13 Supersymmetric sigma-Models in $1+1$ dimensions

The non-linear sigma models have been introduced at the end of the sixties [55] to describe the infrared properties in $d>2$ spacetime dimensions of systems with sponteneously broken symmetry. This lead to the study of models the fields of which take their values in coset spaces. Later on sigma models have been discussed in $d=2$ dimensions to investigate the propagation of strings in curved target spaces [56]. In this case the target space may be a more general Riemannian manifold.
From the point of view of perturbation theory, $d=2$ is particularly interesting, since non-linear sigma models are power-counting renormalizable. Some particular models based on coset spaces, as e.g. the Heisenberg model, have been proved to be really renormalizable in 2 dimensions [57].
A purely bosonic model contains a scalar field ϕ mapping ordinary Minkowski spacetime $\mathcal{M}$ into a target manifold X. If the target space X is a linear space, then a theory with the field ϕ is called a linear σ-model; if the target space is curved, it is a nonlinear σ-model. In susy models there are of course additional spinor fields.
In this section we study supersymmetric nonlinear σ-models with target space metric g_{ij}. As a preparation we collect some useful formulae for nonlinear σ-model calculations in curved space time.

13.1 Geometric preliminaries

We consider a (pseudo)Riemannian manifold X with line element

$$ds^2 = g_{ij} d\phi^i d\phi^j.$$

We recall the definition of the Christoffel symbols,

$$\begin{aligned}
\Gamma^p_{ij} &= g^{pq}\Gamma_{pij}, \qquad \Gamma_{pij} = \tfrac{1}{2}\big(g_{pi,j}+g_{pj,i}-g_{ij,p}\big) = \Gamma_{pji} \\
\Rightarrow \qquad & \nabla_i V^i = \partial_i V^i + \Gamma^j{}_{ij} V^j = \frac{1}{\sqrt{g}}\,\partial_i(\sqrt{g}\,V^i).
\end{aligned} \tag{13.1}$$

which are needed to calculate covariant derivatives,

$$\partial_i V^j = \partial_i V^j + \Gamma^j{}_{ik} V^k \quad \text{and} \quad \partial_i V_j = \partial_i V_j - \Gamma^k{}_{ij} V_k.$$

and the curvature tensor, Ricci tensor and Ricci scalar:

$$R^i{}_{jpq} = \Gamma^i{}_{qj,p} - \Gamma^i{}_{pj,q} + \Gamma^i{}_{pr}\Gamma^r{}_{qj} - \Gamma^i{}_{qr}\Gamma^r{}_{pj}, \qquad R_{jq} = R^i_{jiq} \quad \text{and} \quad R = g^{jq}R_{jq}.$$

By using that the connection is metric, $g_{pq;m} = 0$, it follows

$$\begin{aligned} R_{ijpq} &= \Gamma_{iqj,p} - \Gamma_{ipj,q} + \Gamma_{riq}\Gamma^r{}_{pj} - \Gamma_{rip}\Gamma^r{}_{qj} \\ &= -\tfrac{1}{2}\big(g_{ip,jq} + g_{jq,ip} - g_{iq,jp} - g_{jp,iq}\big) - \Gamma_{rip}\Gamma^r{}_{jq} + \Gamma_{riq}\Gamma^r{}_{jp} \end{aligned} \tag{13.2}$$

The symmetry properties of this tensor are

$$R_{ijpq} = -R_{ijqp}, \quad R_{ijpq} = -R_{jipq}, \quad R_{ijpq} = R_{pqij},$$

and the Bianchi identity reads

$$R_{ijpq} + R_{iqjp} + R_{ipqj} = 0.$$

In particular, for a two-dimensional target space we may (locally) introduce isothermal coordinates

$$g_{ij} = e^{2\sigma}\delta_{ij} \tag{13.3}$$

such that

$$\Gamma^p{}_{ij} = \partial_j\sigma\delta_{pi} + \partial_i\sigma\delta_{pj} - \partial_p\sigma\delta_{ij} \tag{13.4}$$

and

$$R_{1212} = -2e^{2\sigma}\partial^2\sigma, \quad R_{11} = R_{22} = -R_{12} = -2\partial^2\sigma \quad \text{and} \quad R = -4e^{-2\sigma}\partial^2\sigma. \tag{13.5}$$

