2 Poincaré groups and algebras

In what follows M is the d-dimensional Minkowski spacetime with metric $\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$. Events are characterized by their coordinates

$$x = (x^\mu), \quad \mu = 0, \dots, d-1. \tag{2.1}$$

The scalar product is

$$(\xi, \eta) = \sum_{\mu\nu} \eta_{\mu\nu} \xi^\mu \eta^\nu \equiv \xi^\mu \eta_\mu.$$

Poincaré transformations are linear transformations

$$x^\mu \longrightarrow \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \text{or} \quad \tilde{x} = \Lambda x + a, \qquad d\tilde{x} = \Lambda dx$$

with $(d\tilde{x}, d\tilde{x}) = (dx, dx)$. They form a group:

$$iL = \Big\{ (\Lambda, a) \big| \, a \in \mathbb{R}^d, \Lambda \in L(\mathbb{R}^d), \Lambda^T \eta \Lambda = \eta \Big\}. \tag{2.2}$$

The group multiplication is

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2).$$

The unit element is $(\mathbb{1}, 0)$ and the inverse of (Λ, a) reads

$$(\Lambda, a)^{-1} = \big(\Lambda^{-1}, -\Lambda^{-1} a\big).$$

Note, that iL is the semi-direct product of translations and Lorentz transformations. iL has the dimension

$$\tfrac{1}{2} d(d-1) + d = \tfrac{1}{2} d(d+1).$$

For example, in 4(5) spacetime dimensions the Poincaré group has 10(15) generators. 10 is also the number of generators of the (anti)de Sitter group in 4 spacetime dimensions and the dimension of the conformal group in 3 spacetime dimensions. Similarly, the AdS group in 5 dimensions is isomorphic to the conformal group in 4 dimensions. This equivalence is at the heart of the so-called AdS-CFT-correspondence.
The subgroup of translations is Abelian and the subgroup of Lorentz transformations

$$L = \Big\{\Lambda \in L(\mathbb{R}^d) \,\big|\, \Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad \text{or} \quad \Lambda^T \eta \Lambda = \eta\Big\} \tag{2.3}$$

form the (non-compact) simple Lie group $O(1, d-1)$. From (2.3) it follows at once, that

$$\det \Lambda^T \det \Lambda = 1 \Longrightarrow \det \Lambda = \pm 1.$$

The transformations Λ with determinant one form the subgroup $SO(1, d-1)$ of proper Lorentz transformations. Every Λ maps the forward light cone

$$V_+ = \{\xi^0 > 0, (\xi, \xi) > 0\}$$

into itself or into the past light cone V_-. In the second case time is reversed. The Lorentz groups are not (simply) connected, e.g.

$$\pi_1\big(SO_e(1,2)\big) = \mathbb{Z}, \qquad \pi_0\big(O(1,3)\big) = \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \pi_0\big(SO_e(1,3)\big) = \mathbb{Z}_2 \quad \text{and} \quad \pi_1\big(SO_e(1,3)\big) = \mathbb{Z}_2.$$

2.1 The infinitesimal generators

The Lie algebra of the Poincaré group is generated by $d(d+1)/2$ generators: the $d(d-1)/2$ generators $M_{\mu\nu} = -M_{\nu\mu}$ of the Lorentz group and the d generators P_μ of the group of translations,

$$(\Lambda, a) = \exp\Big\{\tfrac{i}{2}\omega^{\mu\nu} M_{\mu\nu} + i a^\mu P_\mu\Big\} = \exp\Big\{\tfrac{i}{2}(\omega, M) + i(a, P)\Big\}. \tag{2.4}$$

The small Lorentz transformations are $\Lambda = \mathbb{1} + i(\omega, M)/2$, where $M_{\mu\nu}$ is an imaginary $d \times d$-matrix. The condition (2.3) on Λ implies that

$$(\omega, M)^T \eta + \eta(\omega, M) = 0 \Longleftrightarrow (\omega, M)_{\rho\sigma} = -(\omega, M)_{\sigma\rho}$$

for all ω and therefore each *matrix* $M_{\mu\nu}$ must be antisymmetric if both its indices ρ and ρ are *lower* ones:

$$(M_{\mu\nu})_{\rho\sigma} = -(M_{\mu\nu})_{\sigma\rho}. \tag{2.5}$$

As basis for the antisymmetric matrices we choose

$$\big(M_{\mu\nu}\big)_{\rho\sigma} = -i\big(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\nu\rho}\eta_{\mu\sigma}\big) \Longrightarrow \tfrac{i}{2}(\omega, M)^\rho{}_\sigma = \omega^\rho{}_\sigma.$$

These generators have the following commutation relations

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}]_{\lambda\xi} &= (M_{\mu\nu})_{\lambda\tau}(M_{\rho\sigma})^\tau{}_\xi - (M_{\rho\sigma})_{\lambda\tau}(M_{\mu\nu})^\tau{}_\xi \\
&= i\big(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}\big)_{\lambda\xi}.
\end{aligned}$$

The spacetime translations form an Abelian subgroup of iL and hence

$$[P_\mu, P_\nu] = 0, \qquad \mu, \nu = 0, \ldots, d-1.$$

To find the commutation relations of the infinitesimal Lorentz transformations and translations we observe that

$$(\Lambda,0)(\mathbb{1},a)(\Lambda,0)^{-1}=(\mathbb{1},\Lambda a)\quad\text{or}\quad e^{i(\omega,M)/2}e^{i(a,P)}e^{-i(\omega,M)/2}=e^{i(\Lambda a,P)}$$

For infinitesimal translations this implies

$$e^{i(\omega,M)/2}(a,P)e^{-i(\omega,M)/2}=\big(\Lambda a,P\big)\quad\text{or}\quad \omega^{\mu\nu}[M_{\mu\nu},(a,P)]=\omega^{\mu\nu}\big(M_{\mu\nu}a,P\big),$$

where we have set $\Lambda\sim\mathbb{1}+i(\omega,M)/2$. This holds true for any a,ω and we conclude

$$[M_{\mu\nu},P_\rho]=(M_{\mu\nu})^\sigma{}_\rho P_\sigma=-i\big(\eta_{\mu\rho}P_\nu-\eta_{\nu\rho}P_\mu\big).$$

To summarize, the Poincaré algebra is given by

$$\begin{aligned}
[M_{\mu\nu},M_{\rho\sigma}] &= i\big(\eta_{\mu\rho}M_{\nu\sigma}+\eta_{\nu\sigma}M_{\mu\rho}-\eta_{\mu\sigma}M_{\nu\rho}-\eta_{\nu\rho}M_{\mu\sigma}\big)\\
[P_\rho,M_{\mu\nu}] &= -i\big(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu\big)\\
[P_\mu,P_\nu] &= 0
\end{aligned}\tag{2.6}$$

One of the main task will be to extend this algebra to supersymmetric algebras containing commutators and anticommutators. Now we turn to the particular and most important case $d=4$.

2.2 The Lorentz-algebra in 4 dimensions

The explicit forms of the generators $(M_{\mu\nu})^\rho{}_\sigma$ of the Lorentz group are

$$(M_{\mu\nu})=\begin{pmatrix}0&K_1&K_2&K_3\\-K_1&0&-J_3&J_2\\-K_2&J_3&0&-J_1\\-K_3&-J_2&J_1&0\end{pmatrix}\quad\text{or}\quad M_{0i}=K_i,\ M_{ij}=-\epsilon_{ijk}J_k$$

with infinitesimal antihermitean boost operators

$$K_1=\begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\qquad K_2=\begin{pmatrix}0&0&i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0\end{pmatrix}\qquad K_3=\begin{pmatrix}0&0&0&i\\0&0&0&0\\0&0&0&0\\i&0&0&0\end{pmatrix}$$

and infinitesimal hermitean rotations

$$J_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}\qquad J_2=\begin{pmatrix}0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0\end{pmatrix}\qquad J_3=\begin{pmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{pmatrix}.$$

They fulfill the following commutation relations

$$[J_i,J_j]=i\epsilon_{ijk}J_k,\quad [K_i,K_j]=-i\epsilon_{ijk}J_k\quad\text{and}\quad [J_i,K_j]=[K_i,J_j]=i\epsilon_{ijk}K_k.\tag{2.7}$$

In a next step we introduce the following generators of the complexified Lie algebra:

$$S_i = \tfrac{1}{2}(J_i + iK_i) \quad \text{and} \quad A_i = \tfrac{1}{2}(J_i - iK_i). \tag{2.8}$$

They obey the simple commutation relations of the $A_1 \times A_1$ Lie algebra

$$[S_i, S_j] = i\epsilon_{ijk} S_k, \quad [A_i, A_j] = i\epsilon_{ijk} A_k \quad \text{and} \quad [S_i, A_j] = 0. \tag{2.9}$$

We conclude, that

$$so(4, \mathbb{C}) \sim sl(2, \mathbb{C}) \times sl(2, \mathbb{C}),$$

so that every irreducible representation of $so(1,3)$ is uniquely determined by (n, m), where the half-integers n and m characterize the $sl(2, \mathbb{C})$ representations. Note that the two $sl(2, \mathbb{C})$ subalgebras are not independent. They can be interchanged by the *parity operation*. Parity acts on the rotation - and boost - generators as follows:

$$J_i \to J_i \quad \text{and} \quad K_i \to -K_i,$$

and this shows that parity transforms S_i into A_i and A_i into S_i. If the generators J_k and K_i are representated by hermitean operators, then parity operation is equivalent to hermitean conjugation. As an example we consider the following representations:

- $(0, 0)$: this one-dimensional scalar representation has total spin zero.
- $(\frac{1}{2}, 0)$: this two-dimensional spinor representation is called the left-handed representation. Left-handed spinors transform trivial under one $sl(2)$ and according to the spin $\frac{1}{2}$-representation with respect to the other $sl(2)$.
- $(0, \frac{1}{2})$: this two-dimensional spinor representation is called the right-handed representation.

