14 Fourdimensional (supersymmetric) sigma models

In 4 dimensions a real superfield is not irreducible and therefore we used a chiral superfield to construct the Wess-Zumino model. The Wess-Zumino model is just a linear σ-model and we expect that nonlinear σ-models also make use of chiral superfields. This is indeed the case as we shall demonstrate now.

14.1 Superfield formulation of sigma models

We start with the simplest sigma models in 4 dimensions, namely the ones for which the target space is Kähler manifold with one complex dimension.
Let us discress and explain the notion of a Kähler manifold. One conveniently starts with an *almost complex manifold*: A $2n$-dimensional real manifolds M is called almost complex manifold if we can assign to each point $x \in M$ a linear and bijective map

$$J : TM_x \longrightarrow TM_x$$

of the tangential space at x with the property

$$J^2 = -\mathbb{1}.$$

Almost complex manifolds are always orientable. A hermitean metric on an almost complex manifold M is a Riemannian metric with the additional property that

$$g_x(Jv, Jw) = g_x(v, w) \qquad \forall v, w \in TM_x$$

and for all $x \in M$. Next one may define a 2-form ϕ by

$$\phi_x(v, w) = g_x(v, Jw) \qquad \forall v, w \in TM_x, \quad x \in M.$$

It is called the *fundamental form* of the hermitean metric. If this 2-form is closed,

$$d\phi = 0,$$

the g is called a *Kähler metric* and M almost Kähler manifold.
Now let Φ be a chiral superfield, the body of which propagates on a Kähler manifold,

$$\begin{aligned}
\Phi &= A(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \\
&= A + i\theta\sigma^\mu\bar\theta\partial_\mu A - \tfrac{1}{4}\theta^2\bar\theta^2\Box A + \sqrt{2}\theta\psi + i\sqrt{2}\theta\sigma^\mu\bar\theta\theta\partial_\mu\psi + \theta^2 F = A + \delta\Phi
\end{aligned}$$

such that

$$\begin{aligned}\Phi^\dagger &= A^*(z)+\sqrt{2}\bar\theta\bar\psi(z)+\bar\theta^2F^*(z)\\ &= A^*-i\theta\sigma^\mu\bar\theta\partial_\mu A^*-\tfrac14\theta^2\bar\theta^2\Box A^*+\sqrt{2}\bar\theta\bar\psi-i\sqrt{2}\theta\sigma^\mu\bar\theta\bar\theta\partial_\mu\bar\psi+\bar\theta^2F^*\equiv A^*+\delta\Phi^\dagger.\end{aligned}$$

The argument of the component fields in the last two lines of the previous formulae is x. Now we expand the superfield

$$K(\Phi,\Phi^\dagger)=K(A+\delta\Phi,A^*+\delta\Phi^\dagger)$$

in powers of $\delta\Phi$ and $\delta\Phi^\dagger$. In the course of calculation one needs the Fierz identity

$$(\theta\sigma^\mu\bar\theta)(\theta\psi)(\bar\theta\bar\psi)=\tfrac14\theta^2\,\bar\theta^2\,(\psi\sigma^\mu\bar\psi).$$

After some algebra one finds

$$\begin{aligned}K(\Phi,\Phi^\dagger)&=K+\sqrt{2}(K_A\theta\psi+K_{\bar A}\bar\theta\bar\psi)\\ &+\theta^2\big(K_AF-\tfrac12K_{AA}\psi^2\big)+\bar\theta^2\big(K_{\bar A}F^*-\tfrac12K_{\bar A\bar A}\bar\psi^2\big)+i\theta\sigma^\mu\bar\theta\big(K_A\partial_\mu A-K_{\bar A}\partial_\mu A^*+K_{A\bar A}\psi\sigma_\mu\bar\psi\big)\\ &+i\sqrt{2}\,\theta\sigma^\mu\bar\theta\Big(\partial_\mu\big\{K_A\theta\psi-K_{\bar A}\,\bar\theta\bar\psi\big\}+2K_{A\bar A}\big\{\partial_\mu A\,\bar\theta\bar\psi-\partial_\mu A^*\theta\psi\big\}\Big)\\ &+\sqrt{2}\,\theta^2\,\bar\theta\bar\psi\big(K_{A\bar A}F-\tfrac12K_{AA\bar A}\psi^2\big)+\sqrt{2}\,\bar\theta^2\,\theta\psi\big(K_{A\bar A}F^*-\tfrac12K_{\bar A\bar AA}\bar\psi^2\big)\\ &+\tfrac14\theta^2\bar\theta^2\Big(-\Box K+4K_{A\bar A}\big\{\partial_\mu A\partial^\mu A^*-\tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi+\tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi+FF^*\big\}\\ &\qquad+2K_{AA\bar A}\big\{i\psi\sigma^\mu\bar\psi\partial_\mu A-\psi^2F^*\big\}-2K_{\bar A\bar AA}\big\{i\psi\sigma^\mu\bar\psi\partial_\mu A^*+\bar\psi^2F\big\}+K_{AA\bar A\bar A}\psi^2\bar\psi^2\Big),\end{aligned}$$

where we have used

$$\begin{aligned}\partial_\mu(K_A\theta\psi)&=K_A\theta\partial_\mu\psi+K_{AA}\partial_\mu A\theta\psi+K_{A\bar A}\partial_\mu A^*\theta\psi\\ \Box K&=K_{AA}\partial_\mu A\partial^\mu A+K_{\bar A\bar A}\partial_\mu A^*\partial^\mu A^*+K_A\Box A+K_{\bar A}\Box A^*+2K_{A\bar A}\partial_\mu A\partial^\mu A^*\end{aligned}$$

to simplify the terms trilinear and quartic in θ. Now we may read off the D-term

$$\begin{aligned}\tfrac12K(\Phi,\Phi^\dagger)|_D&=-\tfrac18\Box K+\tfrac12K_{A\bar A}\big(\partial_\mu A^*\partial^\mu A+\tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi-\tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi+F^*F\big)\\ &+\tfrac14K_{AA\bar A}(i\psi\sigma^\mu\bar\psi\partial_\mu A-\psi^2F^*)-\tfrac14K_{A\bar A\bar A}(i\psi\sigma^\mu\bar\psi\partial_\mu A^*+\bar\psi^2F)\\ &+\tfrac18K_{AA\bar A\bar A}\psi^2\bar\psi^2,\end{aligned}$$

The corresponding action should be invariant under

$$\delta A=\sqrt{2}\zeta\psi,\quad \delta\psi=\sqrt{2}\zeta F+i\sqrt{2}(\sigma^\mu\bar\zeta)\partial_\mu A,\quad \delta F=i\sqrt{2}\bar\zeta\tilde\sigma^\mu\partial_\mu\psi.$$

The algebraic equations of motion for the auxiliary field can easily be solved,

$$F=\tfrac12\partial_A\log(K_{A\bar A})\psi^2\quad\text{or}\quad F^*=\tfrac12\partial_{\bar A}\log(K_{A\bar A})\bar\psi^2.$$

14.1.1 Example: The O(3)-Model

Let us now have a closer look at the $O(3)$ sigma model with target manifold S^2. S^2 is a Kähler manifold with Kähler potential

$$K=\log(1+\bar uu),$$

where the complex scalar field is denoted by u instead of A. We need

$$K_u = \frac{\bar{u}}{1+\bar{u}u}, \quad K_{u\bar{u}} = \frac{1}{(1+\bar{u}u)^2}, \quad K_{uu\bar{u}} = -2\frac{\bar{u}}{(1+\bar{u}u)^3}, \quad K_{uu\bar{u}\bar{u}} = 2\frac{2\bar{u}u-1}{(\bar{u}u+1)^4}.$$

Clearly, $K_{\bar{u}}$ is just the complex conjugate of K_u. Now we can easily solve for the auxiliary field,

$$F = -\frac{\bar{u}}{1+\bar{u}u}\psi^2$$

and insert this solution into the supersymmetry transformations and the Lagrangian density. On shell the susy transformations simplify to

$$\delta u = \sqrt{2}\zeta\psi, \quad \delta\psi = -\sqrt{2}\,\frac{\bar{u}\psi^2}{1+\bar{u}u}\zeta + i\sqrt{2}(\sigma^\mu\bar{\zeta})\partial_\mu u,$$

and the Lagrangian takes the form

$$\begin{aligned}
\mathcal{L} &= +\tfrac{1}{2}\frac{1}{(1+\bar{u}u)^2}\Big(\partial_\mu u\partial^\mu\bar{u} + \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} - \tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi}\Big)\\
&\quad +\tfrac{i}{2}\frac{\psi\sigma^\mu\bar{\psi}}{(1+\bar{u}u)^3}\big(u\partial_\mu\bar{u} - \bar{u}\partial_\mu u\big) - \tfrac{1}{4}\frac{1}{(1+u\bar{u})^4}\psi^2\bar{\psi}^2.
\end{aligned}$$

Similarly as in 2 dimensions we find a 4-Fermi interaction and a coupling of the current $\sim \psi\sigma^\mu\bar{\psi}$ to the bosonic field u.