Next we consider a geodesic on a compact n-dimensional Riemannian manifold X, parametrized by $\phi^i(s)$ and satisfying the differential equation

$$\frac{d^2\phi^i}{ds^2} + \Gamma^i{}_{jk}(\phi(s))\,\frac{d\phi^j}{ds}\frac{d\phi^k}{ds} = 0, \qquad \phi(0) = \phi_0, \quad \dot{\phi}(0) = \xi, \tag{13.6}$$

where s is the arc length and $\Gamma^i{}_{jk}$ denotes the Christoffel symbol for the Levi-Civita connection. Any integral curve of (13.6) is completely determined by a point ϕ_0 and a direction defined by the tangent vector $\xi^i = \dot{\phi}^i(\phi_0)$. The power series solution of (13.6) is:

$$\phi^i(s) = \phi^i_0 + \sum_{a=1}^{\infty}\frac{1}{a!}\big(\frac{d^a\phi^i}{ds^a}\big)_p s^a, \qquad \big(\frac{d\phi^i}{ds}\big)(0) = \xi^i. \tag{13.7}$$

Inserting this expansion into the differential equation (13.6) yields

$$\sum_{a=2}^{\infty}\frac{s^{a-2}}{(a-2)!}\frac{d^a\phi^i}{ds^a} + \Gamma^i{}_{jk}\Big(\phi_0 + \sum_{b=1}^{\infty}\frac{s^b}{b!}\frac{d^b\phi}{ds^b}\Big)\sum_{c,d=1}^{\infty}\frac{s^{c+d-2}}{(c-1)!\,(d-1)!}\;\frac{d^c\phi^j}{ds^c}\frac{d^d\phi^k}{ds^d},$$

where all derivatives are to be evaluated at the point p. In the lowest orders we obtain

$$\big(\frac{d^2\phi^i}{ds^2}\big)_p \;=\; -\Gamma^i{}_{jk}(p)\,\xi^j\xi^k$$

$$
\begin{aligned}
\Big(\frac{d^3\phi^i}{ds^3}\Big)_p &= -\Gamma^i{}_{jk}(p)\cdot\big(\ddot{\phi}^j\xi^k+\xi^j\ddot{\phi}^k\big)_p+\Gamma^i{}_{jk,l}(p)\,\xi^l\xi^j\xi^j \\
&= \big(2\Gamma^i{}_{kj}\Gamma^j{}_{mn}-\Gamma^i{}_{km,n}\big)_p\,\xi^m\xi^n\xi^k.
\end{aligned}
$$

This infinite set of equations define the generalized Christoffel symbols

$$
\phi^i=\phi^i_0+s\xi^i-\frac{s^2}{2}\Gamma^i{}_{jk}(p)\,\xi^i\xi^k-\sum_{n=3}^{\infty}\frac{1}{n!}\,\Gamma^i{}_{i_1i_2\dots i_n}(p)\,\xi^{i_1}\xi^{i_2}\cdots\xi^{i_a}s^n, \tag{13.8}
$$

which are found to be

$$
\begin{aligned}
\Gamma^i_{i_1\dots i_n} &= \tilde{\nabla}_{(i_1\dots}\;\tilde{\nabla}_{i_{n-2}}\Gamma^i_{i_{n-1}i_n)}, \quad \text{e.g.} \\
\Gamma^i_{i_1i_2i_3} &= \Gamma^i{}_{(i_2i_3,i_1)}-2\Gamma^i{}_{j(i_1}\Gamma^j{}_{i_2i_3)}=\tilde{\nabla}_{(i_1}\Gamma^i_{i_2i_3)}
\end{aligned} \tag{13.9}
$$

where $\tilde{\nabla}$ indicates that the covariant derivative is to be taken with respect to the lower indices only. The domain of convergence depends on the metric and the values of ξ^i. However, for sufficiently small values of s it defines an integral curve of (13.6).

13.1.1 Background field expansion

Now we are ready to expand

$$
\mathcal{L}=\tfrac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j
$$

about a background field ϕ_0:

$$
\mathcal{L}=\tfrac{1}{2}g_{ij}\big(\phi^k_0+\xi^k-\tfrac{1}{2}\Gamma^k_{mn}\xi^m\xi^n\big)\;\partial_\mu\big(\phi^i_0+\xi^i-\tfrac{1}{2}\Gamma^i_{pq}\xi^p\xi^q\big)\partial^\mu\big(\phi^j_0+\xi^j-\tfrac{1}{2}\Gamma^j_{rs}\xi^r\xi^s\big)+O(\xi^3).
$$