Since parity exchanges S_i and A_i the left(right) handed spinors are not eigenstates of parity. The representation $(\frac{1}{2}, 0)$ transforms under parity into the $(0, \frac{1}{2})$ representation and vice-versa. To obtain a representation such that parity acts as a linear transformation, one needs to combine the two spinor representation to

$$(\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})$$

which yields the Dirac spinor representation.
Any representation can be generated from the left- and right-handed spinor representations. For example, their tensor product

$$(\tfrac{1}{2}, 0) \otimes (0, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2})$$

yields a 4-component spin 1 (vector) representation. Also,

$$(\tfrac{1}{2}, 0) \otimes (\tfrac{1}{2}, 0) = (0, 0) \oplus (1, 0)$$

yields the scalar representation and a spin-one representation, given by self-dual antisymmetric second rank tensors. More generally, for tensors of rank two we find

$$\begin{aligned} T^{\mu\nu} &\in (\tfrac{1}{2}, \tfrac{1}{2}) \otimes (\tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, 0) \otimes (0, \tfrac{1}{2}) \otimes (0, \tfrac{1}{2}) \otimes (\tfrac{1}{2}, 0) \\ &= \big[(1, 0) \oplus (0, 0)\big] \otimes \big[(0, 1) \oplus (0, 0)\big] = (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0). \end{aligned}$$

Every second rank tensor contains a scalar part $(0,0)$, a spin-1 part $(1,0)\oplus(0,1)$ and a spin-2 part $(1,1)$. This decomposition corresponds to

$$T^{\mu\nu} = a\eta^{\mu\nu} + A^{\mu\nu} + S^{\mu\nu} \quad \text{with} \quad a = \tfrac{1}{4}T^{\rho}_{\ \rho}, \quad A^{\mu\nu} = -A^{\nu\mu}, \quad S^{\mu\nu} = S^{\nu\mu}, \quad S^{\rho}_{\ \rho} = 0.$$

The antisymmetric part can further be decomposed into its selfdual and anti-selfdual pieces.

2.3 Representations of the 4-dimensional Poincaré algebra

There are two cases to be distinguished:

1. There are massive particles. Then the symmetry consists of spacetime translations P_μ, Lorentz transformations $M_{\mu\nu}$ and an algebra G of internal symmetries commuting with the spacetime symmetries.
2. There are only massless particles. Then there is the extra possibility of having conformal symmetry.

First we find two Casimir operators of the Poincaré algebra. It is not difficult to prove, that

Lemma 1 *$P^2 = P_\mu P^\mu$ is a Casimir operator for the Poincaré algebra.*

Proof: Clearly, $[P^2, P_\mu] = 0$.
Also, using the Poincaré algebra (2.6)

$$[P_\mu P^\mu, M_{\rho\sigma}] = P_\mu[P^\mu, M_{\rho\sigma}] + [P_\mu, M_{\rho\sigma}]P^\mu = 0.$$

The second Casimir-operator of can be constructed from the Pauli-Ljubanski polarization vector, defined by

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \qquad \epsilon_{0123} = 1. \tag{2.10}$$

Clearly, this vector is orthogonal to the 4-momentum,

$$W_\mu P^\mu = 0. \tag{2.11}$$

In the $1+3$ split the Pauli-Ljubanski vector takes the form

$$W_0 = -P^i J_i, \quad W_i = P^0 J_i - \epsilon_{ijk}P^j K_k. \tag{2.12}$$

Now we shall prove the

Lemma 2 *$W^2 = W_\mu W^\mu$ is a Casimir operator for the Poincaré algebra.*

Proof: Since

$$[W_\mu, P_\xi] = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu[M^{\rho\sigma}, P_\xi] = i\epsilon_{\mu\nu\rho\xi}P^\nu P^\rho = 0 \tag{2.13}$$

holds true, W^2 commutes with the P_σ.
To continue, we introduce the Lorentz-invariant inner product between $\vec{K}$ and $\vec{J}$:

$$I = \frac{1}{8i}\epsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma} \equiv \frac{1}{i}\vec{K}\cdot\vec{J}, \qquad [I, M_{\mu\nu}] = 0. \tag{2.14}$$

I transforms the 4-momentum into the Pauli-Ljubanski vector,

$$[I, P_\xi] = \frac{1}{8i}\epsilon_{\mu\nu\rho\sigma}\big(M^{\mu\nu}[M^{\rho\sigma}, P_\xi] + [M^{\mu\nu}, P_\xi]M^{\rho\sigma}\big) = \tfrac{1}{2}\epsilon_{\xi\mu\rho\sigma}P^\mu M^{\rho\sigma} = W_\xi,$$

where we used the second set of commutation rules in (2.6). Using the Jacobi identity we can now calculate

$$[W_\rho, M_{\mu\nu}] = \big[[I, P_\rho], M_{\mu\nu}\big] = -\big[[P_\rho, M_{\mu\nu}], I\big] = i\big[\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu, I\big] = -i(\eta_{\rho\mu}W_\nu - \eta_{\rho\nu}W_\mu).$$

As expected, W_ρ transforms under Lorentz transformation the same way as P_ρ. Hence W^2 commutes with the $M_{\mu\nu}$ for the same reason as P^2 does and we conclude that P^2 and W^2 are Casimir operators of the Poincaré algebra. All physical states (fields, particles) in a quantum field theory are classified by the eigenvalues of these two Casimir operators.
The infinitesimal Lorentz transformations consists of two parts, namely the external orbital and internal spin part[1]:

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}, \qquad L_{\mu\nu} = -x_\mu P_\nu + x_\nu P_\mu, \quad [P_\mu, x^\nu] = i\delta_\mu^{\ \nu}.$$

Since the cyclic sum ov $P^\nu L^{\rho\sigma}$ vanishes the Pauli-Ljubanski vector simplify to

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu\Sigma^{\rho\sigma}.$$

2.3.1 Massive particles

A massive particle has a non-vanishing rest mass,

$$P^2 = m^2 > 0 \tag{2.15}$$

and we may transform to its rest frame, defined by

$$P^\mu = (m, \vec{0}).$$

In this rest frame we have

$$W_0 = 0 \quad \text{and} \quad W_i = -\frac{m}{2}\epsilon_{ijk}\Sigma^{jk} = mS_i,$$

where the operators

$$S_i = -\tfrac{1}{2}\epsilon_{ijk}\Sigma^{jk}, \qquad [S_i, S_j] = i\epsilon_{ijk}S_k$$

generate the spin rotations. We see that W^μ is the relativistic generalisation of the spin. Since

$$W^2 = -\vec{W}^2 = -m^2\vec{S}^2,$$

[1] See section (3.2).

we seee that the massive representations are labeled by the mass m and the spin s:

$$P^2 = m^2 \quad \text{and} \quad W^2 = -m^2 s(s+1), \quad s = 0, \tfrac{1}{2}, 1, \ldots.$$

The representations are characterized by the mass and the spin of the particle. Massive particles fall into $(2s+1)$-dimensional multiplets. The different states of an irreducible representation are distinguished by the third component of the spin and the continuous eigenvalues of $\vec{P}$. Examples of massive states are the electron with spin $\frac{1}{2}$ and the pion with spin 0.