14.1.2 Chiral superfield revisited

Here I want to rewrite a chiral superfield in terms of Majorana spinors and real fields for an easier comparison with the lower dimensional cases. Let α denote the Majorana parameter with corresponding Weyl component θ and recall that

$$\bar{\alpha}\gamma^\mu\gamma_5\alpha = 2\theta\sigma^\mu\bar{\theta}, \qquad \bar{\alpha}\alpha = \theta^2 + \bar{\theta}^2, \quad \bar{\alpha}\gamma_5\alpha = \bar{\theta}^2 - \theta^2$$

from which immediately follows that a chiral superfield has the expansion

$$\begin{aligned}
\Phi &= \Big(A + \frac{1}{\sqrt{2}}(\bar{\alpha}\psi - \bar{\alpha}\gamma_5\psi) + \tfrac{1}{2}(\bar{\alpha}\alpha - \bar{\alpha}\gamma_5\alpha)F\Big)\big(x^\mu + \tfrac{i}{2}\bar{\alpha}\gamma^\mu\gamma_5\alpha\big)\\
&= A + \tfrac{i}{2}\bar{\alpha}\gamma^\mu\gamma_5\alpha\partial_\mu A - \tfrac{1}{8}(\bar{\alpha}\gamma_\mu\gamma_5\alpha)(\bar{\alpha}\gamma_\nu\gamma_5\alpha)\partial_\mu\partial_\nu A\\
&\quad +\frac{1}{\sqrt{2}}(\bar{\alpha}\psi - \bar{\alpha}\gamma_5\psi) + \frac{i}{2\sqrt{2}}(\bar{\alpha}\gamma_\mu\gamma_5\alpha)(\bar{\alpha}\partial_\mu\psi - \bar{\alpha}\gamma_5\partial_\mu\psi) + \tfrac{1}{2}(\bar{\alpha}\alpha - \bar{\alpha}\gamma_5\alpha)F.
\end{aligned}$$

To continue we use the Fierz identities

$$\begin{aligned}
\tfrac{3}{2}\bar{\alpha}\gamma_{(\mu}\gamma_5\alpha\,\bar{\alpha}\gamma_{\nu)}\gamma_5\alpha &= \tfrac{1}{4}\big((\bar{\alpha}\alpha)^2 - (\bar{\alpha}\gamma_5\alpha)^2 + \bar{\alpha}\gamma^\sigma\gamma_5\alpha\bar{\alpha}\gamma_\sigma\gamma_5\alpha\big)\eta_{\mu\nu}\\
\bar{\alpha}\gamma^\sigma\gamma_5\alpha\,\bar{\alpha}\gamma_\sigma\gamma_5\alpha &= 2(\bar{\alpha}\alpha)^2 - 2(\bar{\alpha}\gamma_5\alpha)^2 \Longrightarrow\\
\bar{\alpha}\gamma_{(\mu}\gamma_5\alpha\,\bar{\alpha}\gamma_{\nu)}\gamma_5\alpha &= \tfrac{1}{2}\big((\bar{\alpha}\alpha)^2 - (\bar{\alpha}\gamma_5\alpha)^2\big)\eta_{\mu\nu}
\end{aligned}$$

and rewrite the chiral field as follows

$$\begin{aligned}\Phi &= A + \tfrac{i}{2}\bar\alpha\gamma^\mu\gamma_5\alpha\partial_\mu A - \tfrac{1}{8}(\bar\alpha\alpha)^2\Box A\\ &\quad +\sqrt{2}\bar\alpha P_-\psi + \frac{i}{\sqrt{2}}(\bar\alpha P_-\alpha)(\bar\alpha\not{\partial}P_-\psi) + \bar\alpha P_-\alpha F\end{aligned}$$

The corresponding representation for a antichiral superfield reads

$$\begin{aligned}\Phi^\dagger &= A^* - \tfrac{i}{2}\bar\alpha\gamma^\mu\gamma_5\alpha\partial_\mu A^* - \tfrac{1}{8}(\bar\alpha\alpha)^2\Box A^*\\ &\quad +\frac{1}{\sqrt{2}}(\bar\alpha\psi + \bar\alpha\gamma_5\psi) - \frac{i}{2\sqrt{2}}(\bar\alpha\gamma_\mu\gamma_5\alpha)(\bar\alpha\partial_\mu\psi + \bar\alpha\gamma_5\partial_\mu\psi) + \tfrac{1}{2}(\bar\alpha\alpha + \bar\alpha\gamma_5\alpha)F^*\end{aligned}$$

The sum is a real superfield

$$\begin{aligned}\Phi + \Phi^\dagger &= A_1 - \tfrac{i}{2}\bar\alpha\gamma^\mu\gamma_5\alpha\partial_\mu A^* - \tfrac{1}{16}\big((\bar\alpha\alpha)^2 - (\bar\alpha\gamma_5\alpha)^2\big)\Box A^*\\ &\quad +\frac{1}{\sqrt{2}}(\bar\alpha\psi + \bar\alpha\gamma_5\psi) - \frac{i}{2\sqrt{2}}(\bar\alpha\gamma_\mu\gamma_5\alpha)(\bar\alpha\partial_\mu\psi + \bar\alpha\gamma_5\partial_\mu\psi) + \tfrac{1}{2}(\bar\alpha\alpha + \bar\alpha\gamma_5\alpha)F^*\end{aligned}$$

14.1.3 Higher derivatives

Let Φ be a chiral superfield. On finds the following results for the supercovariant derivatives of Φ:

$$\begin{aligned}D_\alpha\Phi &= 2i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\Phi(y) + \sqrt{2}\psi_\alpha(y) + 2\theta_\alpha F(y)\\ D^\alpha\Phi &= -2i(\bar\theta\tilde\sigma^\mu)^\alpha\partial_\mu\Phi(y) + \sqrt{2}\psi^\alpha(y) + 2\theta^\alpha F(y).\end{aligned}$$

Similarly, for the nonvanishing supercovariant derviatives of $\Phi^\dagger$ we obtain

$$\begin{aligned}\bar D_{\dot\alpha}\Phi &= -2i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\Phi^\dagger(z) + \sqrt{2}\bar\psi_{\dot\alpha}(z) + 2\bar\theta_{\dot\alpha}\bar F(z)\\ \bar D^{\dot\alpha}\Phi^\dagger &= 2i(\tilde\sigma^\mu\theta)^{\dot\alpha}\partial_\mu\Phi^\dagger(z) + \sqrt{2}\bar\psi^{\dot\alpha}(z) + 2\bar\theta^{\dot\alpha}\bar F(z).\end{aligned}$$

It follows that

$$\begin{aligned}D^\alpha\Phi D_\alpha\Phi &= 2\psi^2 + 4\sqrt{2}F\theta\psi + 4\theta^2F^2\\ &\quad +4\sqrt{2}i(\psi\sigma^\mu\bar\theta)\partial_\mu\Phi + 8i(\theta\sigma^\mu\bar\theta)F\partial_\mu\Phi + 4\bar\theta^2\partial_\mu\Phi\partial^\mu\Phi,\\ \bar D_{\dot\alpha}\Phi^\dagger\bar D^{\dot\alpha}\Phi^\dagger &= 2\bar\psi^2 + 4\sqrt{2}\bar F\bar\theta\bar\psi + 4\bar\theta^2\bar F^2\\ &\quad -4\sqrt{2}i(\theta\sigma^\mu\bar\psi)\partial_\mu\Phi^\dagger - 8i(\theta\sigma^\mu\bar\theta)\bar F\partial_\mu\Phi^\dagger + 4\theta^2\partial_\mu\Phi^\dagger\partial^\mu\Phi^\dagger\end{aligned}$$

where the arguments of the fields in the first and second formula are y and z, respectively.