When using

$$
g_{mn,p}=\Gamma_{mnp}+\Gamma_{nmp}
$$

one finds

$$
\begin{aligned}
\mathcal{L}(\phi) &= \mathcal{L}(\phi_0)+g_{ij}\partial^\mu\phi^i_0\nabla_\mu\xi^j \\
&\quad+\tfrac{1}{2}g_{ij}\partial_\mu\xi^i\partial^\mu\xi^j-\tfrac{1}{2}g_{ij}\Gamma^j{}_{mn,k}\partial_\mu\phi^i_0\partial^\mu\phi^k_0\xi^m\xi^n-\Gamma_{jmn}\partial_\mu\phi^j_0\partial^\mu\xi^m\xi^n \\
&\quad+g_{ij,k}\partial_\mu\phi^i_0\partial^\mu\xi^j\xi^k+\tfrac{1}{4}\big(g_{ij,kl}-g_{ij,s}\Gamma^l{}_{kl}\big)\partial_\mu\phi^i_0\;\partial^\mu\phi^j_0\xi^k\xi^l,
\end{aligned}
$$

where the metric, Christoffel symbols and their derivatives are to be evaluated at the background field ϕ_0. We have introduced the covariant derivative

$$
\nabla_\mu\xi^i=\partial_\mu\xi^i+\Gamma^i{}_{jk}\partial_\mu\phi^j_0\;\xi^k \tag{13.10}
$$

of the vectorfield ξ^i. To rearrange these rather complicated terms into covariant ones we use

$$
g_{ij}\partial^\mu\xi^i\partial_\mu\xi^j=g_{ij}\nabla^\mu\xi^i\nabla_\mu\xi^j-2\Gamma_{imn}\partial^\mu\phi^m_o\partial_\mu\xi^i\xi^n-\Gamma_{ipq}\Gamma^i{}_{mn}\partial_\mu\phi^p_0\,\partial^\mu\phi^m_0\xi^q\xi^n
$$

and

$$\left(\tfrac{1}{2}g_{pm,qn} - \tfrac{1}{2}g_{pm,s}\Gamma^s{}_{qn} - g_{ps}\Gamma^s_{qn,m} - \Gamma^i_{pq}\Gamma_{imn}\right)\partial_\mu\phi_0^p\partial^\mu\phi_0^m\xi^q\xi^n = R_{pqnm}\partial_\mu\phi_0^p\partial^\mu\phi_0^m\xi^q\xi^n.$$

This way we end up with the following covariant expansion in ξ:

$$\mathcal{L}(\phi) = \mathcal{L}(\phi_0) + g_{ij}\partial^\mu\phi_0^i\nabla_\mu\xi^j + \tfrac{1}{2}g_{ij}\nabla^\mu\xi^i\nabla_\mu\xi^j - \tfrac{1}{2}R_{pqmn}\partial_\mu\phi_0^p\,\partial^\mu\phi_0^m\,\xi^q\xi^n + O(\xi^3). \tag{13.11}$$

We read off the field equation for the background field

$$\nabla_\mu\partial^\mu\phi_0^i = 0 \tag{13.12}$$

and the fluctuation operator for the vector fluctuations

$$(S''\xi)_i = g_{ij}\nabla_\mu\nabla^\mu\xi^j - R_{p(in)m}\partial_\mu\phi_0^p\,\partial^\mu\phi_0^m\,\xi^n. \tag{13.13}$$

13.1.2 Riemann normal coordinates

Now let us simplify the calculation by choosing Riemann normal coordinates. In Riemann normal coordinates (RNC) the geodesics through p are straight lines defined by:

$$\bar{\phi}^i = \xi^i\,s. \tag{13.14}$$

This coordinates cover a neighbourhood of the point p. Substituting (13.14) into (13.7) one has

$$\phi^i = \phi_0^i + \bar{\phi}^i - \tfrac{1}{2}\Gamma^i{}_{jk}\bar{\phi}^i\bar{\phi}^j - \sum_{n=3}^{\infty}\frac{1}{n!}\left(\Gamma^i_{i_1\ldots i_n}\right)_p\bar{\phi}^{i_1}\cdots\bar{\phi}^{i_n}. \tag{13.15}$$

The Jacobian

$$\lfloor\frac{\partial\phi^i}{\partial\bar{\phi}^j}\rfloor \neq 0$$

and thus the series (13.15) can be inverted. For this local system of coordinates the geodesic eqation can be written as

$$\frac{d^2}{ds^2}\bar{\phi}^i + \bar{\Gamma}^i{}_{jk}(\bar{\phi})\,\frac{d\bar{\phi}^j}{ds}\frac{d\bar{\phi}^k}{ds} = \bar{\Gamma}^i{}_{jk}(\bar{\phi})\,\frac{d\bar{\phi}^j}{ds}\frac{\bar{\phi}^k}{ds} = 0,$$

where $\bar{\Gamma}$ are the Christoffel symbols in Riemann normal coordinates. Since $\bar{\phi} = \xi s$, the Taylor expansion in s implies that all symmetric derivatives of the affine connection vanish at the origin in RNC:

$$\bar{\Gamma}^i{}_{(i_1 i_2\,,\,j_3\ldots i_n)}(p) = 0.$$

In field theory we employ the background field expansion in terms of normal coordinates. If ϕ^i is close to ϕ_0^i, there is a unique geodesic which connects ϕ_0^i to ϕ^i and reaches ϕ^i in unit proper time.

