nu}\psi \\ [\delta_\alpha,\delta_\omega]\mathcal{G} &= -\tfrac{i}{2}\omega_{\mu\nu}\bar{\alpha}\gamma_5\not{\partial}\,\Sigma^{\mu\nu}\psi & [\delta_\alpha,\delta_\omega]\psi &= \tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\delta_\alpha\psi. \end{aligned}$$

Now the generator of susy transformations Q_α is a four component Majorana spinor, which we define by the requirement that

$$\delta_\alpha A = \bar\alpha Q A, \quad \delta_\alpha B = \bar\alpha Q B, \dots.$$

Consistency with (5.14) requires

$$\begin{aligned} [\delta_{\alpha_1}, \delta_{\alpha_2}] &= [\bar\alpha_1 Q, \bar\alpha_2 Q] = [\bar Q\alpha_1, \bar\alpha_2 Q] = \left(\bar Q^\beta \alpha_{1\beta}\bar\alpha_2^\alpha Q_\alpha - \bar\alpha_2^\alpha Q_\alpha \bar Q^\beta \alpha_{1\beta}\right) \\ &= \alpha_{1\beta}\bar\alpha_2^\alpha\left(\bar Q^\beta Q_\alpha + Q_\alpha \bar Q^\beta\right) = -2i(\bar\alpha_2\gamma^\mu\alpha_1)\partial_\mu = 2i\alpha_{1\beta}\bar\alpha_2^\alpha(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu. \end{aligned} \tag{5.15}$$

Comparing the last with the third to last expression yields

$$\{Q_\alpha, \bar Q^\beta\} = 2i(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu = 2(\gamma^\mu)_\alpha^{\ \beta}P_\mu. \tag{5.16}$$

Since $\bar Q^\beta = -Q_\gamma(\mathcal{C}^{-1})^{\gamma\beta}$ and $\mathcal{C}^T = -\mathcal{C}$ we may rewrite this commutation relation as

$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu\mathcal{C})_{\alpha\beta}P_\mu. \tag{5.17}$$

Also, since the susy transformations commute with the translations, we have

$$[Q_\alpha, P_\mu] = 0.$$

Finally, since

$$[\delta_\alpha, \delta_w]A = \tfrac{i}{2}\bar\alpha^\alpha\omega^{\mu\nu}[Q_\alpha, J_{\mu\nu}]A = -\tfrac{i}{2}\omega^{\mu\nu}\bar\alpha^\alpha\left(\Sigma_{\mu\nu}\psi\right)_\alpha$$

we conclude, that

$$[Q_\alpha, J_{\mu\nu}]A = -(\Sigma_{\mu\nu}Q)_\alpha A.$$

The same holds true for the other fields, such that

$$[J_{\mu\nu}, Q_\alpha] = (\Sigma_{\mu\nu}Q)_\alpha. \tag{5.18}$$

The supercharges transform as spin-$\frac{1}{2}$ fields.

5.2.2 On-shell formulation

The Lagrangian and equations of motion in their present form are not very illuminating. Notice, however, that the fields $\mathcal{F}$ and $\mathcal{G}$ are just Lagrangian multiplier fields. These auxiliary (dummy) fields satisfy the algebraic equations of motion

$$0 = \frac{\partial\mathcal{L}}{\partial\mathcal{F}} = \mathcal{F} + mA + g(A^2 - B^2) \quad \text{and} \quad 0 = \frac{\partial\mathcal{L}}{\partial\mathcal{G}} = \mathcal{G} + mB + 2gAB$$

and can be eliminated from the Lagrangian and from the equation of motion. With

$$\begin{aligned} &\tfrac{1}{2}(\mathcal{F}^2 + \mathcal{G}^2) + m(\mathcal{F}A + \mathcal{G}B) + g\left\{\mathcal{F}(A^2 - B^2) + 2\mathcal{G}AB\right\} \\ &= -\tfrac{1}{2}m^2(A^2 + B^2) - mgA(A^2 + B^2) - \tfrac{1}{2}g^2(A^2 + B^2)^2 \end{aligned}$$

the 'on-shell' Lagrangian density reads

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{1}{2}\partial_\mu B\partial^\mu B - \tfrac{1}{2}m^2(A^2+B^2) + \tfrac{i}{2}\bar\psi\partial\!\!\!/\,\psi - \tfrac{1}{2}m\bar\psi\psi \\ &\quad -mgA(A^2+B^2) - \tfrac{1}{2}g^2(A^2+B^2)^2 - g\bar\psi(A-i\gamma_5 B)\psi.\end{aligned} \tag{5.19}$$