14.2 Four-dimensional Sigma-models with Faddeev-Niemi term

The purely bosonic model has the Lagrangian

$$\mathcal{L} = \frac{m^2}{2}\partial_\mu\vec n\,\partial^\mu\vec n - \frac{1}{4g^2}H_{\mu\nu}H^{\mu\nu}, \quad \text{where} \quad H_{\mu\nu} = \vec n(\partial_\mu\vec n\wedge\partial_\nu\vec n) = \epsilon_{abc}n^a\partial_\mu n^b\partial_\nu n^c,$$

and the target manifold is S^2, i.e. $\vec{n}\cdot\vec{n}=1$. The constant m has the dimension of a mass and g is dimensionaless. To proceed, we define the Cho-connection[1]

$$A_\mu=\epsilon_{abc}\,n^b\partial_\mu n^c\tau_a\quad\text{or}\quad A=\tfrac{1}{2i}[n,dn],\quad n=n^a\tau_a,$$

where we took as basis for the Lie algebra the Pauli-matrices, so that

$$[\tau_a,\tau_b]=2i\epsilon_{abc}\tau_c,\quad \mathrm{Tr}\,(\tau_a\tau_b)=2\delta_{ab},\qquad \mathrm{Tr}\,(nn)=2\vec{n}\cdot\vec{n}.$$

We find

$$\partial_\mu A_\nu-\partial_\nu A_\mu=2\epsilon_{abc}\partial_\mu n^b\partial_\nu n^c\tau_a\quad\text{and}\quad [A_\mu,A_\nu]=2in\,\epsilon_{abc}n^a\partial_\mu n^b\partial_\nu n^c,$$

so that

$$\mathrm{Tr}\,n(\partial_\mu A_\nu-\partial_\nu A_\mu)=4\epsilon_{abc}n_a\partial_\mu n_b\partial_\nu n_c=-i\mathrm{Tr}\,n([A_\mu,A_\nu])\equiv 4H_{\mu\nu}$$

Next we rewrite the potential and field strength in conformally flat coordinates on the target space S^2. To do that we introduce coordinates on $\mathbb{C}$ related to $\vec{n}$ by the stereographic projection defined by

$$n^a=\frac{1}{1+\bar{u}u}\begin{pmatrix}2u^1\\2u^2\\1-\bar{u}u\end{pmatrix}\Longleftrightarrow n=n^a\tau_a=\frac{1}{1+\bar{u}u}\begin{pmatrix}1-\bar{u}u & 2\bar{u}\\ 2u & \bar{u}u-1\end{pmatrix},\qquad u=u^1+iu^2.$$

The metric on on $\mathbb{C}$ reads

$$ds^2=dn^adn^a=g_{ab}du^adu^b,\qquad g_{ab}=\sqrt{g}\,\delta_{ab},\quad \sqrt{g}=\frac{4}{(1+\bar{u}u)^2}$$

In terms of the u-coordinates the Cho-connection reads

$$A=\frac{2}{i}\frac{1}{(1+\bar{u}u)^2}\begin{pmatrix}\bar{u}du-ud\bar{u} & d\bar{u}+\bar{u}^2du\\ -du-u^2d\bar{u} & ud\bar{u}-\bar{u}du\end{pmatrix},$$

so that

$$dA=\frac{4}{i}\frac{1}{(1+\bar{u}u)^3}\begin{pmatrix}1-\bar{u}u & 2\bar{u}\\ 2u & \bar{u}u-1\end{pmatrix}d\bar{u}\wedge du=\frac{4}{i}\frac{1}{(1+\bar{u}u)^2}n\,d\bar{u}\wedge du.$$

The Field strength introduced by Faddeev and Niemi becomes then reads

$$\begin{aligned}\mathrm{Tr}\,(ndA)\; &= \;\frac{8}{i}\frac{1}{(1+\bar{u}u)^2}d\bar{u}\wedge du=2H_{\mu\nu}dx^\mu\wedge dx^\nu\\ H_{\mu\nu}\; &= \;\frac{2}{i}\frac{1}{(1+\bar{u}u)^2}\big(\partial_\mu u\partial_\nu\bar{u}-\partial_\nu u\partial_\mu\bar{u}\big),\quad H=\tfrac12 H_{\mu\nu}dx^\mu dx^\nu.\end{aligned}$$

The Faddeev-Niemi term reads

$$H_{\mu\nu}H^{\mu\nu}=-\frac{8}{(1+\bar{u}u)^4}\Big((\partial_\mu\bar{u}\,\partial^\mu\bar{u})(\partial_\nu u\partial^\nu u)-(\partial_\mu\bar{u}\partial^\mu u)^2\Big).$$

[1] actually, this is only part of the Cho-connection

For static configurations

$$H_{ij}H_{ij} = \frac{8}{(1+\bar{u}u)^4}\Big((\nabla\bar{u}\cdot\nabla u)^2 - (\nabla\bar{u})^2(\nabla u)^2\Big).$$

Now we are ready to rewrite the full Lagrangian density in terms of the u field: With

$$\partial_\mu\vec{n}\,\partial^\mu\vec{n} = \frac{4\partial_\mu u\partial^\mu\bar{u}}{(1+u\bar{u})^2}$$

we obtain

$$\mathcal{L} = 2m\frac{\partial_\mu u\,\partial^\mu\bar{u}}{(1+u\bar{u})^2} - \frac{2}{g^2}\frac{1}{(1+\bar{u}u)^4}\Big((\partial_\mu\bar{u}\partial^\mu\bar{u})(\partial_\nu u\partial^\nu u) - (\partial_\mu\bar{u}\partial^\mu u)^2\Big). \tag{14.1}$$

The energy density for static configurations is just

$$\mathcal{E} = 2m\frac{\nabla u\,\nabla\bar{u}}{(1+u\bar{u})^2} + \frac{2}{g^2}\frac{1}{(1+\bar{u}u)^4}\Big((\nabla\bar{u}\cdot\nabla u)^2 - (\nabla\bar{u})^2(\nabla u)^2\Big). \tag{14.2}$$

14.2.1 Systematic rewriting the invariants

In the derivative expansion of the $\vec{n}$-field model the following terms are present in fourth order

$$(\vec{n}\partial_\mu\partial_\mu\vec{n})(\vec{n}\partial_\mu\partial_\nu n^a), \qquad (\vec{n}\Box\vec{n})(\vec{n}\Box\vec{n}) \quad \text{and} \quad (\Box\vec{n},\Box\vec{n}).$$

We use the previous parametrization of the n field, such that

$$\partial_\mu\vec{n} = \frac{2}{N}\begin{pmatrix}\partial_\mu u^1\\ \partial_\mu u^2\\ 0\end{pmatrix} - \frac{4(u,\partial_\mu u)}{N^2}\begin{pmatrix}u^1\\ u^2\\ 1\end{pmatrix}, \qquad N = 1+(u,u),$$

where for the moment beeing we switch to the real representation for the u-field, $u = (u^1, u^2)$. A straightforward calculation shows that

$$\vec{n}\,\partial_\mu\partial_\nu\vec{n} = -g_{ab}\partial_\mu u^a\partial_\nu u^b$$

holds true. To continue we observe that for a conformally flat metric in 2 dimensional target space

$$\Gamma^a_{bc} = \frac{1}{2e}\big(\delta^a_b\partial_c e + \delta^a_c\partial_b e - \delta^{ad}\delta_{bc}\,\partial_d e\big), \qquad e = \sqrt{g}$$

so that

$$\nabla_\mu\partial^\mu u^a = \partial\partial u^a + \Gamma^a_{bc}\partial_\mu u^b\partial^\mu u^c = \partial\partial u^a + \partial_\mu\log(e)\,\partial^\mu u^a - \tfrac{1}{2}\delta^{ad}\,\partial_d\log(e)\,(\partial u,\partial u).$$