If ξ^i represents the tangent vector to the geodesic at ϕ_0^i we can parametrize ϕ in terms of ξ. Since ξ is a vector under coordinate changes the covariance of the expansion is ensured.
I particular, an arbitary second rank tensor field (e.g. g_{ij}) can be expanded according to

$$\bar{T}_{ij}(\bar{\phi}) = \bar{T}_{ij}(s\xi) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \bar{T}_{ij\,,\,i_1\ldots i_n}(p)\, \xi^{i_1} \cdots \xi^{i_n}. \tag{13.16}$$

The coefficients are tensors and can be expressed in terms of the components $R^i{}_{jpq}$ of the Riemann tensor and the covariant derivatives $\nabla_i T_{jk}$ and $\nabla_i R^j{}_{kpq}$. To find the explicit form for these coefficients we first express the symmetric derivatives of the Christoffel symbols,

$$\bar{\Gamma}^i_{j(j_1\,,\,j_2\ldots j_n)},$$

in terms of the Riemannian curvature tensor and its covariant derivatives. For example,

$$\bar{R}^i{}_{(pq)j} = \tfrac{1}{2}(\bar{R}^i{}_{pqj} + \bar{R}^i{}_{qpj}) = \tfrac{1}{2}(\bar{R}^i{}_{pqj} + \bar{R}^i{}_{qpj}) + 3\bar{\Gamma}^i{}_{(pq\,,\,j)} = 3\bar{\Gamma}^i{}_{j(p,q)}$$

or

$$\bar{\nabla}_{(p}\bar{R}^i{}_{qr)j} = \partial_{(p}\bar{R}^i{}_{qr)j} = \bar{\Gamma}^i{}_{j(p,qr)} - \bar{\Gamma}^i{}_{(pq,r)j} = \bar{\Gamma}^i{}_{j(p,qr)} - \bar{\Gamma}^i{}_{(pr,q)r} - \tfrac{1}{2}\bar{\Gamma}^i{}_{(jp,qr)} = -2\bar{\Gamma}^i{}_{(pq,r)j}.$$

For the identities containing higher order derivatives of the connection in RNC I refer to [54].
Now we may use these results to find the covariant form of the expansion coefficients in (13.16).
For example, in RNC we have

$$\begin{aligned}
\bar{\nabla}_{i_1}\bar{T}_{ij} &= \bar{T}_{ij,i_1} \\
\bar{\nabla}_{(i_1}\bar{\nabla}_{i_2)}\bar{T}_{ij} &= \bar{T}_{ij,(i_1 i_2)} - \bar{\Gamma}^k{}_{i(i_2,i_1)}\bar{T}_{kj} - \bar{\Gamma}^k{}_{j(i_2,i_1)}\bar{T}_{ik} \\
&= \bar{T}_{ij,(i_1 i_2)} - \tfrac{1}{3}\bar{R}^k{}_{(i_1 i_2)i}\bar{T}_{kj} - \tfrac{1}{3}\bar{R}^k{}_{(i_1 i_2)j}\bar{T}_{ik},
\end{aligned}$$

and hence (for $s = 1$)

$$\begin{aligned}
T_{ij}(\bar{\phi}) &= \bar{T}_{ij} + \bar{\nabla}_{i_1}\bar{T}_{ij}\xi^{i_1} \\
&\quad + \tfrac{1}{2}\big(\bar{\nabla}_{(i_1}\bar{\nabla}_{i_2)}\bar{T}_{ij} + \tfrac{1}{3}\bar{R}^k{}_{(i_1 i_2)i}\bar{T}_{kj} + \tfrac{1}{3}\bar{R}^k{}_{(i_1 i_2)j}\bar{T}_{ik}\big)\xi^{i_1}\xi^{i_2} + \ldots.
\end{aligned} \tag{13.17}$$

All quantities on the right are to be evaluated at the point p with coordinates $\bar{\phi}_0$. Since this is an identity between tensors (recall that the ξ transform as a vector) it holds in any coordinate system. For example, we obtain

$$g_{ij}(\phi) = g_{ij}(\phi_0) + \tfrac{1}{3}R_{i(i_1 i_2)j}(\phi_0)\, \xi^{i_1}\xi^{i_2} + O(\xi^3). \tag{13.18}$$