2.3.2 Massless particles

Massless particles have a vanishing rest mass,

$$P^2 = 0$$

and there is no system in which they are at rest. But we can always transform into a system where

$$P_\mu = (P_0, 0, 0, P_3), \quad P_0^2 - P_3^2 = 0.$$

Since $P_\mu W^\mu = 0$ we conclude, that in this system

$$W_\mu = (\lambda P_0, W_1, W_2, \lambda P_3) \Longrightarrow W^2 = -W_1^2 - W_2^2 \leq 0.$$

For $W^2 = 0$ the 4-momentum and Pauli-Ljubanski vector become linearly dependent,

$$W_\mu = \lambda P_\mu,$$

and this holds in any inertial system. The factor λ is a Lorentz scalar and must commute with all generators of the Lorentz group. Hence we have

$$[\lambda, P^\mu] = [\lambda, M^{\mu\nu}] = 0.$$

Hence there is an additional Casimir operator in the massless case. Since $W_0 = -P^i J_i$ we conclude, that

$$\lambda = -\frac{P^i J_i}{P_0} \tag{2.16}$$

which is the helicity for a massless particle. The helicity must be half-integer[2]. Examples of particles falling into this category are the photon with spin 1 and helicity states ± 1 and a massless neutrino with spin $\frac{1}{2}$ and helicity states $\pm\frac{1}{2}$.
The remaining case $P^2 = 0$ and $W^2 < 0$ describes a particle of zero rest mass with an infinite number of polarization states. These representations do not seem to be realized in nature.

[2]See S. Weinberg, The Quantum Theory of Fields, I, p.90

2.4 Anti-de Sitter algebras

Consider $\mathbb{R}^{d+1}$ with signature $(2, d-1)$ and metric $\eta_{mn} = (+,-,\dots,-,+)$. The coordinates are denoted by ξ^m and

$$ds^2 = (d\xi^0)^2 - \sum_{i=1}^{d-1}(d\xi^i)^2 + (d\xi^d)^2.$$

One obtains the adS spaces as submanifold of $\mathbb{R}^{d+1}$ defined by the $SO(2, d-1)$-invariant condition

$$(\xi^0)^2 - \sum_{i=1}^{d-1}(\xi^i)^2 + (\xi^d)^2 = R^2 \Longleftrightarrow \sum_{\mu=0}^{d-2}\xi^\mu\xi_\mu + \xi^-\xi^+ = R^2, \quad \xi^\pm = \xi^d \pm \xi^{d-1}. \tag{2.17}$$

Note that every point p is invariant under rotations around the vector pointing from the origin to this point. This little group is $SO(1, d-1)$ and we see that

$$adS_d \sim \frac{SO(2, d-1)}{SO(1, d-1)}.$$

We choose the horospherical coordinates (z, x^μ) defined by

$$\xi^- = \frac{R^2}{z}, \quad \xi^\mu = \frac{Rx^\mu}{z}, \quad \mu = 0, \dots, d-2 \Longrightarrow \xi^+ = z - \frac{x_\mu x^\mu}{z}. \tag{2.18}$$

The induced metric is just

$$ds^2 = \sum_0^{d-2} d\xi^\mu d\xi_\mu + d\xi^+ d\xi^- = \frac{R^2}{z^2}\left(dx^\mu dx_\mu - dz^2\right) \tag{2.19}$$

and has signature $(1, d-1)$. Since spacetime is a hyperboloid it has constant curvature. It is called the Anti-de Sitter spacetime. Note that

$$g_{\nu\nu} = e^{2\sigma}\eta_{\mu\nu}, \qquad e^\sigma = \frac{R}{z} \tag{2.20}$$

and for such a conformally flat metric

$$\Gamma^\rho_{\mu\nu} = \Big(\delta^\rho_{\ \mu}\sigma,_\nu + \delta^\rho_{\ \nu}\sigma,_\mu - \eta_{\mu\nu}\sigma^{,\rho}\Big). \tag{2.21}$$

From this one finds the Ricci tensor and Ricci scalar:

$$\begin{aligned} R_{\mu\nu} &= (2-d)\Big(\sigma_{;\mu\nu} - \sigma,_\mu \sigma,_\nu + \eta_{\mu\nu}(\nabla\sigma)^2\Big) - \eta_{\mu\nu}\triangle\sigma \\ R &= -(d-1)e^{-2\sigma}\Big[2\triangle\sigma + (d-2)(\nabla\sigma)^2\Big]. \end{aligned} \tag{2.22}$$

For the AdS-metric

$$R_{\mu\nu} = \frac{d-1}{R^2} g_{\mu\nu} \quad \text{and} \quad R = \frac{d(d-1)}{R^2}. \tag{2.23}$$

Since the Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\frac{(d-1)(d-2)}{2R^2} g_{\mu\nu}$$

we see, that AdS solves the Einstein equation with cosmological constant

$$G_{\mu\nu} = \Lambda g_{\mu\nu}, \qquad \Lambda = -\frac{(d-1)(d-2)}{2R^2}. \tag{2.24}$$

Anti-de Sitter space is a spacetime with a maximal number of isometries and a negative cosmological constant.
The isometry group $SO(2, d-1)$ is linearly realized in the embedding space and the ξ^m transform as

$$\delta \xi^m = \omega^m{}_n \xi^n. \tag{2.25}$$

Since

$$\delta\xi^+ = \omega^+{}_+\xi^+ + \omega^+{}_\mu\xi^\mu, \quad \delta\xi^- = \omega^-{}_-\xi^- + \omega^-{}_\mu\xi^\mu, \quad \delta\xi^\mu = \omega^\mu{}_\nu\xi^\nu + \omega^\mu{}_+\xi^+ + \omega^\mu{}_-\xi^- \tag{2.26}$$

When raising and lowering indices, one should keep in mind that $\eta_{+-} = 1/2$ such that

$$\begin{aligned} V^+ &= V^d + V^{d-1} = 2V_- = V_d - V_{d-1} \\ V^- &= V^d - V^{d-1} = 2V_+ = V_d + V_{d-1}, \end{aligned}$$

from which follows, that $\omega^-{}_- = -\omega^+{}_+$ and $\omega^+{}_-$ and $\omega^-{}_+$ vanish. From (2.18) one now obtains

$$\begin{aligned} \delta z &= -\omega^-{}_- z - \omega^{-\mu} x_\mu z/R \\ \delta x^\mu &= R\omega^\mu{}_- + \omega^\mu{}_\nu x^\nu - \omega^-{}_- x^\mu - \tfrac{1}{2}\omega^{-\mu}(z^2 - x_\nu x^\nu)/R - \omega^{-\nu} x_\nu x^\mu/R \end{aligned} \tag{2.27}$$

The Lie derivatives are

$$\begin{aligned} \omega^-{}_- &: \; -z\partial_z - x^\mu\partial_\mu & \omega^{-\mu} &: \; -\tfrac{1}{R}x^\mu z\partial_z - \tfrac{1}{2}(z^2 - x^2)\partial_\mu - \tfrac{1}{R}x_\mu x^\nu\partial_\nu \\ \omega^\mu{}_\nu &: \; \omega^\mu{}_\nu x^\nu \partial_\mu & \omega^\mu{}_- &: \; R\partial_\mu. \end{aligned}$$

They form an $so(2, d-1)$ algebra, as expected (exercise!).
Suppose that you only know the metric (2.19). Then to determine the symmetries, you should solve the Killing equation

$$(L_X g)_{\mu\nu} = X^\rho g_{\mu\nu,\rho} + g_{\rho\nu} X^\rho_{,\mu} + g_{\mu\rho} X^\rho_{,\nu} = 0.$$

One can prove, that the transformations (2.27) are the only solutions.

3 Spinors

Now we shall study the Clifford algebras and spinors in arbitrary dimensions. We allow for arbitrary signatures of spacetime.