For the 2-sphere $e = 4/N^2$ and

$$\partial_d\log e = -\frac{4}{N}u^d \quad \text{and} \quad \partial_\mu\log e = -\frac{4}{N}(u,\partial_\mu u),$$

so that

$$
\begin{aligned}
g_{ab}\nabla^2 u^a \nabla^2 u^b &= \frac{4}{N^2}(\Box u,\Box u) + \frac{16}{N^3}\Big((\partial u)^2(\partial u)^2 - 2(\Box u,\partial_\mu u)(u,\partial^\mu u) + (\Box u,u)(\partial u)^2\Big) \\
&+ \frac{16}{N^4}\Big(4(u,\partial_\mu u)(u,\partial_\nu u)(\partial^\mu u,\partial^\nu u) - (\partial u)^2(\partial u)^2 - 4(u,\partial u)^2(\partial u)^2\Big)
\end{aligned}
\tag{14.3}
$$

We want to compare with the invariant constructed out of

$$
\Box\vec{n} = \Big(\frac{16}{N^3}(u,\partial u)^2 - \frac{4}{N}\big\{(\partial u)^2 + (u,\Box u)\big\} + \frac{2}{N}\Box - \frac{8}{N^2}(u,\partial_\mu u)\,\partial^\mu\Big)\begin{pmatrix}u^1\\ u^2\\ 1\end{pmatrix}.
$$

After a rather lengthy calculation one finds

$$
\Big(\Box\vec{n} - (\vec{n},\Box\vec{n})\vec{n},\Box\vec{n} - (\vec{n},\Box\vec{n})\vec{n}\Big) = \big(\Box\vec{n},\Box\vec{n}\big) - (\vec{n},\Box\vec{n})^2 = g_{ab}\nabla^2 u^a \nabla^2 u^b. \tag{14.4}
$$

Let us summarize our findings: Up to fourth derivatives there are four invariant actions operators, and their form in the u-variables are

$$
\begin{aligned}
S_1 &= \int \big(\partial\vec{n},\partial\vec{n}\big) = \int g_{ab}\partial_\mu u^a \partial^\mu u^b \\
S_2 &= \int \big(\Box\vec{n},\Box\vec{n}\big) = \int g_{ab}\nabla^2 u^a\,\nabla^2 u^b + \int \Big(g_{ab}\partial_\mu u^a\partial^\mu u^b\Big)^2 \\
S_3 &= \int \big(\vec{n}\Box\vec{n},\vec{n}\Box\vec{n}\big) = \int \Big(g_{ab}\partial_\mu u^a\partial^\mu u^b\Big)^2 \\
S_4 &= \int \big(\vec{n}\partial_\mu\partial_\nu\vec{n},\vec{n}\partial^\mu\partial^\nu\vec{n}\big) = \int g_{ab}\partial_\mu u^a\partial_\nu u^b\; g_{cd}\partial^\mu u^c\partial^\nu u^d.
\end{aligned}
$$

The celebrated Faddeev-Niemi term is just the difference $S_3 - S_4$:

$$
S_{FN} = \int H_{\mu\nu}H^{\mu\nu} = S_3 - S_4 = \int \Big(\big(g_{ab}\partial_\mu u^a\partial^\mu u^b\big)^2 - g_{ab}\partial_\mu u^a\partial_\nu u^b\; g_{cd}\partial^\mu u^c\partial^\nu u^d\Big).
$$

Not suprisingly these expression find a simpler form in term of the complex field $u = u^1 + iu^2$, since the target space is Kähler. First we rewrite the covariant Laplacian. With

$$
(\nabla^2 u)^1 + i(\nabla^2 u)^2 = \partial^2 u - \frac{4}{N}(u,\partial^\mu u)\partial_\mu u + \frac{2}{N}(\partial u,\partial u)u
$$

and

$$
(u,\partial^\mu u) = \tfrac{1}{2}\partial^\mu(u\bar{u}) \quad \text{and} \quad (\partial u,\partial u) = \partial_\mu u\partial^\mu\bar{u}
$$

we obtain

$$
\begin{aligned}
(\nabla^2 u)^1 + i(\nabla^2 u)^2 &= \partial^2 u - \frac{2}{N}\partial^\mu(u\bar{u})\partial_\mu u + \frac{2}{N}(\partial u\partial^\mu\bar{u})u = \partial^2 u - \frac{2}{N}(\partial u)^2\,\bar{u} \\
(\nabla^2 u)^1 - i(\nabla^2 u)^2 &= \partial^2 \bar{u} - \frac{2}{N}\partial^\mu(u\bar{u})\partial_\mu \bar{u} + \frac{2}{N}(\partial u\partial^\mu\bar{u})\bar{u} = \partial^2 \bar{u} - \frac{2}{N}(\partial\bar{u})^2\,u.
\end{aligned}
\tag{14.5}
$$

Hence that the above invariants can be rewritten as

$$
\begin{aligned}
S_1 &= \int e\, \partial u \partial \bar{u} & S_2 - S_3 &= \int e^2\, \left|\partial^2 u - \tfrac{2}{N}(\partial u)^2 \bar{u}\right|^2 \\
S_3 &= \int e^2\, \left(\partial u \partial \bar{u}\right)^2 & S_4 - \tfrac{1}{2} S_3 &= \tfrac{1}{2} \int e^2\, (\partial u \partial u)(\partial \bar{u} \partial \bar{u}).
\end{aligned}
\tag{14.6}
$$

The Faddeev-Niemi term has the simple form

$$
S_{FN} = \tfrac{1}{2} \int e^2\, \Big((\partial u \partial \bar{u})(\partial u \partial \bar{u}) - (\partial u \partial u)(\partial \bar{u} \partial \bar{u}) \Big). \tag{14.7}
$$

14.2.2 Equations of motion and fluctuations

We slightly generalize the model and assume that spacetime is curved with metric $h_{\mu\nu}$ in which case

$$
\mathcal{L} = \frac{m^2}{2} \sqrt{-h}\, h^{\mu\nu} g_{ab} \partial_\mu u^a \partial_\nu u^b - \frac{1}{4e^2} \sqrt{-h}\, h^{\mu\alpha} h^{\nu\beta} H_{\mu\nu} H_{\alpha\beta} = \mathcal{L}_\sigma + \mathcal{L}_{FN}
$$

where we have introduced the antisymmetric tensor

$$
H_{\mu\nu} = \eta_{ab}\, \partial_\mu u^a \partial_\nu u^b.
$$

Actually, the have now changed the sign of $H_{\mu\nu}$! We made use of the skewsymmetric pseudotensor

$$
\eta_{ab} = \sqrt{g}\, \epsilon_{ab}, \quad \epsilon_{12} = 1
$$

which is related to the closed volume form η on the target manifold via

$$
\eta = \tfrac{1}{2} \eta_{ab} du^a \wedge du^b.
$$

We are going to expand the action

$$
S[u + tv] = S[u] + t(S', v) + \tfrac{1}{2} t^2 (v, S'' v) + O(t^3)
$$

to determine the first and second derivative of S. The first derivative yields the equations of motion and the second derivative is necessary for the stability analysis and the semiclassical quantisation. First we vary the sigma-model term S_σ:

$$
\begin{aligned}
S_\sigma[u + v] &= \frac{m^2}{2} \int \sqrt{-h} h^{\mu\nu} g_{ab}(u + v) \partial_\mu (u^a + v^a) \partial_\nu (u^b + v^b) \\
&= \frac{m^2}{2} \int \sqrt{-h} \big(g_{ab} + g_{ab,c} v^c + \tfrac{1}{2} g_{ab,cd} v^c v^d \big) \big\{ \partial_\mu u^a \partial^\mu u^b + 2 \partial_\mu v^a \partial^\mu u^b + \partial_\mu v^a \partial^\mu v^b \big\} \\
&= S_\sigma[u] + \frac{m^2}{2} \int \sqrt{-h}\, h^{\mu\nu} \big\{ 2 g_{ab}\, \partial_\mu v^a \partial_\nu u^b + g_{ab,c} v^c \partial_\mu u^a \partial_\nu u^b \big\} \\
&\qquad + \frac{m^2}{2} \int \sqrt{-h}\, h^{\mu\nu} \big\{ g_{ab} \partial_\mu v^a \partial_\nu v^b + 2 g_{ab,c} v^c \partial_\mu v^a \partial_\nu u^b + \tfrac{1}{2} g_{ab,cd}\, v^c v^d \partial_\mu u^a \partial_\nu u^b \big\}
\end{aligned}
$$