The field equations are

$$\begin{aligned}(\Box+m^2)A &= -mg(3A^2+B^2) - 2g^2A(A^2+B^2) - g\bar\psi\psi \\ (\Box+m^2)B &= -2mgAB - 2g^2B(A^2+B^2) + ig\bar\psi\gamma_5\psi \\ (i\partial\!\!\!/ - m)\psi &= 2g(A-i\gamma_5 B)\psi.\end{aligned} \tag{5.20}$$

Several features of this Lagrangian are characteristic of supersymmetric theories. The masses of the scalars and the fermions are equal. There are cubic and quartic couplings between the scalar fields and there is a Yukawa-type interaction between the fermion and the scalar fields. In total there are only two free parameters: m and g. The relation between boson and fermion masses and couplings is a common feature of SUSY and it is stable under renormalization. This has been found to be so, since there exists a (Pauli-Villars) regularization which preserves supersymmetry for the model [30]

Indeed, the Wess-Zumino Model has also some remarkable renormalization properties. Despite the presence of scalar fields, there is no renormalization of the mass and coupling constant. There is only a wave-function renormalization. The divergences arising from boson loops are cancelled by those from fermion loops which have the opposite sign. These powerful non-renormalization theorems make SUSY particularly attractive.

The on-shell transformations read

$$\begin{aligned}\delta_\alpha A &= \bar\alpha\psi, \qquad \delta_\alpha B = i\bar\alpha\gamma_5\psi \\ \delta_\alpha\psi &= -\big\{i\partial\!\!\!/ + m + g(A+i\gamma_5 B)\big\}\big\{A+i\gamma_5 B\big\}\alpha,\end{aligned} \tag{5.21}$$

so that

$$\delta_\alpha\bar\psi = \bar\alpha\big\{i\overleftarrow{\partial\!\!\!/}\,(A-i\gamma_5 B) - \big(m+g(A+i\gamma_5 B)\big)\big(A+i\gamma_5 B\big)\big\}.$$

Note that these transformations have become dependent on the model (here they depend on m and g), and there is no part of the Lagrangian which separately transforms as a density under them. Of course, the action is invariant under these transformation. For example, for $m = g = 0$ we have

$$\delta_\alpha\mathcal{L} = \partial_\mu\bar\alpha V^\mu, \qquad V^\mu = \tfrac{1}{2}\gamma^\mu\partial\!\!\!/\,(A+i\gamma_5 B)\psi.$$

This is just the result for the off-shell formulation in which one replaces the auxiliary fields by their equation of motion. Now we shall see, that the susy algebra only closes on-shell. It closes off-shell on the fields A and B, but not on the fermion field,

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = -\big\{i\partial\!\!\!/ + m + 2g(A+i\gamma_5 B)\big\}\big\{M-\gamma_5 M\gamma_5\big\}\psi,$$

where $M = \alpha_2\bar\alpha_1 - \alpha_1\bar\alpha_2$ has already been used in (5.12). Since

$$M - \gamma_5 M\gamma_5 = -\gamma_\rho(\bar\alpha_1\gamma^\rho\alpha_2) \quad \text{and} \quad \partial\!\!\!/\,\gamma_\rho = -\gamma_\rho\partial\!\!\!/ - 2\partial_\rho$$

we can rewrite the commutator of 2 susy-transformations on the fermi-field as

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\psi = -2(\bar{\alpha}_2\gamma^\rho\alpha_1)\partial_\rho\psi + \gamma_\rho(\bar{\alpha}_2\gamma^\rho\alpha_1)\{i\partial\!\!\!/ - m - 2g(A - i\gamma_5 B)\}\psi. \tag{5.22}$$

Only if we impose the field equation for the spinor field, then the last term vanishes and

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\psi = -2(\bar{\alpha}_2\gamma^\rho\alpha_1)\partial_\rho\psi.$$