13.1.3 An alternative version for the background field expansion

In the following exposition of an efficient method to arrive at a covariant background field expansion we follow Mukhi [58]. We begin by recalling that

$$\nabla_k g_{ij} = 0, \quad [\nabla_p, \nabla_q]f = 0, \qquad [\nabla_p, \nabla_q]V^i = R^i{}_{jpq}V^j \tag{13.19}$$

for a scalar field f and a vector field V^i. As earlier we define an interpolating field with

$$\phi^i(0) = \phi^i_0, \quad \frac{d\phi^i}{ds}\Big|_{s=0} = \xi^i \quad \text{and} \quad \phi^i(1) = \phi^i \tag{13.20}$$

that satisfies the geodesic equation. Now we study the s-dependency of the Lagrangian along the geodesic

$$\mathcal{L}(s) = \tfrac{1}{2}g_{ij}(\phi(s))\partial_\mu\phi^i(s)\;\partial^\mu\phi^j(s), \tag{13.21}$$

so that

$$\mathcal{L}(\phi) = L(1) = L(0) + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^nL(s)}{ds^n}\Big|_{s=0} = L(0) + \sum_{n=1}^{\infty}\frac{1}{n!}(\nabla_s)^nL(s)\Big|_{s=0}, \tag{13.22}$$

where

$$\nabla_s = \dot{\phi}^i(s)\nabla_i \tag{13.23}$$

is the covariant derivative along the curve $\phi(s)$. From (13.19) it follows that

$$\begin{aligned}
\nabla_s g_{ij} &= 0, \qquad \nabla_s\dot{\phi}^i(s) = 0 \\
\nabla_s\partial_\mu\phi^i &= \dot{\phi}^k\big(\partial_k\partial_\mu\phi^i + \Gamma^i{}_{jk}\partial_\mu\phi^j\big) = \partial_\mu\dot{\phi}^i + \Gamma^i{}_{jk}\partial_\mu\phi^j\dot{\phi}^k \equiv \nabla_\mu\nabla_s\phi^i \\
[\nabla_s, \nabla_\mu]V^i &= R^i{}_{jpq}V^j\dot{\phi}^p\partial_\mu\phi^q.
\end{aligned} \tag{13.24}$$

Now we are ready to calculate the leading term in the series (13.21):

$$\begin{aligned}
\mathcal{L}(\phi) &= \mathcal{L}(\phi_0) + g_{ij}(\phi_0)\nabla_\mu\xi^i\partial^\mu\phi^j_0 + \tfrac{1}{2}\Big(g_{ij}\nabla_\mu\xi^i\nabla^\mu\xi^j + g_{ij}\nabla_s\nabla_\mu\dot{\phi}^i\partial^\mu\phi^j_0\Big) \\
&= \mathcal{L}(\phi_0) + g_{ij}(\phi_0)\nabla_\mu\xi^i\partial^\mu\phi^j_0 + \tfrac{1}{2}g_{ij}\nabla_\mu\xi^i\nabla^\mu\xi^j - \tfrac{1}{2}R_{jipq}\partial^\mu\phi^i_0\partial_\mu\phi^p_0\xi^j\xi^q.
\end{aligned}$$

13.2 Supersymmetric sigma models in 2 dimensions

As in section (11.1) we start with a real superfield, but now with several components,

$$\Phi^i = A^i + \bar{\alpha}\psi^i + \tfrac{1}{2}\bar{\alpha}\alpha F^i.$$

The real A^i, F^i take their values in the curved target space. We calculate

$$\begin{aligned}
\bar{D}\Phi^i D\Phi^j &= \bar{\psi}^i\psi^j + \bar{\alpha}(\psi^iF^j + \psi^jF^i) - i\bar{\alpha}\gamma^\mu\psi^j\partial_\mu A^i - i\bar{\alpha}\gamma^\mu\psi^i\partial_\mu A^j \\
&\quad +\bar{\alpha}\alpha\big(\partial_\mu A^i\partial^\mu A^j - \tfrac{i}{2}\bar{\psi}^i\partial\!\!\!/\,\psi^j + \tfrac{i}{2}\partial_\mu\bar{\psi}^i\gamma^\mu\psi^j + F^iF^j\big).
\end{aligned}$$