Next we replace the fluctuation v^a by the tangent ξ^a to the geodesic from u to $u + v$ (we set $t = 1$):

$$
v^a = \xi^a - \tfrac{1}{2} \Gamma^a_{\ pq} \xi^p \xi^q + O(\xi^3)
$$

Inserting $v(\xi)$ the first order terms partly transforms into the second order terms. With the identities used in section (13.1.1) we arrive at the following expansion

$$\begin{aligned} S_\sigma[u+v] &= S_\sigma[u] + \frac{m^2}{2}\int \sqrt{-h}\, h^{\mu\nu}\big\{g_{ab}(\partial_\mu\xi^a\partial_\nu u^b + \partial_\mu u^a\partial_\nu\xi^b) + g_{ab,c}\xi^c\partial_\mu u^a\partial_\nu u^b\big\} \\ &\quad + \frac{m^2}{2}\int \sqrt{-h}\, h^{\mu\nu}\big(g_{ab}\nabla_\mu\xi^a\nabla_\nu\xi^b - R_{acbd}\partial_\mu u^a\partial_\nu u^b\xi^c\xi^d\big). \end{aligned}$$

We read off the first derivative of S_σ as follows

$$(S'_\sigma,\xi) \;=\; m^2\int\sqrt{-h}g_{ab}\nabla_\mu\xi^a\partial^\mu u^b, \quad \text{where} \quad \nabla_\mu\xi^a = \partial_\mu\xi^a + \Gamma^a_{\ bc}\partial_\mu u^b\xi^c.$$

Here $\Gamma^a_{\ bc}$ are the Christoffel symbols of the target manifold. We need the following formula: If A^a_μ is a vector with respect to spacetime and target space diffeomorphisms, e.g. $A^a_\mu = \partial_\mu u^a$, then the following formula holds up to surface terms,

$$\int\sqrt{-h}\, g_{ab}\,\nabla_\mu\xi^a A^{b\nu} = -\int\sqrt{-h}\, g_{ab}\,\xi^a\nabla_\mu A^{b\nu}, \quad \nabla_\mu A^{b\nu} = \partial_\mu A^{b\mu} + \gamma^\nu_{\ \mu\alpha}A^{b\alpha} + \Gamma^b_{\ cd}\partial_\mu u^c A^{d\nu}.$$

Here $\gamma^\nu_{\ \mu\alpha}$ are the Christoffel symbols corresponding to the spacetime metric. I leave the proof for an exercise in partial integration. Hence we may partially integrate in (S'_σ,ξ) which results in

$$(S'_\sigma,\xi) = -m^2\int\sqrt{-h}\, g_{ab}\xi^a\nabla_\mu\partial^\mu u^b = -m^2\int\sqrt{-h}\, g_{ab}\xi^a\big(\Box u^b + \Gamma^b_{\ cd}h^{\mu\nu}\partial_\mu u^c\partial_\nu u^d\big).$$

Without Faddeev-Niemi term the field equations read

$$\Box u^a + \Gamma^a_{\ bc}\,\partial_\mu u^b\partial^\mu u^c = 0. \tag{14.8}$$

Hence u would be a harmonic map from spacetime to the 2-dimensional target manifold. Let us see that this equation is covariant: under spacetime diffeomorphism $x\to y$ the equation is covariant since the $\partial_\mu u^a$ are vector fields,

$$\frac{\partial u^a}{\partial x^\mu} = \Big(\frac{\partial y^\nu}{\partial x^\mu}\Big)\frac{\partial u^a}{\partial y^\nu}$$

Under field transformations

$$u^a \longrightarrow u^a(w^1,w^2).$$

the objects $\partial_\mu u^a$ are vectors as well,

$$\partial_\mu u^a = \Big(\frac{\partial u^a}{\partial w^b}\Big)\partial_\mu w^b$$

and the metric transforms as usual

$$g_{ab}(u)\partial_\mu u^a\partial_\nu u = \frac{\partial u^a}{\partial w^c}\frac{\partial u^b}{\partial w^d}\, g_{ab}(u)\,\partial_\mu w^c\partial_\nu w^d = \tilde g_{cd}(w)\,\partial_\mu w^c\partial_\nu w^d.$$

Now it is a easy exercise to prove, that

$$D_\mu\partial_\nu u^a, \qquad D_\mu v^a_\nu = \partial_\mu v^a_\nu - \gamma^\alpha_{\ \mu\nu}v^a_\alpha + \Gamma^a_{\ bc}\partial_\mu u^b v^c_\nu,$$

where γ are the Christoffel sympols of spacetime and Γ are those of the target space, transforms as a 2 tensor under spacetime diffeomorphism and as vector under target space diffeomorphism, provided the target space connection transforms properly,

$$\tilde{\Gamma}^a_{\ bc} = \tau^a_{\ p}\big(t^p_{\ b}t^q_{\ c}\Gamma^p_{\ pq} + \frac{\partial^2 u^p}{\partial w^b \partial w^c}\big), \qquad \tau^a_{\ p} = \frac{\partial w^a}{\partial u^p}, \quad t^p_{\ b} = \frac{\partial u^p}{\partial w^b}.$$

It follows that the wave operator compatibel with both set of diffeomorphism is

$$D^2 u^a = h^{\mu\nu} D_\mu \partial_\nu u^a = \Box u^a + \Gamma^a_{\ bc}\partial_\mu u^b \partial^\mu u^c.$$

Remark: The equations (14.8) should be compared with equation

$$\ddot{x}^a + \Gamma^a_{\ bc}\dot{x}^b \dot{x}^c = 0$$

for the geodesic motion of a particle on a curved target space.
Now we turn to the second derivative of the sigma-model term

$$\begin{aligned}(\xi, S''\xi) &= m^2 \int \sqrt{-h}\, h^{\mu\nu}\big(g_{ab}\nabla_\mu \xi^a \nabla_\nu \xi^b - R_{cadb}\partial_\mu u^c \partial_\nu u^d \xi^a \xi^b\big) \\ &= -m^2 \int \sqrt{-h}\, \xi^a \big(g_{ab}\nabla^\mu \nabla_\mu \xi^b + R_{cadb}\partial_\mu u^c \partial^\mu u^d \xi^b\big).\end{aligned}$$

We read off the differential equations for fluctuation about a background field u^a:

$$\nabla^\mu \nabla_\mu \xi^a - R^a_{\ cdb}\,\partial_\mu u^c\,\partial^\mu u^d\,\xi^b = \lambda \xi^a. \tag{14.9}$$

Now we turn to the variation of the Faddeev-Niemi term: We use the elegant method of Mukhi, see section (13.1.3), to find the expansion of $\mathcal{L}_{FN}$. First we recall that the antisymmetric pseudo tensor η_{ab} is covariantly conserved. In 2 dimensions the proof is elementary:

$$\nabla_c \eta_{ab} = \partial_c(\sqrt{g})\epsilon_{ab} - \Gamma^d_{\ ca}\eta_{db} - \Gamma^d_{\ cb}\eta_{ad} = \sqrt{g}\big(\Gamma^d_{\ cd}\,\epsilon_{ab} - \Gamma^d_{\ ca}\epsilon_{db} - \Gamma^d_{\ cb}\epsilon_{ad}\big).$$

We only need to check that this vanishes for $a = 1, b = 2$:

$$\nabla_c \eta_{12} = \sqrt{g}\big(\Gamma^d_{\ cd} - \Gamma^1_{\ c1} - \Gamma^2_{\ c2}\big) = 0.$$

Again we consider a geodesic $u(s)$ which interpolates between $u = u(0)$ and $u(1) = u + v$ and calculate the change of $H^2(s) = H_{\mu\nu}(s)H^{\mu\nu}(s)$ along this curve:

$$\begin{aligned}H^2(s) &= H^2(u) + s\nabla_s H^2(s)|_{s=0} + \tfrac{1}{2}s^2(\nabla_s)^2 H^2(s)|_{s=0} + O(s^3) \\ &= H^2(u) + 2sH^{\mu\nu}(u)\nabla_s H_{\mu\nu}|_{s=0} + s^2\big(\nabla_s H_{\mu\nu}\nabla_s H^{\mu\nu} + H^{\mu\nu}(\nabla_s)^2 H_{\mu\nu}\big).\end{aligned}$$