We see, that after the elimination of the auxiliary fields the susy algebra closes only if the equations of motion hold. This could be disastrous for quantum corrections: there the fields must be taken off-shell, away from their classical paths through configuration space. However, if there is (as for the WZ-model) some off-shell version, there should be no problem. But what happens if the theory were intrinsically only on-shell supersymmetric (as $N = 4$ and $N = 8$ theories may well be) or if there several off-shell versions (as there are for $N = 1$ supergravity), is unclear.
Counting degrees of freedom: A heuristic understanding for the necessity of auxiliary fields in the off-shell formulation is based on a counting of degrees of freedom. In a supersymmetric theory the fermionic degrees of freedom must match the bosonic ones. In 4 dimensions a real field $A(x)$ describes one neutral scalar particle and the Dirac field $\psi(x)$ has 8 real components, but describes only 4 states of a spin-$\frac{1}{2}$ particle. Thus, in going from the field to the states we have lost some dimensions of our representation space, but differently so for different spins. Supersymmetric models for which the number of fermions is the number of bosons somehow take care of this, and they do so by means of auxiliary field whose off-shell degrees of freedom disappear completely on-shell. For the Wess-Zumino-Model

$$\text{off-shell: } (A, B, \mathcal{F}, \mathcal{G}) \longleftrightarrow \psi_C = \psi \quad , \quad \text{on-shell: } (A, B) \longleftrightarrow \psi_C = \psi.$$

5.2.3 Noether currents and supercharges

We proceed as we did in the last chapter. For space-times symmetries the Lagrangian density is invariant up to a total divergence and the Noether currents

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_X\phi - V^\mu$$

acquire an additional term $-V^\mu$. Here we are mainly interested in the conserved charges and not so much in the currrents. Hence we do not attempt to improve the Noether currents. For example, the conserved energy momentum current

$$T^\mu_\nu = \partial^\mu A\partial_\nu A + \partial^\mu B\partial_\nu B + \tfrac{i}{2}\bar{\psi}\gamma^\mu\partial_\nu\psi - \delta^\mu{}_\nu\mathcal{L}$$

is not symmetric and not traceless for vanishing mass. But it leads to the following conserved energy-momentum

$$P_\nu = \int_{x^0} \big(\pi_A\partial_\nu A + \pi_B\partial_\nu B + i\psi^\dagger\partial_\nu\psi - \delta^0{}_\nu\mathcal{L}\big).$$

To construct the supercurrent, i.e. the Noether current associated to the supersymmetry, we must determine

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\alpha\phi.$$

Because $\bar\psi\overleftrightarrow{\partial}\psi = -\partial_\mu\bar\psi\gamma^\mu\psi$ we obtain

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\alpha\phi &= \partial^\mu A\bar\alpha\psi + i\partial^\mu B\bar\alpha\gamma_5\psi - \tfrac{i}{2}\delta_\alpha\bar\psi\gamma^\mu\psi \\ &= \bar\alpha\big\{2\partial^\mu(A+i\gamma_5B) - \tfrac{1}{2}\gamma^\mu\partial\!\!\!/\,(A+i\gamma_5B) - \tfrac{i}{2}(\mathcal{F}-i\gamma_5\mathcal{G})\big\}\psi.\end{aligned}$$

Now we must subtract V^μ in (5.9) to obtain the conserved current

$$\begin{aligned}\bar\alpha J^\mu &= \bar\alpha\big\{\partial\!\!\!/\,(A-i\gamma_5B)\gamma^\mu\psi + im\gamma^\mu(A-i\gamma_5B)\psi + ig\gamma^\mu(A-i\gamma_5B)^2\big\}\psi \\ &= -i\delta_\alpha\bar\psi\gamma^\mu\psi = i\bar\psi\gamma^\mu\delta_\alpha\psi.\end{aligned} \tag{5.23}$$

In the elegant last two forms for the conserved supercurrent we must replace the velocities by the momenta. This supercurrent is indeed conserved due to the equations of motion. Note, that the auxiliary fields have gone in the expression for the Noether current.
To arrive at the symplectic structure we first recall the fundamental equal time (anti)commutators

$$[A(x),\pi_A(y)] = i\delta(\vec x-\vec y) \quad , \quad [B(x),\pi_B(y)] = i\delta(\vec x-\vec y) \tag{5.24}$$

and

$$\{\psi_\alpha(x),\psi^{\beta\dagger}(y)\} = \delta_\alpha{}^\beta\delta(\vec x-\vec y) \quad \text{or} \quad \{\psi_\alpha(x),\bar\psi^\beta(y)\} = (\gamma^0)_\alpha^{\ \beta}\delta(\vec x-\vec y). \tag{5.25}$$