3.1 Gamma matrices and their symmetries

We consider arbitrary spacetime dimensions $d = t + s$, with t timelike directions and s spacelike directions, and consider the general facts of Clifford algebras and spinors, as has been done first in [20].
The Clifford algebra is

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \equiv \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \, , \tag{3.1}$$

where $\eta = \mathrm{diag}(+\ldots+-\ldots-)$, writing first the timelike directions and then the spacelike ones.
First we give a representation of the Clifford algebra for signature $(d, 0)$ (only timelike) in terms of σ matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.2}$$

The only relevant properties are $\sigma_1\sigma_2 = \mathrm{i}\sigma_3$ and cyclic, that the σ_i square to $\mathbb{1}_2 \equiv \sigma_0$ and are that they are hermitian.

$$\begin{array}{rclccc} \gamma_0 & = & \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \ldots & \gamma_2 = \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \ldots & \gamma_4 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots \\ \gamma_1 & = & \sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \ldots & \gamma_3 = \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \ldots & \ldots . \end{array} \tag{3.3}$$

To show, that these matrices furnish a representation of the Clifford algebra, one uses

$$\{A_1 \otimes \ldots \otimes A_p, B_1 \otimes \ldots \otimes B_p\} = A_1B_1 \otimes \ldots \otimes A_pB_p + B_1A_1 \otimes \ldots \otimes B_pA_p .$$

If $[A_i, B_i] = 0$ for all i but $i = q$, then

$$\{A_1 \otimes \ldots \otimes A_p, B_1 \otimes \ldots \otimes B_p\} = A_1B_1 \otimes \ldots \otimes \{A_q, B_q\} \otimes \ldots \otimes A_pB_p . \tag{3.4}$$

This is for even dimensions a representation of dimension $2^{d/2}$. For odd dimensions, e.g. $d = 5$, one does not need the last σ_1 factor in γ_4, and the dimension is still $2^{(d-1)/2}$. These matrices are all hermitian. If there are s spacelike directions, we just have to multiply the last s matrices in the representation above by i. Thus we get as hermiticity property for timelike and spacelike directions

$$\gamma^\dagger_{\mu<t} = \gamma_\mu\,, \qquad \gamma^\dagger_{\mu\geq t} = -\gamma_\mu. \tag{3.5}$$

To define the Dirac conjugate we introduce the unitary $\mathcal{A} = \gamma_0\gamma_1\cdots\gamma_{t-1}$ which obeys

$$\mathcal{A}^\dagger = (-)^{t(t-1)/2}\mathcal{A} \quad \text{or} \quad \mathcal{A}^* = (-)^{t(t-1)/2}\mathcal{A}^T. \tag{3.6}$$

Then

$$\mathcal{A}\gamma_{\mu<t}\mathcal{A}^{-1} = -(-)^t\gamma_\mu \quad \text{and} \quad \mathcal{A}\gamma_{\mu\geq t}\mathcal{A}^{-1} = (-)^t\gamma_\mu \Longrightarrow \mathcal{A}\gamma_\mu\mathcal{A}^{-1} = -(-)^t\gamma^\dagger_\mu. \tag{3.7}$$

Of course the realization given above is not unique. Equivalent representations preserving (3.1) are

$$\gamma' = U\gamma U^{-1}\,. \tag{3.8}$$

To respect also (3.5), U has to be unitary. It is clear, that $\pm\gamma^*$ can also be used as representations for the Clifford algebra. For even dimensions these are unitary equivalent to the original one. In odd dimensions one finds two unitary inequivalent representations. For example,

$$\gamma_0 = \sigma_1\,, \quad \gamma_1 = \sigma_2 \quad \text{and} \quad \gamma_2 = \pm\sigma_3$$

are two inequivalent representation of the three dimensional Clifford algebra.
Another important fact is that **in even dimensions** a complete set of $2^{d/2}\times 2^{d/2}$ matrices is provided by $\{\gamma^{(n)}\}$ with $n = 0, 1, \ldots d$, and

$$\gamma^{(n)} = \gamma_{\mu_1\ldots\mu_n} \equiv \gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_n]} \qquad \binom{d}{n} \quad \text{matrices.}$$

The last matrix $\gamma^{(d)}$, we write as

$$\gamma_* = (-\mathrm{i})^{d/2+s}\gamma_0\ldots\gamma_{d-1} = \underbrace{\sigma_3\otimes\sigma_3\otimes\ldots\otimes\sigma_3}_{d/2} \qquad \gamma_*\gamma_* = 1, \tag{3.9}$$

where the normalization is chosen such that the last equation holds in any case. γ_* is independent on the signature of spacetime, in the sense that when we first introduce the γ matrices in Euclidean signature, and then go to an arbitrary signature, we multiply the last s gamma matrices by i. This drops out again by the multiplication by $(-\mathrm{i})^t$ in (3.9). For even dimensions we have

$$\gamma_{\mu_1\ldots\mu_n} = \frac{1}{(d-n)!}\varepsilon_{\mu_1\ldots\mu_d}\mathrm{i}^{d/2+s}\gamma_*\gamma^{\mu_d\cdots\mu_{n+1}}\,. \tag{3.10}$$

Note that γ_* anti-commutes with γ_μ and can thus be used as γ_d in the next, **odd dimension**. In odd dimensions the product of all γ matrices gives ±1 or $\pm\mathrm{i}$, and a basis is formed by $\gamma^{(n)}$ with $n = 0, 1, \ldots, (d-1)/2$.

3.2 Lorentz transformations

We study the transformation property of Dirac spinors under Lorentz transformations on spaces with dimension $d = t + s$. For that we introduce the $d(d-1)/2$ matrices

$$\Sigma^{\mu\nu} = -\Sigma^{\nu\mu} = \frac{1}{2i}\gamma^{\mu\nu} \equiv \frac{1}{4i}[\gamma^\mu, \gamma^\nu]. \tag{3.11}$$

They have the following commutation relations with the γ-matrices and with themselves:

$$\begin{aligned} [\Sigma^{\mu\nu}, \gamma^\rho] &= i\big(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu\big) \\ [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] &= i\big(\eta_{\mu\rho}\Sigma_{\nu\sigma} + \eta_{\nu\sigma}\Sigma_{\mu\rho} - \eta_{\mu\sigma}\Sigma_{\nu\rho} - \eta_{\nu\rho}\Sigma_{\mu\sigma}\big). \end{aligned} \tag{3.12}$$

Hence, the Σ furnish a $2^{[d/2]}$-dimensional representation of the Lorentz algebra. Let

$$\Gamma^\rho(s) = S^{-1}(s)\gamma^\rho S(s) \quad \text{with} \quad S(s) = e^{i\frac{s}{2}(\omega,\Sigma)} = e^{\frac{s}{4}\omega_{\mu\nu}\gamma^{\mu\nu}}, \quad (\omega,\Sigma) = \omega_{\mu\nu}\Sigma^{\mu\nu}$$

be a one-parameter group of transformations with $S(0) = \mathbb{1}$. With (3.12) we obtain the differential equation

$$\frac{d}{ds}\Gamma^\rho(s) = -\frac{i}{2}\omega_{\mu\nu}S^{-1}(s)[\Sigma^{\mu\nu}, \gamma^\rho]S(s) = \omega^\rho_{\ \sigma}\Gamma^\sigma(s)$$

the solution of which is

$$\Gamma^\rho(s) = S^{-1}(s)\gamma^\rho S(s) = (e^{s\omega})^\rho_{\ \sigma}\gamma^\sigma. \tag{3.13}$$

Since $\omega_{\mu\nu} = -\omega_{\nu\mu}$ the matrix e^ω is a Lorentz transformation. Setting $s = 1$ we conclude

$$S^{-1}\gamma^\rho S = \Lambda^\rho_{\ \sigma}\gamma^\sigma, \quad \text{where} \quad S = e^{\frac{i}{2}(\omega,\Sigma)}, \quad \Lambda = e^\omega. \tag{3.14}$$

To find the action of infinitesimal Lorentz transformation on spinors, we recall that a Dirac spinor transforms as

$$\psi(x) \longrightarrow S(\omega)\psi\big(\Lambda(\omega)^{-1}x\big) = e^{\frac{i}{2}(\omega,\Sigma)}\psi(e^{-\omega}x). \tag{3.15}$$

The infinitesimal form of this transformation is

$$\delta\psi(x) = \tfrac{i}{2}\omega_{\mu\nu}\Big(L^{\mu\nu} + \Sigma^{\mu\nu}\Big)\psi(x), \tag{3.16}$$

where

$$L_{\mu\nu} = \frac{1}{i}\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big).$$

The $M_{\mu\nu}$ and $\Sigma_{\mu\nu}$ both satisfy the commutation relations (2.6) of the Lorentz algebra so that

$$J_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}, \tag{3.17}$$

also generate Lorentz transformations of Dirac spinors.

To construct tensor fields which are bilinear in the spinor fields we note, that (3.7) implies

$$\mathcal{A}[\gamma_\mu,\gamma_\nu]\mathcal{A}^{-1}=[\gamma_\mu^\dagger,\gamma_\nu^\dagger]=-[\gamma_\mu,\gamma_\nu]^\dagger.$$

It follows at once that

$$\mathcal{A}^{-1}S^\dagger\mathcal{A}=\exp\big(\tfrac{1}{4}\mathcal{A}^{-1}(\omega^{\mu\nu}\gamma^\dagger_{\mu\nu})\mathcal{A}\big)=\exp\big(-\tfrac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\big)=S^{-1}. \tag{3.18}$$

Now we are ready to define the Dirac conjugate:

$$\bar\psi\equiv\psi^\dagger\mathcal{A}. \tag{3.19}$$

It transforms as

$$\bar\psi\longrightarrow\psi^\dagger S^\dagger\mathcal{A}=\psi^\dagger\mathcal{A}S^{-1}=\bar\psi S^{-1}. \tag{3.20}$$

Hence, with (3.14) the objects

$$\bar\psi\gamma_{\mu_1\ldots\mu_n}\psi\longrightarrow\bar\psi\,S^{-1}\gamma_{\mu_1\ldots\mu_n}S\,\psi=\Lambda_{\mu_1}^{\ \nu_1}\cdots\Lambda_{\mu_n}^{\ \nu_n}\,\bar\psi\gamma_{\nu_1\ldots\nu_n}\psi$$

transform as antisymmetric tensor field.