Clearly we need the variation of $H_{\mu\nu}$ along the geodesic: The first derivative is

$$\nabla_s H_{\mu\nu}(s) = \eta_{ab}\nabla_s(\partial_\mu u^a \nabla_\nu u^b) = \eta_{ab}\big(\nabla_\mu \dot{u}^a \partial_\nu u^b + \partial_\mu u^a \nabla_\nu \dot{u}^b\big)$$

and its value at $s = 0$ becomes

$$\nabla_s H_{\mu\nu}(0) = \eta_{ab}\big(\nabla_\mu \xi^a \partial_\nu u^b + \partial_\mu u^a \nabla_\nu \xi^b\big). \tag{14.10}$$

The second derivative is found to be

$$\begin{aligned}(\nabla_s)^2 H_{\mu\nu}(s) &= \eta_{ab}\nabla_s\big(\nabla_\mu \dot{u}^a \partial_\nu u^b + \partial_\mu u^a \nabla_\nu \dot{u}^b\big) \\ &= \eta_{ab}\big([\nabla_s, \nabla_\mu]\dot{u}^a \partial_\nu u^b + \partial_\mu u^a [\nabla_s, \nabla_\nu]\dot{u}^b + \nabla_\mu \dot{u}^a \nabla_\nu \dot{u}^b + \nabla_\mu \dot{u}^a \nabla_\nu \dot{u}^b\big)\end{aligned}$$

so that its value at the origin reads

$$(\nabla_s)^2 H_{\mu\nu}(0) = \eta_{ab}\big(R^a{}_{cdp}\partial_\mu u^p \partial_\nu u^b \xi^c\xi^d + R^b{}_{cdp}\partial_\nu u^p \partial_\mu u^a \xi^c\xi^d + 2\nabla_\mu\xi^a\nabla_\nu\xi^b\big) \tag{14.11}$$

Now we are ready to calculate the first variation of the Faddeev-Niemi term

$$(S'_{FN},\xi) = -\frac{1}{e^2}\int \sqrt{-h}H^{\mu\nu}\eta_{ab}\nabla_\mu\xi^a\nabla_\nu u^b = \frac{1}{e^2}\int \sqrt{-h}\nabla_\mu\big(\eta_{ab}H^{\mu\nu}\nabla_\nu u^b\big)\xi^a.$$

Together with the previous result we obtain the formula for the first order variations of the complete action:

$$(S',\xi) = \int \sqrt{-h}g_{bc}\Big(-m^2\big(\Box u^c + \Gamma^c{}_{pq}\partial_\mu u^p\partial_\nu u^q\big) + \frac{1}{e^2}\,\nabla_\nu\big(H^{\mu\nu}\partial_\mu u^a\big)\eta_a{}^c\Big)\xi^b$$

and this gives rise to the following equations of motion

$$m^2\big(\Box u^c + \Gamma^c{}_{pq}\partial_\mu u^p\partial_\nu u^q\big) = \frac{1}{e^2}\,\nabla_\nu\big(H^{\mu\nu}\partial_\mu u^a\big)\eta_a{}^c \tag{14.12}$$

Now we come to the most difficult part, the calculation of the second order terms induced by the FN-action. We find

$$\begin{aligned}\nabla_s H_{\mu\nu}\nabla_s H^{\mu\nu} + (\nabla_s)^2 H_{\mu\nu}H^{\mu\nu} &= 2\eta_{ab}\eta_{pq}\big(\partial_\nu u^b\partial^\nu u^q\,\nabla^\mu\xi^p\nabla_\mu\xi^a + \partial_\mu u^a\partial^\nu u^q\,\nabla^\mu\xi^p\nabla_\nu\xi^b\big)\\ &+\ 2\eta_{ab}H^{\mu\nu}\big(R^a_{\ cdp}\partial_\mu u^p\partial_\nu u^b\xi^c\xi^d + \nabla_\mu\xi^a\nabla_\nu\xi^b\big)\end{aligned}$$

Adding the second variation of the sigma-model term results in

$$\begin{aligned}(\xi S'',\xi) &= -m^2\int\sqrt{-h}\,\xi^a\big(g_{ab}\nabla^\mu\nabla_\mu\xi^b + R_{cadb}\partial_\mu u^c\partial^\mu u^d\xi^b\big)\\ &-\ \frac{1}{e^2}\int\sqrt{-h}\Big(\eta_{ab}\eta_{pq}\big(\partial_\nu u^b\partial^\nu u^q\,\nabla^\mu\xi^p\nabla_\mu\xi^a + \partial_\mu u^a\partial^\nu u^q\,\nabla^\mu\xi^p\nabla_\nu\xi^b\big)\\ &\qquad + \eta_{ab}\,H^{\mu\nu}\big(R^a_{\ cdp}\partial_\mu u^p\partial_\nu u^b\xi^c\xi^d + \nabla_\mu\xi^a\nabla_\nu\xi^b\big)\Big)\end{aligned} \tag{14.13}$$

or after partial integration

$$\begin{aligned}(\xi S'',\xi) &= -m^2\int\sqrt{-h}\,\xi^a\big(g_{ab}\nabla^\mu\nabla_\mu\xi^b + R_{cadb}\partial_\mu u^c\partial^\mu u^d\xi^b\big)\\ &+\ \frac{1}{e^2}\int\sqrt{-h}\xi^a\eta_{ab}\eta_{cd}\nabla^\mu\Big(\partial_\mu u^c\partial^\nu u^b\nabla_\nu\xi^d - \partial_\nu u^b\partial^\nu u^c\nabla_\mu\xi^d\Big)\\ &+\ \frac{1}{e^2}\int\sqrt{-h}\xi^a\Big(\eta_{ab}\nabla_\mu(H^{\mu\nu}\partial_\nu\xi^b) - \eta_{bc}H^{\mu\nu}R^b_{\ adp}\partial_\mu u^p\partial_\nu u^c\xi^d\Big),\end{aligned} \tag{14.14}$$

where we used that the covariant derivative of the pseudotensor η_{ab} vanishes. This then yields the following equations for the fluctuations

$$\begin{aligned}
&-m^2\nabla^2\xi^a + m^2 R^a_{bcd}\partial_\mu u^b \partial^\mu u^c \xi^d \\
+ \quad & \frac{1}{e^2}\eta^a{}_b\eta_{cd}\nabla^\mu\Big(\partial_\mu u^c\partial^\nu u^b\nabla_\nu\xi^d - \partial_\nu u^b\partial^\nu u^c\nabla_\mu\xi^d\Big) \\
+ \quad & \frac{1}{e^2}\eta^a{}_b\nabla_\mu(H^{\mu\nu}\partial_\nu\xi^b) - \frac{1}{e^2}\eta_c{}^b H^{\mu\nu}R^a_{bdp}\partial_\mu u^p\partial_\nu u^c\xi^d = \lambda^2\xi^a,
\end{aligned} \tag{14.15}$$

14.2.3 Hopfinvariant

In this subsection we are going to express the Hopfinvariant in terms of the u-field. For that we need a one-form, denoted by a, the differential of which is the abelian field strength H,

$$da = H.$$

Actually, we may easily write down such an abelian potential,

$$a = d\theta + \frac{1}{i}\frac{\bar{u}du - ud\bar{u}}{1+\bar{u}u}.$$

Indeed, we find

$$da = \frac{2}{i}\frac{1}{(1+\bar{u}u)^2}d\bar{u}\wedge du = H.$$

For the Hopf-density we find

$$h \sim a\wedge H = \frac{2}{i}\frac{1}{(1+\bar{u}u)^2}\, d\theta\wedge d\bar{u}\wedge du$$

At this point I do not know what to insert for θ. To understand this apparent ambiguity we switch to yet another parametrization. We use that the Hopf invariant invariant has a simple geometric interpretation in terms of $g\in SU(2)$ defined (up to a $U(1)$-factor) by

$$n = g\tau_3 g^{-1}, \quad g = \frac{1}{(1+\bar{u}u)^{1/2}}\begin{pmatrix} 1 & -\bar{u} \\ u & 1\end{pmatrix}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{pmatrix} = \frac{1}{(1+\bar{u}u)^{1/2}}\begin{pmatrix} e^{i\phi} & -\bar{u}e^{-i\phi} \\ ue^{i\phi} & e^{-i\phi}\end{pmatrix}.$$