For a Majorana spinor, the last anti-commutator implies

$$\{\psi_\alpha(x),\psi_\beta(y)\} = -(\gamma^0\mathcal{C})_{\alpha\beta}\delta(\vec x-\vec y), \qquad (\gamma^0\mathcal{C})^T = \gamma^0\mathcal{C}. \tag{5.26}$$

The Hamiltonian is

$$\begin{aligned} H &= H_0 + H_m + H_g, \quad \text{where} \quad H_m + H_g = -\int_{x^0}(\mathcal{L}_m+\mathcal{L}_g) \quad \text{and} \\ H_0 &= \tfrac{1}{2}\int_{x^0}\big\{\pi_A^2 + (\nabla A)^2 + \pi_B^2 + (\nabla B)^2 - i\psi^\dagger\alpha^i\partial_i\psi - (\mathcal{F}^2+\mathcal{G}^2)\big\}.\end{aligned} \tag{5.27}$$

As familiar from the Dirac-theory, the hermitian matrices $\alpha^i = \gamma^0\gamma^i$ appeared in the Hamiltonian. Using

$$[AB,C] = A\{B,C\} - \{A,C\}B$$

in the time-evolution equation

$$\dot O = i[H,O]$$

one finds the equation of motion for the fields and their conjugate momenta. For example, for the free massless model

$$\begin{aligned}\dot A(x) &= i[H_0,A(x)] = \pi_A, \quad \dot\pi_A = i[H_0,\pi_A] = \triangle A \Longrightarrow \Box A = 0 \\ \dot\psi_\alpha(x) &= i[H_0,\psi_\alpha(x)] = \tfrac{1}{2}(\alpha^i)_\beta^{\ \gamma}\int\big(\psi^{\dagger\beta}\{\partial_i\psi_\gamma,\psi_\alpha(x)\} - \{\psi^{\dagger\beta},\psi_\alpha(x)\}\partial_i\psi_\gamma\big) \\ &= -\tfrac{1}{2}(\alpha^i\partial_i\psi)_\alpha - \tfrac{1}{2}(\alpha^i)_\beta^{\ \gamma}\int\partial_i\psi^{\dagger\beta}\{\psi_\gamma,\psi_\alpha(x)\} = -(\alpha^i\partial_i\psi)_\alpha,\end{aligned}$$

which are just the field equations of the massless free Wess-Zumino model. The supercharge is

$$\bar{\alpha}Q = \bar{\alpha}\int_{x^0}\{\pi_A + i\gamma_5\pi_B - \alpha^i\partial_i(A+i\gamma_5 B) + im(A+i\gamma_5 B)\gamma^0 + ig(A+i\gamma_5 B)^2\gamma^0\}\psi,$$

and generates the following transformations

$$\begin{aligned} i[\bar{\alpha}Q, A] &= \bar{\alpha}\psi, \quad i[\bar{\alpha}Q, B] = i\bar{\alpha}\gamma_5\psi \\ i[\bar{\alpha}Q, \psi_\beta] &= -i\left(\gamma^0(\pi_A + i\pi_B\gamma_5) + \gamma^i\partial_i(A+i\gamma_5 B) + m(A+i\gamma_5 B) + g(A+i\gamma_5 B)^2\right)\alpha, \end{aligned}$$

which are just the on-shell susy-transformations. We used the identities

$$\begin{aligned} &\gamma_5\mathcal{C} = \mathcal{C}\gamma_5^T, \quad (\bar{\alpha}\gamma^0\mathcal{C})^T = \gamma^0\alpha, \quad (\bar{\alpha}\gamma_5\gamma^0\mathcal{C})^T = \gamma^0\gamma_5\alpha \\ &(\bar{\alpha}\alpha^i\gamma^0\mathcal{C})^T = -\gamma^i\alpha \quad \text{and} \quad (\bar{\alpha}\alpha^i\gamma_5\gamma^0\mathcal{C})^T = \gamma_5\gamma^i\alpha. \end{aligned}$$

For the infinitesimal transformation of the momenta one finds

$$i[\bar{\alpha}Q, \pi_A] = \bar{\alpha}\dot{\psi} \quad \text{and} \quad i[\bar{\alpha}Q, \pi_B] = i\bar{\alpha}\gamma_5\dot{\psi},$$

where we used the abbreviation

$$\dot{\psi} = -\big(\alpha^i\partial_i + i\gamma^0(m + 2g(A - i\gamma_5 B)\big)\psi.$$