3.3 Charge conjugation

To motivate the properties of the charge conjugation we shall discuss below, we consider the Dirac equation

$$\big(i\gamma^\mu(\partial_\mu-ieA_\mu)-m\big)\psi=0$$

in d dimensions. Now we multiply this equation from the left with the matrix $\mathcal{A}$ we introduced in (3.7):

$$\Big(i\mathcal{A}\gamma^\mu\mathcal{A}^{-1}(\partial_\mu-ieA_\mu)-m\Big)\mathcal{A}\psi=\Big(-i(-)^t\gamma^{\mu\dagger}(\partial_\mu-iA_\mu)-m)\Big)\mathcal{A}\psi=0.$$

The complex conjugate of this equation is

$$\Big(i(-)^t\gamma^{\mu T}(\partial_\mu+ieA_\mu)-m)\Big)(\mathcal{A}\psi)^*=0.$$

Finally we multiply with a charge conjugation matrix $\mathcal{C}$ and obtain

$$\Big(i(-)^t\mathcal{C}\gamma^{\mu T}\mathcal{C}^{-1}(\partial_\mu+ieA_\mu)-m)\Big)\mathcal{C}(\mathcal{A}\psi)^*=\big(i\gamma^\mu(\partial_\mu+ieA_\mu)-m\big)\mathcal{C}\mathcal{A}^*\psi^*=0,,$$

if only

$$\gamma_\mu^T=(-)^t\mathcal{C}^{-1}\gamma_\mu\mathcal{C} \tag{3.21}$$

holds. Since (see (3.6))

$$\mathcal{A}\mathcal{A}^\dagger=\mathbb{1}\quad\text{implies}\quad\mathcal{A}^2=(-)^{t(t-1)/2}\mathbb{1}\Longrightarrow\mathcal{A}^*=(-)^{t(t-1)/2}\mathcal{A}^T$$

we take for the charge conjugated spinor

$$\psi_C = \mathcal{B}\psi^* = \mathcal{C}\bar{\psi}^T, \qquad \mathcal{B} = \mathcal{C}\mathcal{A}^T. \tag{3.22}$$

The Dirac conjugate has been defined in (3.19). ψ_C fulfills the Dirac equation with reversed electric charge. In *Euclidean space* we have $\mathcal{A} = \mathbb{1}$ and

$$\gamma_\mu^T = \mathcal{C}^{-1}\gamma_\mu\mathcal{C}, \qquad \psi_C = \mathcal{C}\psi^* = \mathcal{C}\bar{\psi}^T$$

and in *Minkowski space* we have $\mathcal{A} = \gamma^0$ and

$$\gamma_\mu^T = -\mathcal{C}^{-1}\gamma_\mu\mathcal{C}, \quad \psi_C = \mathcal{C}\gamma_0^*\psi^* = \mathcal{C}\bar{\psi}^T.$$

In the last equality we used, that γ_0 is hermitian, $\gamma_0^* = \gamma_0^T$. If there exists a Majorana representation in which all γ are imaginary we may choose $\mathcal{C} = \gamma_0$ such that

$$\psi_C = \psi^*.$$

In such a representation a Majorana spinor, which is invariant under charge conjugations is real. Now we are going to show, that there exists always a '**charge conjugation matrix**' $\mathcal{C}$, such that

$$\gamma_\mu^T = -\eta\,\mathcal{C}^{-1}\gamma_\mu\mathcal{C} \quad \text{and} \quad \mathcal{C}^T = -\varepsilon\mathcal{C} \tag{3.23}$$

for $\varepsilon = \pm 1$ and $\eta = \pm 1$. A formal proof can be found in [19, 20], but I prove it here by giving two possibilities in the representation (3.3) for even dimensions. We use

$$\sigma_1\sigma_1\sigma_1 = \sigma_1^T \quad \sigma_1\sigma_2\sigma_1 = \sigma_2^T \quad \sigma_1\sigma_3\sigma_1 = -\sigma_3^T \quad \text{and} \quad \sigma_2\sigma_i\sigma_2 = -\sigma_i^T$$

to show that

$$\begin{aligned} \mathcal{C}_+ &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \dots & \eta = +1 \\ \mathcal{C}_- &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \dots \propto \mathcal{C}_+\gamma_* & \eta = -1 \end{aligned} \tag{3.24}$$

are possible charge conjugation matrices. Note that they satisfy $\mathcal{C} = \mathcal{C}^\dagger = \mathcal{C}^{-1}$. For odd dimensions, only one of the two can be used, see below. If one changes the representation as in (3.8), then, to preserve (3.23), the charge conjugation matrix should transform as

$$\mathcal{C}' = U\mathcal{C}U^T. \tag{3.25}$$

Therefore you see that the unitarity, $\mathcal{C}^\dagger = \mathcal{C}^{-1}$, is true in any representation, but the charge conjugation not necessarily squares to $\mathbb{1}$.
We will now deduce which signs of the two variables ε and η are allowed for each value of t and s. First observe that

$$\begin{aligned} \left(\mathcal{C}^{-1}\gamma_{\mu_1}\cdots\gamma_{\mu_n}\right)^T &= -\varepsilon\gamma_{\mu_n}^T\cdots\gamma_{\mu_1}^T\mathcal{C}^{-1} = -\varepsilon(-\eta)^n\mathcal{C}^{-1}\gamma_{\mu_n}\cdots\gamma_{\mu_1} \\ &= -\varepsilon(-)^{n(n-1)/2}(-\eta)^n\mathcal{C}^{-1}\gamma_{\mu_1}\cdots\gamma_{\mu_n}. \end{aligned}$$

We assumed, that all μ_i are different. It follows, that $\mathcal{C}^{-1}\gamma_{\mu_1}\cdots\gamma_{\mu_n}$ is either symmetric or antisymmetric. In particular

$$\left(\mathcal{C}^{-1}\gamma^{(n)}\right)^T = -\varepsilon(-)^{[n/2]}(-\eta)^n \mathcal{C}^{-1}\gamma^{(n)}. \tag{3.26}$$

This has a periodicity of $n \to n+4$. This periodicity implies that the number of symmetric matrices should be the sum of lines of the following formulae

$$\begin{aligned}
s_0 &= \binom{d}{0} + \binom{d}{4} + \ldots = 2^{d-2} + 2^{d/2-1}\cos\frac{d\pi}{4} \\
s_1 &= \binom{d}{1} + \binom{d}{5} + \ldots = 2^{d-2} + 2^{d/2-1}\sin\frac{d\pi}{4} \\
s_2 &= \binom{d}{2} + \binom{d}{6} + \ldots = 2^{d-2} - 2^{d/2-1}\cos\frac{d\pi}{4} \\
s_3 &= \binom{d}{3} + \binom{d}{7} + \ldots = 2^{d-2} - 2^{d/2-1}\sin\frac{d\pi}{4}\,.
\end{aligned} \tag{3.27}$$

To prove these identities one uses

$$\begin{aligned}
&(1+1)^d = 2^d = \sum_{k=0}^{d}\binom{d}{k} && (1-1)^d = 0 = \sum(-)^k\binom{d}{k} \\
&(1+i)^d = \sqrt{2}e^{i\pi d/4} = \sum i^k\binom{d}{k} && (1-i)^d = \sqrt{2}e^{-i\pi d/4} = \sum(-i)^k\binom{d}{k}.
\end{aligned}$$

For example, the sum of these four series leads to the first identity in (3.27). We list the sums for the lowest dimensions

dim.	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7	d=8
s_0	1	1	1	1	2	6	16	36	72
s_1	0	1	2	3	4	6	12	28	64
s_2	0	0	1	3	6	10	16	28	56
s_3	0	0	0	1	4	10	20	36	64

We can count how many symmetric and antisymmetric matrices exist in d dimensions:

$$\text{symmetric:}\quad 2^{[d/2]-1}\left(2^{[d/2]}+1\right) \qquad \text{antisymmetric:}\quad 2^{[d/2]-1}\left(2^{[d/2]}-1\right). \tag{3.28}$$

The dimensions of the representation, number of symmetric and antisymmetric matrices for the lowest dimensions are given in the following table:

spacetime dimension	d=1	d=2,3	d=4,5	d=6,7	d=8,9	d=10,11
dimension of repr.	1	2	4	8	16	32
symmetric matrices	1	3	10	36	136	528
antisymmetric ones	0	1	6	28	120	496