This bilinear form is contracted with the metric g_{ij} evaluated at Φ^k which has the following component expansion:

$$g_{ij}(\Phi^k) = g_{ij} + g_{ij,k}\bar{\alpha}\psi^k + \tfrac{1}{2}\bar{\alpha}\alpha\big(g_{ij,k}F^k - \tfrac{1}{2}g_{ij,kl}\bar{\psi}^k\psi^l\big),$$

where the argument of the metric and its derivatives on the right is the body A of the superfield. After using the identities

$$g_{ij,k}\bar{\psi}^k\gamma^\mu\psi^j\partial_\mu A^i = \tfrac{1}{2}(g_{ij,k} - g_{jk,j})\bar{\psi}^k\gamma^\mu\psi^j\partial_\mu A^i = -\Gamma_{kij}\bar{\psi}^k\gamma^\mu\psi^i\partial_\mu A^i$$

and

$$(g_{ij,k} + g_{ji,k} - g_{ki,j})\bar{\psi}^k\psi^i F^j = (g_{ij,k} + g_{jk,i} - g_{ki,j})\bar{\psi}^k\psi^i F^j = 2\Gamma_{jki}\bar{\psi}^k\psi^i F^j$$

we find the following result in superspace

$$\tfrac{1}{2}g_{ij}(\Phi)\bar{D}\Phi^i\, D\Phi^j = \tfrac{1}{2}g_{ij}\bar{\psi}^i\psi^j + \bar{\alpha}\big(g_{ij}(\psi^i F^j - i\partial\!\!\!/\, A^i\psi^j) + \tfrac{1}{2}g_{ij,k}\psi^k(\bar{\psi}^i\psi^j)\big) + \bar{\alpha}\alpha\mathcal{L}$$

giving rise to the Langrangian density

$$\begin{aligned}\mathcal{L} \;=\;& \tfrac{1}{2}g_{ij}\Big(\partial_\mu A^i\partial^\mu A^j - \tfrac{i}{2}\bar{\psi}^i\partial\!\!\!/\,\psi^j + \tfrac{i}{2}\partial_\mu\bar{\psi}^i\gamma^\mu\psi^j + F^iF^j\Big)\\ &-\tfrac{i}{2}\Gamma_{kij}\bar{\psi}^k\gamma^\mu\psi^j\partial_\mu A^i - \tfrac{1}{2}\Gamma_{jki}\bar{\psi}^k\psi^i F^j - \tfrac{1}{8}g_{ij,kl}(\bar{\psi}^k\psi^l)(\bar{\psi}^i\psi^j).\end{aligned} \tag{13.25}$$

We may elminiate the auxiliary field by its algebraic equation of motion

$$F^i = \tfrac{1}{2}\Gamma^i{}_{pq}\bar{\psi}^p\psi^q$$

which leads to the following density for the dynamical fields

$$\tfrac{1}{2}g_{ij}(\Phi)\bar{D}\Phi^i\, D\Phi^j = \tfrac{1}{2}g_{ij}\bar{\psi}^i\psi^j + \tfrac{1}{2}\bar{\alpha}\big(g_{ip,q} + g_{iq,p} + g_{pq,i}\big)\psi^i\,\bar{\psi}^p\psi^q - ig_{ij}\bar{\alpha}\gamma^\mu\psi^j\partial_\mu A^i + \bar{\alpha}\alpha\mathcal{L}$$

with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}g_{ij}\big(\partial_\mu A^i\partial^\mu A^j - i\bar{\psi}^i\gamma^\mu(D_\mu\psi)^j\big) - \tfrac{1}{8}\big(\Gamma_{rmn}\Gamma^r{}_{pq} + g_{pq,mn}\big)(\bar{\psi}^m\psi^n)(\bar{\psi}^p\psi^q), \tag{13.26}$$

where we have introduced the covariant derivative

$$(D_\mu\psi)^j = \partial_\mu\psi^j + \Gamma^j{}_{pq}\partial_\mu A^p\psi^q,$$

which takes into account that ψ^j transforms in the tangent space of X. We have used, that

$$\partial_\mu g_{ij}\bar{\psi}^i\gamma^\mu\psi^j = 0,$$

on account of the antisymmetry of the fermionic bilinear, so that the terms containing derivatives of the Majorana spinors are equal, up to a total derivative. To make further progress we need the symmetry properties of the four-fermi term $(\bar{\psi}^m\psi^n)(\bar{\psi}^p\psi^q)$. Clearly, this term is symmetric in (m,n) and symmetric in (p,q). We can also prove the following property for the cyclic sum of three indices:

$$(\bar\psi^m\psi^n)(\bar\psi^p\psi^q)+(\bar\psi^m\psi^q)(\bar\psi^n\psi^p)+(\bar\psi^m\psi^p)(\bar\psi^q\psi^n)=0. \tag{13.27}$$

This is shown with the help of the Fierz identity (11.1):

$$\begin{aligned} \dots &= (\bar\psi^m\psi^n)(\bar\psi^p\psi^q)+(\bar\psi^n\psi^p)(\bar\psi^q\psi^m)+(\bar\psi^m\psi^p)(\bar\psi^q\psi^n)\\ &= (\bar\psi^m\psi^n)(\bar\psi^p\psi^q)-\tfrac12\bar\psi^n\big(\bar\psi^q\psi^p+\gamma_\mu\bar\psi^q\gamma^\mu\psi^p+\gamma_*\bar\psi^q\gamma_*\psi^p\big)\psi^m\\ &\quad -\tfrac12\bar\psi^m\big(\bar\psi^q\psi^p+\gamma_\mu\bar\psi^q\gamma^\mu\psi^p+\gamma_*\bar\psi^q\gamma_*\psi^p\big)\psi^n=0. \end{aligned}$$