The Hamiltonian equation for the fields and their conjugate momentum fields are equivalent to the field equations, as expected. The spinorial charges are

$$\begin{aligned} Q &= \int\{\pi_A + i\gamma_5\pi_B - \alpha^i\partial_i(A+i\gamma_5 B) + i\big(m + g[A+i\gamma_5 B]\big)\big(A+i\gamma_5 B\big)\gamma^0\}\psi \\ \bar{Q} &= \int\bar{\psi}\{\pi_A + i\gamma_5\pi_B + \alpha^i\partial_i(A+i\gamma_5 B) - i\gamma^0\big(m + g[A+i\gamma_5 B]\big)\big(A+i\gamma_5 B\big)\}. \end{aligned} \tag{5.28}$$

To finally calculate the anticommutator of the supercharges we use

$$\{B_1F_1, F_2\} = B_1\{F_1, F_2\} - [B_1, F_2]F_1, \quad \{F_1, B_2F_2\} = B_2\{F_1, F_2\} + [F_1, B_2]F_2 \tag{5.29}$$

so that

$$\{B_1F_1, B_2F_2\} = B_1B_2\{F_1, F_2\} + B_1[F_1, B_2]F_2 - [B_1, B_2]F_2F_1 - B_2[B_1, F_2]F_1 \tag{5.30}$$

holds. The B_i are bosonic and the F_i are fermionic operators. Since $Q, \bar{Q}$ have the form

$$Q_\alpha = \int dx B_\alpha^{\ \gamma}(x)\psi_\gamma(x) \quad \text{and} \quad \bar{Q}^\beta = \int dy\bar{\psi}^\delta(y)C_\delta^{\ \beta}(y),$$

where the bosonic operator $B_\alpha^{\ \gamma}$ commutes with ψ. Hence

$$\{Q_\alpha, \bar{Q}^\beta\} = \int dxdy\left(B_\alpha^{\ \gamma}(x)\{\psi_\gamma(x), \bar{\psi}^\delta(y)\}C_\delta^{\ \beta}(y) + \bar{\psi}^\delta(y)[C_\delta^{\ \beta}(y), B_\alpha^{\ \gamma}(x)]\psi_\gamma(x)\right).$$

A rather involved calculations shows, that

$$\{Q_\alpha, \bar{Q}^\beta\} = 2(\gamma^\mu)_\alpha^{\ \beta}P_\mu,$$

as expected. Actually, one gets an additional term

$$-\tfrac{i}{2}\gamma^0[\gamma^i,\gamma^j]\gamma_5\int(\partial_i A\partial_j B-\partial_j A\partial_i B),$$

but the integrand is a total divergence and hence the integral vanishes. One gets the off-shell Hamiltonian P_0 with eliminated auxiliary fields. Also, one automatically obtains the hermitian momentum. For example, the bosonic contribution to the momentum is calculated to be

$$P_i=\tfrac{1}{2}\big(\pi_A\partial_i A+\partial_i A\pi_A+\pi_B\partial_i B+\partial_i B\pi_B\big).$$

5.2.4 The superpotential

To generalize the results it is convenient to work with Weyl-spinors. The following calculations can be done in any representations. For notational simplicity I prefer to choose the chiral representation introduced in section (3.6.1). In the van der Waerden notation with dotted and un-dotted indices a Dirac spinor has the form

$$\psi=\begin{pmatrix}\psi_A\\ \bar\chi^{\dot A}\end{pmatrix},\quad \bar\psi=\psi^\dagger\gamma^0=(\,\bar\psi_{\dot A},\chi^A\,)\,\gamma^0=(\,\chi^A,\bar\psi_{\dot A}\,),\quad \text{where}\quad \bar\psi_{\dot A}=(\psi_A)^\dagger\ldots.$$

A Majorana spinor ψ can be formed entirely from a left-handed Weyl spinor,

$$\psi=\psi_L+(\psi_L)_C,\quad \psi=P_L\psi,\quad P_L=\tfrac{1}{2}(1-\gamma_5). \tag{5.31}$$