The Hopfinvariant is just the winding number of $g\in SU(2)$ as function of the spatial variables. Hence we assume that g is time-independent and tends to a constant element for large distances from the origin. From

$$g^{-1}dg = i\tau_3 d\phi + \tfrac{1}{2}\frac{1}{1+\bar{u}u}\begin{pmatrix} \bar{u}du - ud\bar{u} & -2e^{-2i\phi}d\bar{u} \\ 2e^{2i\phi}du & ud\bar{u}-\bar{u}du\end{pmatrix}$$

we may calculate the density giving rise to the Hopf invariant. The powers of the potential $g^{-1}dg$ are:

$$\begin{aligned}
(g^{-1}dg)^2 &= \frac{2i}{1+\bar{u}u}\begin{pmatrix} 0 & e^{-2i\phi}d\bar{u} \\ e^{2i\phi}du & 0\end{pmatrix}\wedge d\phi + \frac{1}{(1+\bar{u}u)^2}\begin{pmatrix} -1 & e^{-2i\phi}\bar{u} \\ e^{2i\phi}u & 1\end{pmatrix} d\bar{u}\wedge du \\
(g^{-1}dg)^3 &= -\frac{3i}{(1+\bar{u}u)^2}\mathbb{1}\, d\bar{u}\wedge du\wedge d\phi
\end{aligned}$$

such that the winding number density is proportional to

$$\mathrm{Tr}\,(g^{-1}dg)^3 = -6i\frac{1}{(1+\bar{u}u)^2}d\bar{u}\wedge du\wedge d\phi.$$

To check the normalization we use the standard Hopf map

$$g = i\sin\theta\begin{pmatrix} e^{-i\phi_{34}} & e^{-i\phi_{12}}\cot\theta \\ e^{i\phi_{12}}\cot\theta & -e^{i\phi_{34}} \end{pmatrix} = \begin{pmatrix} x_3+ix_2 & x_1+ix_0 \\ -x_1+ix_0 & x_3-ix_2 \end{pmatrix}.$$

where we introduced the coordinates

$$\begin{aligned} \hat{x}^\mu &= \big(\cos\theta\cos\phi_{12},\cos\theta\sin\phi_{12},\sin\theta\cos\phi_{34},\sin\theta\sin\phi_{34}\big) \\ & \quad 0\le\theta<\pi/2,\quad 0\le\phi_{12},\phi_{34}<2\pi \end{aligned}$$

on the 3-sphere. Comparing with the parametrization of g used above we find the identifications

$$e^{i\phi} = ie^{-i\phi_{34}},\quad \sin^{-2}\theta = 1+\bar{u}u,\quad u=\cot\theta\, e^{i(\phi_{12}+\phi_{34})},$$

such that the winding number density is proportional to

$$\mathrm{Tr}\,(g^{-1}dg)^3 = 6\sin 2\theta d\theta\wedge d\phi_{12}\wedge d\phi_{34}.$$

On the other hand, since

$$\frac{1}{4\pi^2}\int_{S^3}\sin 2\theta d\theta\wedge d\phi_{12}\wedge d\phi_{34} = 1$$

the correct normalization for the Hopf-density is

$$q = \frac{1}{24\pi^2}\mathrm{Tr}\,(g^{-1}dg)^3 = \frac{1}{4i\pi^2}\frac{1}{(1+\bar{u}u)^2}d\bar{u}\wedge du\wedge d\phi.$$

There are problems when one tries to find the Hopfinvariant in terms of u. It is nonolocal, therefore we turn to the

14.3 Alternative formulations

In this section we shall investigate alternative formulations of the models just considered. We begin with the

14.3.1 CP_1 formulation

We introduce

$$n_a = z^\dagger\tau_a z,\quad z=\begin{pmatrix} z_0 \\ z_1\end{pmatrix}\in\mathbb{C}^2,\quad z^\dagger z=1.$$

More explicitly

$$n_1-in_2 = 2\bar{z}_1z_0,\quad n_3=\bar{z}_0z_0-\bar{z}_1z_1,\quad u=\frac{z_1}{z_0},\quad 1+\bar{u}u = 1/|z_0|^2.$$

First we rewrite the σ-model term in terms of the CP_1 field. Since

$$\partial_\mu u = \frac{1}{z_0^2}(z_0\partial_\mu z_1 - z_1\partial_\mu z_0)$$

we find

$$\begin{aligned}\frac{\partial_\mu u\partial^\mu \bar{u}}{(1+\bar{u}u)^2} &= |z_0|^2\partial_\mu z_1\partial^\mu \bar{z}_1 + |z_1|^2\partial_\mu z_0\partial^\mu \bar{z}_0 - z_1\partial_\mu \bar{z}_1\bar{z}_0\partial^\mu z_0 - z_0\partial_\mu \bar{z}_0\bar{z}_1\partial^\mu z_1 \\ &= \partial_\mu z^\dagger\partial_\mu z + (z^\dagger\partial_\mu z)(z^\dagger\partial^\mu z),\end{aligned}$$

where we used that z has length one. Introducing the $U(1)$ gauge potential and covariant derivative as

$$A_\mu = -iz^\dagger\partial_\mu z, \quad D_\mu z = (\partial_\mu - iA_\mu)z,$$

this can be rewritten as

$$\frac{\partial_\mu u\partial^\mu \bar{u}}{(1+\bar{u}u)^2} = (D_\mu z)^\dagger(D^\mu z).$$

The proof is simple,

$$(D_\mu z)^\dagger(D^\mu z) = \partial_\mu z^\dagger\partial^\mu z + A_\mu A^\mu z^\dagger z - iA^\mu\partial_\mu z^\dagger z + iA_\mu z^\dagger\partial^\mu z = \partial_\mu z^\dagger\partial^\mu z - A_\mu A^\mu.$$

Next we wish to express the Faddeev-Niemi term in terms of the CP_1 field. We calculate

$$\begin{aligned}&\frac{\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u}{(1+\bar{u}u)^2} = (\bar{z}_0\partial_\mu \bar{z}_1 - \bar{z}_1\partial_\mu \bar{z}_0)(z_0\partial_\nu z_1 - z_1\partial_\nu z_0) - (\mu\leftrightarrow\nu) \\ &= |z_0|^2\partial_\mu \bar{z}_1\partial_\nu z_1 + |z_1|^2\partial_\mu \bar{z}_0\partial_\nu z_0 - z_1\partial_\mu \bar{z}_1\bar{z}_0\partial_\nu z_0 - z_0\partial_\mu \bar{z}_0\bar{z}_1\partial_\nu z_1 - (\mu\leftrightarrow\nu) \\ &= \left\{\partial_\mu \bar{z}_1\partial_\nu z_1 + \partial_\mu \bar{z}_0\partial_\nu z_0 + (z^\dagger\partial_\mu z)(z^\dagger\partial_\nu z)\right\} - \left\{\mu\leftrightarrow\nu\right\} \\ &= \partial_\mu \bar{z}_1\partial_\nu z_1 + \partial_\mu \bar{z}_0\partial_\nu z_0 - \partial_\nu \bar{z}_1\partial_\mu z_1 - \partial_\nu \bar{z}_0\partial_\mu z_0 = i(\partial_\mu A_\nu - \partial_\nu A_\mu)\end{aligned}$$

Hence we have proven the nice formula

$$\frac{\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u}{(1+\bar{u}u)^2} = iF_{\mu\nu}.$$

Now we proceed to calculate the Faddeev-Niemi density in terms of u. After some algebra one obtains

$$\begin{aligned}d\vec{n}\wedge d\vec{n} &= \frac{8}{(1+\bar{u}u)^2}\vec{n}du_1\wedge du_2 \quad \text{or} \\ \left(d\vec{n}\wedge d\vec{n}\right)_a &= \epsilon_{abc}\partial_\mu n^b\partial_\nu n^b dx^\mu\wedge dx^\nu = \frac{8}{(1+\bar{u}u)^2}\, n_a\, du_1\wedge du_2\end{aligned}$$