For odd dimensions, $\sin d\pi/4 = \pm\cos d\pi/4$ and either $s_0 = s_1, s_2 = s_3$ or $s_0 = s_3, s_1 = s_2$. Since there are more symmetric matrices as antisymmetric ones there is a unique solution. For even dimensions, only two of the lines in (3.27) are equal, and there are thus two possibilities for which matrices are symmetric. This corresponds to the two choices for the charge conjugation matrix in that case. The result is given in the following table:

d mod 8	0		1	2		3	4		5	6		7
symmetric	0,3	0,1	0,1	1,0	1,2	1,2	2,1	2,3	2,3	3,2	3,0	0,3
antisymmetric	2,1	2,3	2,3	3,2	3,0	0,3	0,3	0,1	0,1	1,0	1,2	1,2
ε	-1	-1	-1	-1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1
η	$+1$	-1	-1	-1	$+1$	$+1$	$+1$	-1	-1	-1	$+1$	$+1$

3.4 Irreducible spinors

We still do not know whether the spinor is irreducible. There are two types of projections that one can envisage.
The first is only for even dimensions, where we have γ_* available as in (3.9). We can thus define *left and right spinors.* We take

$$\psi_L = \tfrac{1}{2}(1-\gamma_*)\,\psi \equiv P_L\psi\,, \qquad \psi_R = \tfrac{1}{2}(1+\gamma_*)\,\psi \equiv P_R\psi\,. \tag{3.29}$$

The chirality is preserved under Lorentz transformation,

$$\delta(\psi_L) = \big(\delta\psi\big)_L\,, \qquad \delta\psi = \tfrac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\psi,$$

since γ_* commutes with the infinitesimal Lorentz generators $\gamma_{\mu\nu}$. In four dimensional Minkowski space $(t=1, s=3)$ we define

$$\gamma_5 = \gamma_* = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3. \tag{3.30}$$

The second possible projection is a *reality condition.* Consider first the reality properties of the γ matrices. Combining (3.7) and (3.23), we have

$$\gamma_\mu^* = \eta(-)^t\mathcal{B}^{-1}\gamma_\mu\mathcal{B}\,, \qquad \mathcal{B} = \mathcal{C}\mathcal{A}^T\,. \tag{3.31}$$

As $\mathcal{A}$ and $\mathcal{C}$ are unitary, we find that also $\mathcal{B}$ is unitary. One obtains

$$\mathcal{B}^*\mathcal{B} = -\varepsilon\eta^t(-)^{t(t+1)/2}\,. \tag{3.32}$$

To prove this equation, we note that

$$\mathcal{A}^T = \gamma_{t-1}^T\cdots\gamma_0^T = (-\eta)^t\mathcal{C}^{-1}\gamma_{t-1}\cdots\gamma_0\mathcal{C} = (-\eta)^t\mathcal{C}^{-1}\mathcal{A}^{-1}\mathcal{C} = \eta^t(-)^{t(t+1)/2}\mathcal{C}^{-1}\mathcal{A}\mathcal{C} \tag{3.33}$$

so that

$$\mathcal{B}^*\mathcal{B} = \eta^t(-)^{t(t+1)/2}\mathcal{C}^*\mathcal{A}^\dagger\mathcal{A}\mathcal{C} = -\varepsilon\eta^t(-)^{t(t+1)/2}\mathcal{C}^\dagger\mathcal{C}$$

which is just (3.32), since $\mathcal{C}$ is unitary.
Let us now try to put a *reality condition* on spinors

$$\psi = \tilde{\mathcal{B}}\psi^* \,, \tag{3.34}$$

for some matrix $\tilde{\mathcal{B}}$. First of all we want consistency with Lorentz transformations, i.e. the reality condition should be respected by infinitesimal transformations,

$$\delta\big(\tilde{\mathcal{B}}\psi^*\big) = \tilde{\mathcal{B}}(\delta\psi)^*.$$

With (3.31) this is equivalent to

$$\gamma_{\mu\nu} = \tilde{\mathcal{B}}\gamma^*_{\mu\nu}\tilde{\mathcal{B}}^{-1} = \tilde{\mathcal{B}}\mathcal{B}^{-1}\gamma_{\mu\nu}\mathcal{B}\tilde{\mathcal{B}}^{-1} \tag{3.35}$$

and therefore we take

$$\tilde{\mathcal{B}} = \mathcal{B} = \mathcal{C}\mathcal{A}^T. \tag{3.36}$$

For consistency we must have $\psi^{**} = \psi$ so that $\tilde{\mathcal{B}}^*\tilde{\mathcal{B}} = 1$. Combining this with (3.32) yields the constraint

$$-\varepsilon\eta^t(-)^{t(t+1)/2} = 1 \tag{3.37}$$

which, together with our previous results on the allowed values on η and ε, leads to conditions for the existence of Majorana spinors. If this constraint is fulfilled, then

$$\mathcal{A}^T = -\epsilon\mathcal{C}^{-1}\mathcal{A}\mathcal{C} \quad \text{or} \quad (\mathcal{A}\mathcal{C})^T = (\mathcal{A}\mathcal{C}). \tag{3.38}$$

Let us now discuss the consequences of (3.37). The left hand side of this equation has period 4 in t and yields

$$\varepsilon = -1 \text{ for } t = 0, \quad \varepsilon\eta = 1 \text{ for } t = 1, \quad \varepsilon = 1 \text{ for } t = 2 \quad \text{and} \quad \varepsilon\eta = -1 \text{ for } t = 3.$$

The possible spinors in various dimensions, and for various number of time directions (modulo 4) are listed in the following table. M stands for Majorana spinors. For even (odd) dimensions, $M^\pm$ indicates the sign of η. MW indicates the possibility of Majorana–Weyl spinors. For even dimensions one can always have Weyl spinors. The number indicates the real dimension of the minimal spinor.

d \ t	0		1		2		3	
1	M$^-$	1	M$^-$	1				
2	M$^-$	2	MW	1	M$^+$	2		
3		4	M$^+$	2	M$^+$	2		4
4	SMW	4	M$^+$	4	MW	2	M$^-$	4
5		8		8	M$^-$	4	M$^-$	4
6	M$^+$	8	SMW	8	M$^-$	8	MW	4
7	M$^+$	8		16		16	M$^+$	8
8	MW	8	M$^-$	16	SMW	16	M$^+$	16
9	M$^-$	16	M$^-$	16		32		32
10	M$^-$	32	MW	16	M$^+$	32	SMW	32
11		64	M$^+$	32	M$^+$	32		64
12	SMW	64	M$^+$	64	MW	32	M$^-$	64

Since $s-t=d-2t$, we see that the condition (3.37) is satisfied for[1]

$$\begin{array}{rcll} s-t &=& 0,1,7 \bmod 8 & \\ s-t &=& 2 \bmod 8 \quad \text{with} & \eta=+1 \\ s-t &=& 6 \bmod 8 \quad \text{with} & \eta=-1\,. \end{array} \tag{3.39}$$

In these cases, we can thus define a reality condition as in (3.34), and such spinors will be denoted as 'Majorana spinors'. Let us rewrite the Dirac conjugate of ψ_C:

$$\bar{\psi}_C=\psi_C^\dagger\mathcal{A}=\psi^T\mathcal{B}^\dagger\mathcal{A}=\psi^T\mathcal{A}^*C^{-1}\mathcal{A}=-\epsilon\psi^T\mathcal{A}^*\mathcal{A}^T\mathcal{C}^{-1}=\psi^T(\mathcal{C}^{-1})^T=-\psi^T\mathcal{C}^{-1}, \tag{3.40}$$

where we used (3.38). Hence another way of expressing the Majorana condition is that the 'Majorana conjugate' $\bar{\psi}_C$ equals the 'Dirac conjugate' $\bar{\psi}$:

$$\bar{\psi}_C=\psi^T(\mathcal{C}^{-1})^T=\bar{\psi}. \tag{3.41}$$

Consider now that the right hand side of (3.32) is not 1 but -1. Then there is still another possibility if we have extended supersymmetry. Indeed, then one can define a '*symplectic Majorana condition*'

$$\psi_i=\mathcal{B}\Omega_{ij}\psi_j^*\,, \tag{3.42}$$

where Ω is some antisymmetric matrix, with $\Omega\Omega^*=-1$. This way one gets real spinors for $s-t=4$ mod 8 (see table above), e.g. in $d=6$ Minkowski spacetime.
Having two projections, to chiral spinors and to Majorana spinors, one may ask whether one can

[1]we have listed the odd dimensional cases in the first row, since there is no choice for the sign of η in this case.

define a reality condition respecting the chiral projection. This means, that

$$(1 \pm \gamma_*)\tilde{\mathcal{B}} = \tilde{\mathcal{B}}\big(1 \pm (\gamma_*)^*\big).$$

Using the definition of γ_* in (3.9), the property (3.31) and that d is even this is equivalent to

$$\mathcal{B}(\gamma_*)^*\mathcal{B}^{-1} = i^{d/2+s}\mathcal{B}\gamma_0^* \cdots \gamma_{d-1}^*\mathcal{B}^{-1} = (-1)^{d/2+s}\gamma_*. \tag{3.43}$$

Hence, there are Majorana-Weyl spinors if

$$\text{MW spinor} : 2s - d = 0 \bmod 4 \quad \text{or} \quad s - t = d - 2t = 0 \bmod 4. \tag{3.44}$$

The dimensions and number of timelike directions for which Majorana-Weyl spinors exist, are listed in the above table.