In the chiral representation representation introduced in the last chapter we have

$$\gamma_5=\begin{pmatrix}-\sigma_0&0\\0&\sigma_0\end{pmatrix},\quad B=\begin{pmatrix}0&\varepsilon\\-\varepsilon&0\end{pmatrix},\quad \varepsilon=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=i\sigma_2,\quad \gamma_\mu=\begin{pmatrix}0&\sigma_\mu\\ \tilde\sigma_\mu&0\end{pmatrix}, \tag{5.32}$$

where

$$\sigma_\mu=(\sigma_0,-\sigma_i),\quad \tilde\sigma_\mu=(\sigma_0,\sigma_i)\quad \text{so that}\quad \varepsilon\sigma_\mu\varepsilon=-\tilde\sigma_\mu^T. \tag{5.33}$$

Left and right-handed spinors are simply

$$\psi_L=\begin{pmatrix}\psi_A\\0\end{pmatrix}\quad\text{and}\quad \psi_R=\begin{pmatrix}0\\ \bar\chi^{\dot A}\end{pmatrix}, \tag{5.34}$$

and a Majorana spinor has the form

$$\psi=\psi_L+(\psi_L)_C=\begin{pmatrix}\psi_A\\-(\varepsilon\psi^*)^{\dot A}\end{pmatrix}\equiv\begin{pmatrix}\chi_A\\ \bar\chi^{\dot A}\end{pmatrix}=\psi_C\quad\Longrightarrow\quad \bar\psi=(\psi^A,\bar\psi_{\dot A}). \tag{5.35}$$

The raising and lowering of the indices are performed with the ε-symbols

$$\psi^A=\varepsilon^{AB}\,\psi_B,\quad \psi_A=\psi^B\epsilon_{BA}\quad\text{and}\quad \bar\psi^{\dot A}=\bar\psi_{\dot B}\,\varepsilon^{\dot B\dot A},\quad \bar\psi_{\dot A}=\epsilon_{\dot A\dot B}\bar\psi^{\dot B}$$

where we introduced

$$
\begin{aligned}
(\varepsilon^{AB}) = (\varepsilon_{AB}) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
(\varepsilon^{\dot{A}\dot{B}}) = (\varepsilon_{\dot{A}\dot{B}}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}
$$

It follows that

$$\varepsilon^{AB}\varepsilon_{BC} = -\delta^A_C, \quad \varepsilon_{\dot{A}\dot{B}}\varepsilon^{\dot{B}\dot{C}} = -\delta^{\dot{A}}_{\dot{C}}.$$

For fermionic variables we have

$$\psi^A\chi_A = \varepsilon^{AB}\psi_B\chi_A = \varepsilon^{BA}\chi_A\psi_B = \chi^A\psi_A \equiv \psi\chi, \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{A}}\bar{\chi}^{\dot{A}} = \bar{\chi}_{\dot{A}}\bar{\psi}^{\dot{A}}.$$

Note, however, that

$$\psi^A\chi_A = \varepsilon^{AB}\psi_B\chi^C\varepsilon_{CA} = -\delta^B_C\psi_B\chi^C = -\psi_B\chi^B, \quad \bar{\psi}_{\dot{A}}\bar{\chi}^{\dot{A}} = -\bar{\psi}^{\dot{A}}\bar{\chi}_{\dot{A}}.$$

The mass term reads

$$\bar{\psi}\psi = \psi^A\psi_A + \bar{\psi}_{\dot{A}}\bar{\psi}^{\dot{A}} = \psi\psi + \bar{\psi}\bar{\psi}, \qquad \bar{\psi}\bar{\psi} = (\psi\psi)^\dagger, \tag{5.36}$$

and

$$\bar{\psi}\gamma_5\psi = -\psi\psi + \bar{\psi}\bar{\psi}.$$

The vector current can be written as

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_{\dot{A}}(\tilde{\sigma}^\mu)^{\dot{A}A}\psi_A + \psi^A(\sigma^\mu)_{A\dot{A}}\bar{\psi}^{\dot{A}} = \bar{\psi}\tilde{\sigma}^\mu\psi + \psi\sigma^\mu\bar{\psi} = \bar{\psi}\tilde{\sigma}^\mu\psi + \text{ h.c.},$$

so that the 'kinetic term' for Majorana fermions is

$$\bar{\psi}\not{\partial}\,\psi = \bar{\psi}\tilde{\sigma}^\mu\partial_\mu\psi + \psi\sigma^\mu\partial_\mu\bar{\psi}. \tag{5.37}$$

Now we are ready to rewrite the susy-Lagrangian (5.1) in terms of a single left-handed field ψ, and complex fields ϕ and F for its scalar partners

$$\phi = \frac{1}{\sqrt{2}}(A + iB) \quad \text{and} \quad F = \frac{1}{\sqrt{2}}(\mathcal{F} - i\mathcal{G}). \tag{5.38}$$