In the last step we used the symmetry properties of the bilinears in the Majorana fields. Now we are ready to prove that the last term in (13.26) is proportional to

$$\begin{aligned} R_{mpnq}\bar\psi^m\psi^n\bar\psi^p\psi^q &= -\tfrac12\big(g_{mn,pq}+g_{pq,mn}-g_{mq,pn}-g_{pn,mq}\big)\,\bar\psi^m\psi^n\bar\psi^p\psi^q\\ &\quad -\big(\Gamma_{kmn}\Gamma^k_{\ pq}-\Gamma_{kmq}\Gamma^k_{\ pn}\big)\,\bar\psi^m\psi^n\bar\psi^p\psi^q. \end{aligned}$$

Using the symmetry and cyclicity properties of the four fermi term we may rewrite the terms containig second derivatives of the metric as follows

$$-\tfrac12\big(2g_{mn,pq}-g_{mq,pn}-g_{mp,nq}\big)\,\bar\psi^m\psi^n\bar\psi^p\psi^q=-\tfrac12\big(2g_{mn,pq}+g_{mn,qp}\big)\dots=-\tfrac32 g_{mn,pq}\dots.$$

Similarly, the terms containing the Christoffel symbols simplifiy as follows,

$$\begin{aligned} &\big(\Gamma_{kmn}\Gamma^k_{\ pq}-\Gamma_{kmq}\Gamma^k_{\ pn}\big)\,\bar\psi^m\psi^n\bar\psi^p\psi^q=\tfrac32\big(\tfrac23\Gamma_{kmn}\Gamma^k_{\ pq}-\tfrac13\Gamma_{kmq}\Gamma^k_{\ pn}-\tfrac13\Gamma_{kmp}\Gamma^k_{\ nq}\big)\ \dots\\ &=\tfrac32\big(\tfrac23\Gamma_{kmn}\Gamma^k_{\ pq}+\tfrac13\Gamma_{kmn}\Gamma^k_{\ qp}\big)\dots=\tfrac32\Gamma_{kmn}\Gamma^k_{\ pq}\dots. \end{aligned}$$

If follows that

$$R_{mpnq}\bar\psi^m\psi^n\bar\psi^p\psi^q=-\tfrac32\Big(g_{mn,pq}+\Gamma_{kmn}\Gamma^k_{\ pq}\Big)\bar\psi^m\psi^n\bar\psi^p\psi^q \tag{13.28}$$

so that the density aquires the following rather simple form

$$\tfrac12 g_{ij}(\Phi)\bar D\Phi^i\,D\Phi^j=\tfrac12 g_{ij}\bar\psi^i\psi^j-ig_{ij}\bar\alpha\gamma^\mu\psi^j\partial_\mu A^i+\bar\alpha\alpha\mathcal{L}$$

with Lagrangian density

$$\mathcal{L}=\tfrac12 g_{ij}\big(\partial_\mu A^i\partial^\mu A^j-i\bar\psi^i\gamma^\mu(D_\mu\psi)^j\big)+\tfrac1{12}R_{mpnq}\,(\bar\psi^m\psi^n)(\bar\psi^p\psi^q). \tag{13.29}$$

Clearly, the objects $\partial_\mu A^i$ and ψ^i must transform as target space vectors. In addition, $\partial_\mu A^i$ is a spacetime vector.
To find the susy transformation of $\mathcal{L}$ we note that $g_{ij}(\Phi)\bar D\Phi^i\Phi^j$ is a real superfield, similarly as Φ in (11.5). From (11.7) we read off, that under the (on-shell) susy transformations

$$\delta A^i=\bar\beta\psi^i\quad\text{and}\quad\delta\psi^i=\big(\tfrac12\Gamma^i_{\ pq}\bar\psi^p\psi^q+i\partial\!\!\!/\,A^i\big)\beta$$

the Lagrangian transforms into a derivative, as it must be,

$$\delta\mathcal{L}=\partial_\mu(\bar\beta V^\mu),\qquad V^\mu=\tfrac12 g_{ij}\gamma^\mu\gamma^\nu\psi^i\partial_\nu A^j.$$