3.5 Majorana condition in Minkowski spacetimes

In certain dimensions there exists a representation of the Dirac algebra with antisymmetric γ^0 and symmetric γ^i. Since γ^0 is hermitean and the others are antihermitean, it follows that in this representation all γ^μ are pure imaginary. Let us assume that such an representation exists. Then $\mathcal{C} = -\gamma^0$ is a charge conjugations matrix with $\epsilon = \eta = 1$. From our table above we conclude that in Minkowski spacetime a Majorana representation can only exist, if

$$d \bmod 8 \in \{2, 3, 4\}.$$

In 2 dimensions we may choose

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1 \quad \text{such that} \quad \gamma_* = \sigma_3,$$

in 3 dimensions

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_3.$$

and in 4 dimensions

$$\gamma^0 = \sigma_0 \otimes \sigma_2, \quad \gamma^1 = i\sigma_0 \otimes \sigma_3, \quad \gamma^2 = i\sigma_1 \otimes \sigma_1, \quad \gamma^3 = i\sigma_3 \otimes \sigma_1 \quad \text{such that} \quad \gamma_* = \sigma_2 \otimes \sigma_1.$$

Since in Minkowski spaces $\mathcal{A} = \gamma^0$ we have $\mathcal{B} = C\mathcal{A}^T = \gamma_0^2 = \mathbb{1}$. Hence a Majorana spinor is real,

$$\psi_C = \psi^* = \psi \quad \Longrightarrow \quad \psi \quad \text{is real.}$$

In 8 and 9 dimensions the γ^μ are 16×16 matrices. There are 9 such symmetrical, anticommuting and real matrices which all square to one, namely

$$\sigma_1 \otimes \sigma_0 \otimes \mathbb{1}_4, \quad \sigma_2 \otimes \sigma_2 \otimes \mathbb{1}_4, \quad \sigma_2 \otimes \sigma_0 \otimes \mathbb{1}_4, \quad \sigma_2 \otimes \sigma_1 \otimes \alpha_i, \quad \sigma_2 \otimes \sigma_3 \otimes \beta_i,$$

where we used the six antisymmetric and imaginary 4×4-matrices

$$\begin{aligned} &\alpha_1 = \sigma_2 \otimes \sigma_1, \qquad \alpha_2 = \sigma_0 \otimes \sigma_2, \quad \alpha_3 = \sigma_2 \otimes \sigma_3 \\ &\beta_1 = \sigma_1 \otimes \sigma_2, \qquad \beta_2 = \sigma_2 \otimes \sigma_0, \quad \beta_3 = \sigma_3 \otimes \sigma_2. \end{aligned}$$

In these dimensions there is no antisymmetric $\mathcal{C}$ with $\eta = 1$, because such a $\mathcal{C}$ would anticommute with all symmetric γ^μ:

$$\gamma_\mu^T = \gamma_\mu = -\mathcal{C}^{-1}\gamma_\mu\mathcal{C} \quad \text{or} \quad \{\mathcal{C}, \gamma_\mu\} = 0.$$

In 8 dimensions $\mathcal{C} \sim \gamma_*$ and in 9 dimensions there is no such $\mathcal{C}$. However, the matrix $\mathcal{C} = \mathbb{1}$ is a charge conjugation matrix with $\eta = \epsilon = -1$ in $d = 8$ and $d = 9$. Since γ^0 is symmetric and hence real, we have

$$\psi_C = \gamma_0^T \psi^* = \gamma_0 \psi^* \Longrightarrow (\psi_C)_C = \psi.$$

3.6 4 dimensions

3.6.1 Chiral representation

In another representation that I present now, γ_5 is diagonal. That basis is useful to have manifestly chiral spinors. These are then just spinors that have only two upper components χ^A, while antichiral have two lower components, denoted as $\chi_{\dot{A}}$, (where A and $\dot{A}$ can be 1 or 2). The explicit realization is

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}. \tag{3.45}$$

and with this choice

$$\mathcal{C} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \quad \text{where} \quad \varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{B} = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}. \tag{3.46}$$

A Majorana spinor obeys

$$\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \Psi_C \Longleftrightarrow \chi = \varepsilon\phi^*.$$

This basis makes the connection to the '2-component formalism' easy. In that formulation, indices are raised or lowered with ε in a NW-SE convention

$$\chi^A = \varepsilon^{AB}\chi_B\,, \qquad \chi_A = \chi^B\varepsilon_{BA} \qquad (\varepsilon_{AB}) = (\varepsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which implies $\varepsilon^{AB}\varepsilon_{BC} = -\delta^A{}_C$, and the Majorana condition for a spinor $(\zeta^A, \zeta_{\dot{A}})$ is then

$$\zeta_A = (\zeta_{\dot{A}})^*\,. \tag{3.47}$$

More details about this van der Waerden notation can be found in appendix A.

3.6.2 Dirac representation

A third representation that is often used, has diagonal γ_0.

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}. \tag{3.48}$$

The unitary and antisymmetric charge conjugation matrix must obey

$$\gamma_0 = -\mathcal{C}^{-1}\gamma_0\mathcal{C}, \quad \gamma_1 = \mathcal{C}^{-1}\gamma_1\mathcal{C}, \quad \gamma_2 = -\mathcal{C}^{-1}\gamma_2\mathcal{C}, \quad \gamma_3 = \mathcal{C}^{-1}\gamma_3\mathcal{C}.$$

One takes $\mathcal{C} = \gamma_0\gamma_2 = \mathcal{C}^{-1} = \mathcal{C}^\dagger$ and the Majorana condition amounts to

$$\psi = \gamma_0\gamma_2\gamma_0\psi^* = -\gamma_2\psi^* = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\psi^*. \tag{3.49}$$

3.7 Formulae, technical tips

We collect the relevant formulae we obtained in this section (for the symplectic case some signs change). The charge conjugation matrix $\mathcal{C}$, the $\mathcal{A}$ needed for Dirac conjugation and $\mathcal{B}$ fulfill

$$\begin{aligned}
\mathcal{C} &= \mathcal{C}^\dagger = -\varepsilon\mathcal{C}^T \\
\gamma_\mu^T &= -\eta\mathcal{C}^{-1}\gamma_\mu C \\
(\mathcal{C}^{-1}\gamma^{(n)})^T &= -\varepsilon(-)^{n(n+1)/2}\eta^t\,\mathcal{C}^{-1}\gamma^{(n)} \\
\left(\gamma^n\right)^T &= (-)^{n(n+1)/2}\eta^t\,\mathcal{C}^{-1}\gamma^{(n)}\mathcal{C} \\
\mathcal{A} &= \gamma_0\cdots\gamma_{t-1} = \mathcal{A}^\dagger \\
\mathcal{A}\gamma_\mu\mathcal{A}^{-1} &= -(-)^t\gamma_\mu^\dagger \\
\mathcal{A}^* &= (-)^{t(t-1)/2}\mathcal{A}^T \\
\mathcal{A}^T &= -\epsilon\,\mathcal{C}^{-1}\mathcal{A}\mathcal{C} \Rightarrow (\mathcal{A}\mathcal{C})^T = \mathcal{A}\mathcal{C} \\
\mathcal{B} &= \mathcal{C}\mathcal{A}^T = \mathcal{B}^T \\
\mathcal{B}\mathcal{B}^* &= \mathbb{1}, \quad \gamma_\mu^* = \eta(-)^t\mathcal{B}^{-1}\gamma_\mu\mathcal{B} \\
1 &= -\varepsilon\eta^t(-)^{t(t+1)/2} \\
\mathcal{C} &= (-\eta)^t\mathcal{A}\mathcal{B}
\end{aligned} \tag{3.50}$$

The various operations of spinors are charge conjugation, chiral projection and Dirac conjugate[2]:

$$\psi_C = \mathcal{B}\psi^*, \quad (\psi_C)_C = \psi, \quad \psi_L = \tfrac{1}{2}(1-\gamma_*)\psi \quad \text{and} \quad \bar{\psi} = \psi^\dagger\mathcal{A}. \tag{3.51}$$