With

$$
\begin{aligned}
\partial_\mu\phi^\dagger\partial^\mu\phi &= \tfrac{1}{2}\{\partial_\mu A\partial^\mu A + \partial_\mu B\partial^\mu B\} \quad , \quad F^\dagger F = \tfrac{1}{2}(\mathcal{F}^2 + \mathcal{G}^2) \\
\phi F + (\phi F)^\dagger &= A\mathcal{F} + B\mathcal{G} \quad , \quad \sqrt{2}\{\phi^2 F + (\phi^2 F)^\dagger\} = \mathcal{F}(A^2 - B^2) + 2AB\mathcal{G} \\
\sqrt{2}\{\phi\psi\psi + \phi^\dagger\bar{\psi}\bar{\psi}\} &= \bar{\psi}(A - i\gamma_5 B)\psi.
\end{aligned}
$$

we can rewrite (5.1) as

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^\dagger\partial^\mu\phi + F^\dagger F + \tfrac{1}{2}\{i\bar\psi\tilde\sigma^\mu\partial_\mu\psi + 2m\phi F - m\psi^2 + \text{ h.c.}\} \\ &\qquad\qquad +\sqrt{2}g\{\phi^2 F - \phi\psi\psi + \text{ h.c.}\}.\end{aligned} \tag{5.39}$$

Then, using the equation of motion for F, which gives

$$F^\dagger = -m\phi - \sqrt{2}g\phi^2$$

we can eliminate the auxiliary field so that the Lagrangian density becomes

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - |m\phi + \sqrt{2}g\phi^2|^2 + \{\tfrac{i}{2}\bar\psi\tilde\sigma^\mu\partial_\mu\psi - \tfrac{1}{2}\bar\psi\psi - \sqrt{2}g\phi^2 F + \text{ h.c.}\}. \tag{5.40}$$

It is useful to re-express these Lagrangian in terms of an analytic function

$$W(\phi) = \tfrac{1}{2}m\phi^2 + \tfrac{1}{3}\sqrt{2}g\phi^3, \tag{5.41}$$

known as *superpotential*, as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_K + F^\dagger F + \Big\{F\frac{\partial W}{\partial\phi} + \tfrac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi + \text{ h.c.}\Big\} \\ \mathcal{L} &= \mathcal{L}_K - \Big|\frac{\partial W}{\partial\phi}\Big|^2 + \tfrac{1}{2}\Big\{\frac{\partial^2 W}{\partial\phi^2}\psi\psi + \text{ h.c.}\Big\}.\end{aligned} \tag{5.42}$$

For a renormalizable theory W can be, at most, a cubic function of ϕ. The superpotential is the only free function in the susy Lagrangian and determines both the potential for the scalar fields and the masses and couplings of the fermions and bosons.
In general there may be several chiral multiplets. For example, if the lefthanded spinor fields ψ_i belongs to a representation of $SU(N)$, we will have supermultiplets

$$(\phi_i, \chi_i),$$

where in the fundamental representation $i = 1, 2, \ldots, N$. Now (5.42) is readily generalized to a density that is invariant under the additional symmetry and contains the new supermultiplets. It is

$$\mathcal{L}_{\text{chiral}} = \sum_i \left|\partial_\mu\phi_i\right|^2 + \Big\{\tfrac{i}{2}\sum_i \bar\psi_i\tilde\sigma^\mu\partial_\mu\psi_i - \sum_i \Big|\frac{\partial W}{\partial\phi_i}\Big|^2 + \tfrac{1}{2}\sum_{ij}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j + \text{ h.c.}\Big\}. \tag{5.43}$$

Supersymmetry transformations: we write the Majorana spinor α in terms of its left-handed part,

$$\alpha = \begin{pmatrix}\alpha_A \\ \bar\alpha^{\dot A}\end{pmatrix}, \quad \bar\alpha = (\alpha^A, \bar\alpha_{\dot A}).$$