Since

$$id\bar{u}\wedge du = 2du_1\wedge du_2 \Longrightarrow du_1\wedge du_2 = \tfrac{i}{4}(\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u)dx^\mu\wedge dx^\nu$$

we may rewrite

$$\begin{aligned}\vec{n}(d\vec{n}\wedge d\vec{n}) &= \epsilon_{abc}n^a\partial_\mu n^b\wedge\partial_\nu n^c dx^\mu\wedge dx^\nu \\ &= \frac{8}{(1+\bar{u}u)^2}\tfrac{i}{4}(\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u)dx^\mu\wedge dx^\nu = -2F_{\mu\nu}dx^\mu\wedge dx^\nu.\end{aligned}$$

As a consequence, the antisymmetric tensor $H_{\mu\nu}$ is proportional to the field strength of A_μ,

$$H_{\mu\nu} = \epsilon_{abc} n^a (\partial_\mu n^b \wedge \partial_\nu n^c) = -2F_{\mu\nu}.$$

For the standard Hopf map

$$\vec{n} = \begin{pmatrix} 2(x_1x_3 - x_2x_4) \\ 2(x_1x_4 + x_2x_3) \\ x_3^2 + x_4^2 - x_1^2 - x_2^2 \end{pmatrix} = \begin{pmatrix} \sin 2\theta \cos(\phi_{12} + \phi_{34}) \\ \sin 2\theta \sin(\phi_{12} + \phi_{34}) \\ -\cos 2\theta \end{pmatrix} = \begin{pmatrix} \bar{z}_0 z_1 + \bar{z}_1 z_0 \\ i\bar{z}_1 z_0 - i\bar{z}_0 z_1 \\ \bar{z}_0 z_0 - \bar{z}_1 z_1 \end{pmatrix}$$

we may take

$$z = \begin{pmatrix} \sin\theta\, e^{-i\phi_{12}} \\ \cos\theta\, e^{i\phi_{34}} \end{pmatrix}.$$

For the gauge potential and field strength we obtain

$$A = -i\dot{2}dz = \cos^2\theta\, d\phi_{34} - \sin^2\theta\, d\phi_{12}, \qquad F = -\sin 2\theta\, d\theta \wedge (d\phi_{12} + d\phi_{34}).$$

It follows that

$$A \wedge F = \sin 2\theta\; d\theta \wedge d\phi_{12} \wedge d\phi_{34} \quad \text{and} \quad \frac{1}{4\pi^2} \int_{S^3} A \wedge F = 1.$$

Let us finally summarize what we have gotten so far. The effective Lagrangian density takes the form

$$\mathcal{L} = \frac{m^2}{2} (D_\mu z)^\dagger D^\mu z - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu},$$

where

$$A = -iz^\dagger dz, \quad D_\mu = \partial_\mu - iA_\mu, \quad F = dA.$$

For static fields and in the Weyl gauge we obtain

$$-\mathcal{L} = \frac{m^2}{2} (D_i z)^\dagger D_i z + \frac{1}{4e^2} F_{ij} F^{ij},$$

The properly normalized Hopf invariant reads

$$H = \frac{1}{4\pi^2} \int d^3x A \wedge F = \frac{1}{8\pi^2} \int \epsilon_{ijk} A_i F_{jk}\, d^3x.$$

We may write the energy as a square. For that we notice that

$$z^\dagger D_i z = z^\dagger \partial_i z - iA_i = iA_i - iA_i = 0 = -(D_i z) z^\dagger,$$

such that with

$$\tilde{D}_i z \equiv D_i z - i\epsilon_{ijk}\, F_{jk}$$

we obtain

$$\begin{aligned} |\tilde{D}_i z|^2 &= (D_i z)^\dagger D_i z + \epsilon_{ijk}\epsilon_{ipq} F_{jk} F_{pq} - i\epsilon_{ijk} F_{jk} (D_i z)^\dagger z + i\epsilon_{ipq} F_{pq} z^\dagger D_i z \\ &= (D_i z)^\dagger D_i z + 2F_{ij} F_{ij}. \end{aligned}$$

14.3.2 Yet another formulation

Assume that $z \in \mathbb{C}^2$ with $z^\dagger z = 1$ and A_μ are independent field and consider the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\mathcal{D}_i z)^\dagger \mathcal{D}_i z + 2i\theta\epsilon_{ijk}A_i(\partial_j z^\dagger \partial_k z),$$

where $\mathcal{D}$ is the covariant derivative taken with respect to the apriori independent potential A. The first term becomes

$$(\partial_i z^\dagger + iA_i z^\dagger)(\partial_i z - iA_i z) = \partial_i z^\dagger \partial_i z + A_i A_i + 2iA_i Z_i, \quad \text{where} \quad Z_i = z^\dagger \partial_i z.$$

Later Z_i will be an induced connection. Its fieldstrength is

$$Z_{ij} = \partial_i Z_j - \partial_j Z_i = \partial_i z^\dagger \partial_j z - \partial_j z^\dagger \partial_i z.$$

Hence we obtain

$$\mathcal{L} = \tfrac{1}{2}\partial_i z^\dagger \partial_i z + \tfrac{1}{2}A_i A_i + iA_i Z_i + i\theta\epsilon_{ijk}A_i Z_{jk}.$$

Now we eliminate the field A_i by its algebraic equation of motion, which read

$$A_i + iZ_i + i\theta\epsilon_{ijk}Z_{jk} = 0$$

which yield

$$\mathcal{L} = \tfrac{1}{2}\partial_i z^\dagger \partial_i z + \tfrac{1}{2}(Z_i + \theta\epsilon_{ijk}Z_{jk})^2 = \tfrac{1}{2}(D_i z)^\dagger D_i z + \theta^2 Z_{pq}Z_{pq} - \theta\epsilon_{ijk}Z_i Z_{jk}.$$

Hence we recover the sigma model term, the Faddeev-Nieme term and the Hopf invariant.

14.3.3 Coadjoint orbit variables

We introduce the hermitean and traceless field

$$Q = \tfrac{1}{2}\mathbb{1} - zz^\dagger, \qquad \mathrm{Tr}\, Q = 0, \quad \det Q = -\tfrac{1}{4},$$

such that

$$\partial_\mu Q = -(\partial_\mu z z^\dagger - z\partial_\mu z^\dagger)$$

and

$$\partial_\mu Q \partial^\mu Q = \partial_\mu z z^\dagger \partial^\mu z z^\dagger + \partial_\mu z \partial^\mu z^\dagger + z\partial_\mu z^\dagger \partial^\mu z z^\dagger + z\partial_\mu z^\dagger z \partial^\mu z^\dagger.$$

Using $\mathrm{Tr}\, uv^\dagger = v^\dagger u$ and introducing $Z_\mu = z^\dagger \partial_\mu z$ we obtain

$$\tfrac{1}{2}\mathrm{Tr}\,(\partial_\mu Q \partial^\mu Q) = \partial_\mu z^\dagger \partial^\mu z + Z_\mu Z^\mu.$$

Next we rewrite the Faddeev-Niemi term and the Hopf invariant. With

$$dQdQ = ZdQ + dZ(1 + zz^\dagger)$$

we obtain

$$\mathrm{Tr}\,(dQ)^2 = \mathrm{Tr}\,(dQ)^2 zz^\dagger = 2dZ \quad \text{so that} \quad \mathrm{Tr}\, Q(dQ)^2 = -dZ.$$

Furthermore, using $Z^2 = 0$ we find

$$(dQ)^4 = dZdZ(1 + zz^\dagger)(1 + zz^\dagger) + ZdZdQ(1 + zz^\dagger) + dZZ(1 + zz^\dagger)dQ$$

so that

$$\begin{aligned} \mathrm{Tr}\,(dQ)^4 &= 4dZdZ + ZdZ(\mathrm{Tr}\,dQ + \mathrm{Tr}\,dQzz^\dagger) = 4dZdZ \\ \mathrm{Tr}\big\{(dQ)^4 zz^\dagger\big\} &= 4dZdZ + 4ZdZ\,\mathrm{Tr}\,(dQzz^\dagger) = 4dZdZ \end{aligned}$$

and hence

$$\mathrm{Tr}\big\{Q(dQ)^4\big\} = -2dZdZ = -2(dz^\dagger \wedge dz) \wedge (dz^\dagger \wedge dz) = -2d\big(Z \wedge dZ\big).$$