The Noether current is gotten from

$$\begin{aligned}\sum \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi &= g_{ij}\big(\partial^\mu A^i\delta A^j - \tfrac{i}{2}\bar\psi^i\gamma^\mu\psi^l\Gamma^j{}_{kl}\delta A^k - \tfrac{i}{2}\bar\psi^i\gamma^\mu\delta\psi^j\big)\\ &= \bar\beta g_{ij}\big(\tfrac{3}{2}\partial^\mu A^i\psi^j + \tfrac{1}{2}\gamma_*\psi^i\epsilon^{\mu\nu}\partial_\nu A^j\big),\end{aligned}$$

where we made use of $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \epsilon^{\mu\nu}\gamma_*$. This then leads to the following Noether current,

$$J^\mu = g_{ij}\big(\eta^{\mu\nu} + \gamma_*\epsilon^{\mu\nu}\big)\partial_\nu A^i\psi^j, \tag{13.30}$$

which is almost identical to the one of the supersymmetric Wess-Zumino model (11.11).

13.3 2-dimensional supersymmetric 0(3)-model

If the target manifold is $S^2 = SO(3)/SO(2)$, then the model possesses a global $SO(3)$-invariance and is called $O(3)$ model. The model maybe reformulated in terms of fields propagating on CP^1 and in this formulation it is called CP^1-model.
We use the following conformally flat metric on S^2:

$$g_{ij} = \frac{1}{N^2}\delta_{ij} \quad \text{with} \quad e^{2\sigma} = \frac{1}{N^2} = \frac{1}{(1+u^iu^i)^2}, \quad u^i \in \mathbb{R}^2 \tag{13.31}$$

With $\sigma = -\log(1+\vec u\vec u)$ we may immediately apply the formulae (13.3-13.5) which yield

$$\Gamma^p{}_{ij} = -\frac{2}{N}\big(u^j\delta_{pi} + u^i\delta_{pj} - u^p\delta_{ij}\big) \quad \text{and} \quad \partial^2\sigma = -\frac{1}{N^2}$$

which leads to the following supersymmetric Lagrangian

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\frac{1}{(1+\vec u\vec u)^2}\Big(\partial_\mu u^i\partial^\mu u^i - i\bar\psi^i\partial\!\!\!/\,\psi^i + \frac{2i}{N}\bar\psi^i\gamma^\mu\psi^j\big(u^j\partial_\mu u^i - u^i\partial_\mu u^j\big)\Big)\\ &\quad + \tfrac{1}{2}\frac{1}{(1+\vec u\vec u)^4}\;\bar\psi^1\psi^1\bar\psi^2\psi^2.\end{aligned}$$

The Lagrangian further simlifies if we introduce complex coordinates on S^2 and combine the two Majorana spinor to a Dirac spinor,

$$u = u^1 + iu^2 \quad \text{and} \quad \psi = \psi^1 + i\psi^2, \quad \bar\psi = \bar\psi^1 - i\bar\psi^2.$$

Then

$$\begin{aligned}&\partial_\mu u^i\partial^\mu u^i = \partial_\mu\bar u\partial^\mu u\\ &\bar\psi^i\partial\!\!\!/\,\psi^i = \bar\psi\partial\!\!\!/\,\psi - i\partial_\mu(\bar\psi^1\gamma^\mu\psi^2), \qquad \partial_\mu\bar\psi^i\gamma^\mu\psi^i = \partial_\mu\bar\psi\gamma^\mu\psi - i\partial_\mu(\bar\psi^1\gamma^\mu\psi^2)\\ &\bar\psi^i\gamma^\mu\psi^j\big(u^j\partial_\mu u^i - u^i\partial_\mu u^j\big) = \tfrac{1}{2}\bar\psi\gamma^\mu\psi\big(\bar u\partial_\mu u - u\partial_\mu\bar u\big)\end{aligned}$$

and the Lagrangian density now takes the simple form

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\frac{1}{(1+\bar uu)^2}\Big(\partial_\mu\bar u\partial^\mu u - \tfrac{i}{2}\bar\psi^i\partial\!\!\!/\,\psi^i + \tfrac{i}{2}\partial_\mu\bar\psi\gamma^\mu\psi + \frac{i}{N}\bar\psi\gamma^\mu\psi\big(\bar u\partial_\mu u - u\partial_\mu\bar u\big)\Big)\\ &\quad + \tfrac{1}{4}\frac{1}{(1+\bar uu)^4}\;(\bar\psi\psi)^2.\end{aligned}$$

The supersymmetry transformation of u and ψ are

$$\delta u = \bar{\beta}\psi \quad \text{and} \quad \delta\psi = \frac{2}{N}\bar{u}(\bar{\beta}\psi)\psi + i\partial\!\!\!/\, u\beta,$$

where β is a Majorana spinor (contrary to ψ). Actually, this model admits an extended $N = 2$ supersymmetry.