Then it follows at once that

$$\begin{aligned}\bar\alpha\psi = \alpha\chi + \bar\alpha\bar\chi \quad &, \quad \bar\alpha\gamma_5\psi = -\alpha\psi + \bar\alpha\bar\psi \\ \bar\alpha\gamma^\mu\psi = \bar\alpha\tilde\sigma^\mu\psi + \alpha\sigma^\mu\bar\psi \quad &, \quad \bar\alpha\gamma_5\gamma^\mu\psi = \bar\alpha\tilde\sigma^\mu\psi - \alpha\sigma^\mu\bar\psi,\end{aligned}$$

where we used, that the σ_μ are hermitian. The susy-transformations take now the form

$$\delta_\alpha\phi = \sqrt{2}\alpha\psi, \quad \delta_\alpha F = -i\sqrt{2}\bar\alpha\tilde\sigma^\mu\partial_\mu\psi, \quad \delta_\alpha\psi = \sqrt{2}\big(F\alpha - i\sigma^\mu\partial_\mu\phi\,\bar\alpha\big).$$

5.2.5 Supersymmetry algebra in Weyl-basis

We may recast the supersymmetry algebra for the Majorana-spinor charges Q_α, namely

$$\begin{aligned} \{Q_\alpha, Q_\beta\} = -2(\gamma^\mu \mathcal{C})_{\alpha\beta}P_\mu \quad &\text{or} \quad \{Q_\alpha, \bar Q^\beta\} = 2(\gamma^\mu)_\alpha^{\ \beta}P_\mu \\ [Q_\alpha, P_\mu] = 0 \qquad &\text{and} \qquad [J_{\mu\nu}, Q_\alpha] = (\Sigma_{\mu\nu}Q)_\alpha, \end{aligned} \tag{5.44}$$

plus the Poincaré algebra, in terms of the left-handed charges. For that we expand (5.15) in the lefthanded supercharges

$$\begin{aligned} [\bar Q\alpha, \bar\beta Q] &= \alpha_A\beta^B\{Q^A, Q_B\} + \bar\alpha^{\dot A}\bar\beta_{\dot B}\{\bar Q_{\dot A}, \bar Q^d b\} + \alpha_A\bar\beta_{\dot B}\{Q^A, \bar Q^{\dot B}\} + \bar\alpha^{\dot A}\beta^B\{\bar Q_{\dot A}, Q_B\} \\ &= -2\big\{\beta^B(\sigma^\mu)_{B\dot A}\bar\alpha^{\dot A} + \bar\beta_{\dot B}(\tilde\sigma^\mu)^{\dot BA}\alpha_A\big\}P_\mu. \end{aligned}$$

Comparing the two expressions yields

$$\{Q_A, \bar Q_{\dot B}\} = 2(\sigma^\mu)_{A\dot B}P_\mu \quad \text{or} \quad \{\bar Q^{\dot A}, Q^B\} = 2(\tilde\sigma^\mu)^{\dot AB}P_\mu \tag{5.45}$$

and

$$\{Q_A, Q_B\} = \{\bar Q^{\dot A}, \bar Q^{\dot B}\} = 0.$$

Analogously, when we express the Majorana spinors in

$$[J_{\mu\nu}, \bar\alpha Q] = \frac{\bar\alpha}{2}\begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde\sigma_{\mu\nu}\end{pmatrix} Q, \qquad \sigma_{\mu\nu} = \tfrac{1}{2i}(\sigma_\mu\tilde\sigma_\nu - \sigma_\nu\tilde\sigma_\mu), \quad \tilde\sigma_{\mu\nu} = \tfrac{1}{2i}(\tilde\sigma_\mu\sigma_\nu - \tilde\sigma_\nu\sigma_\mu),$$

in terms of their lefthanded components, we obtain

$$[J_{\mu\nu}, Q_A] = \tfrac{1}{2}(\sigma_{\mu\nu})_A^{\ B}Q_B \quad \text{or} \quad [J_{\mu\nu}, \bar Q^{\dot A}] = \tfrac{1}{2}(\tilde\sigma_{\mu\nu})^{\dot A}_{\ \dot B}\bar Q^{\dot B}, \tag{5.46}$$

Hence, an alternative form of the susy algebra (5.44) reads

$$\begin{aligned} \{Q_A, \bar Q_{\dot B}\} &= 2(\sigma^\mu)_{A\dot B}P_\mu \quad , \quad \{Q_A, Q_B\} = 0 \\ [Q_A, P_\mu] &= 0 \qquad \text{and} \quad [J_{\mu\nu}, Q_A] = \tfrac{1}{2}(\sigma_{\mu\nu})_A^{\ B}Q_B, \end{aligned} \tag{5.47}$$

plus the Poincaré algebra.