We have

$$\psi_C = \mathcal{C}\bar{\psi}^T, \quad \bar{\psi}_C = \psi^T\mathcal{C}^* = -\epsilon\psi^T\mathcal{C}^{-1}, \quad (\bar{\psi}_C)^* = \psi^\dagger\mathcal{C} = (-\eta)^t\bar{\psi}\mathcal{B} \tag{3.52}$$

It follows, that

$$\bar{\psi}_C\gamma^{(n)}\chi_C \quad = \quad -\epsilon\psi^T\mathcal{C}^{-1}\gamma^{(n)}\mathcal{C}\bar{\chi}^T = -\epsilon(-)^{n(n+1)/2}\eta^t\psi^T\left(\gamma^{(n)}\right)^T\bar{\chi}^T$$

[2] In the symplectic case $(\psi_C)_C = -\psi$.

$$= \quad \epsilon(-)^{n(n+1)/2}\eta^t \left(\bar\chi\gamma^{(n)}\psi\right)^T = \epsilon(-)^{n(n+1)/2}\eta^t\bar\chi\gamma^{(n)}\psi.$$

In the second last step we used, that the fermion fields *anticommute.* In particular, for Majorana spinors

$$\bar\psi\gamma^{(n)}\chi = \epsilon(-)^{n(n+1)/2}\eta^t\,\bar\chi\gamma^{(n)}\psi. \tag{3.53}$$

In 4 dimensions and $\epsilon = \eta = 1$ we get

$$\begin{array}{l} \bar\psi\chi = \bar\chi\psi, \quad \bar\psi\gamma_5\chi = \bar\chi\gamma_5\psi, \quad \bar\psi\gamma_\mu\chi = -\bar\chi\gamma_\mu\psi \\ \bar\psi\gamma_5\gamma_\mu\chi = \bar\chi\gamma_5\gamma_\mu\psi \quad \text{and} \quad \bar\psi\gamma_{\mu\nu}\chi = -\bar\chi\gamma_{\mu\nu}\chi. \end{array} \tag{3.54}$$

3.7.1 Complex and hermitean conjugation

We are now equipped to discuss complex conjugation of spinors. First of all, we must decide whether complex conjugation changes the order of fermions or not. In fact, both types of complex conjugation are possible. I introduce a new sign factor to reflect this choice, say β. Thus for bilinears

$$(\psi\chi)^* = \beta\psi^*\chi^* = -\beta\chi^*\psi^*\,, \qquad \beta = \pm 1\,. \tag{3.55}$$

Consider now the complex conjugation of $\bar\chi M\psi$, where M is some matrix in spinor space. With the previous equations one finds

$$\left(\bar\psi M\chi\right)^* = \beta(\bar\psi)^* M^*\chi^* = \beta(-\eta)^t\bar\psi_C\mathcal{B}M^*\mathcal{B}^{-1}\chi_C.$$

We introduce the $\mathcal{C}$-conjugated matrix

$$M_C \equiv \mathcal{B}M^*\mathcal{B}^{-1} = \mathcal{C}(\mathcal{A}M\mathcal{A}^{-1})^*\mathcal{C}^{-1} \tag{3.56}$$

such that

$$(\bar\psi M\chi)^* = \beta(-\eta)^t\,\bar\psi_C M_C\chi_C. \tag{3.57}$$

The $\mathcal{C}$-conjugation of the γ-matrices and of γ_* is

$$\gamma_\mu^C = (-)^t\eta\gamma_\mu \qquad \gamma_*^C = (-)^{d/2+s}\gamma_*, \tag{3.58}$$

the latter for even dimensions. We used (3.7) and (3.9) to prove these two identities.
Result: to perform complex conjugation of a bilinears in spinors, first add a factor $\beta(-\eta)^t$ and then replace fields and matrices by their C-conjugates, for which (3.58) gives the important relations.
Hermitean conjugate:
The hermitean conjugate is *defined as*

$$(\bar\psi\chi)^\dagger \equiv \chi^\dagger\bar\psi^\dagger.$$

Note that the Grassmann-property has not been used. Now it follows, that

$$(\bar\psi M\chi)^\dagger = \chi^\dagger M^\dagger \bar\psi^\dagger = -\bar\psi^* M^* \chi^* = -(-\eta)^t\, \bar\psi_C M_C \chi_C. \qquad (3.59)$$

As an example consider Minkowski spaces where $\eta = \epsilon = 1$, that is the dimensions $d = 2, 3, 4$ mod 8, containing 4, 10 and 11 dimensions. Then (3.59) implies that we can hermitean conjugate by using the $\mathcal{C}$ operation. Majorana spinors are thus considered as just real. Actually, in these dimensions we may choose the γ^μ to be purely imaginary and $C = -\gamma^0 \Rightarrow \mathcal{B} = \mathbb{1}$ so that *Majorana spinors are real in these Majorana representation* and that

$$(\bar\psi M\chi)^\dagger = \bar\psi_C M^* \chi_C. \qquad (3.60)$$

For Majorana spinors

$$\bar\psi\chi,\ \bar\psi\gamma^{\mu\nu}\chi,\ \ldots:\ \text{are hermitean}, \qquad \bar\psi\gamma^\mu\chi,\ \bar\psi\gamma^{\mu\nu\rho}\chi\ \ldots:\ \text{are antihermitean}. \qquad (3.61)$$

3.7.2 Fierz relation

We know that the γ^μ are $\Delta\times\Delta$ matrices with $\Delta = 2^{[d/2]}$, and that a basis of such matrices is given by the set

$$\{\mathbb{1}, \gamma_\mu, \gamma_{\mu_1\nu_2}, \ldots, \gamma_{\mu_1\ldots\mu_D}\}, \quad \text{where} \qquad \begin{cases} D = d & \text{for even } d \\ D = (d-1)/2 & \text{for odd } d\,, \end{cases} \qquad (3.62)$$

of which only the first has nonzero trace. Therefore every $\Delta\times\Delta$-matrix M can be expanded as

$$M = \sum_{p=0}^{D} \sum_{\mu_1,\ldots,\mu_p} a^{\mu_1\ldots\mu_p}\gamma_{\mu_1\ldots\mu_p}.$$

Since

$$\mathrm{Tr}\left(\gamma_{\mu_1\ldots\mu_p}\gamma^{\nu_1\ldots\nu_q}\right) = p!\delta_p^{\,q}\Delta\cdot\delta^{[\,\nu_1}_{[\mu_p}\cdots\delta^{\,\nu_p]}_{\mu_1]}$$

we find

$$\mathrm{Tr}\left(M\gamma^{\nu_1\ldots\nu_q}\right) = p!\,\Delta\sum_\mu a^{\mu_1\ldots\mu_q}\delta^{[\,\nu_1}_{[\mu_q}\cdots\delta^{\,\nu_q]}_{\mu_1]} = p!(-)^{q(q-1)/2}\Delta\, a^{\nu_1\ldots\nu_q}.$$

This implies the general Fierz formula for an arbitrary matrix M in spinor space:

$$M_\alpha^{\ \beta}\Delta = \sum_{p=0}^{[D]} \frac{(-)^{p(p-1)/2}}{p!}\left(\gamma_{\mu_1\ldots\mu_p}\right)_\alpha^{\ \beta}\ \mathrm{Tr}\left(\gamma^{\mu_1\ldots\mu_p}M\right). \qquad (3.63)$$

For two spinors ψ and χ whose components anticommute we have

$$M = \psi\bar\chi \quad \text{or} \quad M_\alpha^{\ \beta} = \psi_\alpha\bar\chi^\beta \Longrightarrow \mathrm{Tr}\left(\gamma^{\mu_1\ldots\mu_p}M\right) = -\bar\chi\gamma^{\mu_1\ldots\mu_p}\psi,$$

and the identity (3.63) implies

$$\Delta\psi\bar\chi = -\sum_p \frac{(-)^{p(p-1)/2}}{p!}\gamma_{\mu_1\ldots\mu_p}\left(\bar\chi\gamma^{\mu_1\ldots\mu_p}\psi\right). \tag{3.64}$$

For example, in 4 dimensions

$$4\psi\bar\chi = -(\bar\chi\psi) - \gamma_\mu(\bar\chi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\chi\gamma^{\mu\nu}\psi) + \tfrac{1}{3!}\gamma_{\mu\nu\rho}(\bar\chi\gamma^{\mu\nu\rho}\psi) - \tfrac{1}{4!}\gamma_{\mu\nu\rho\sigma}(\bar\chi\gamma^{\mu\nu\rho\sigma}\psi). \tag{3.65}$$

Now we use (3.10)

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_{\mu\nu\rho} = i\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma \quad \text{and} \quad \gamma_{\mu\nu\rho\sigma} = i\epsilon_{\mu\nu\rho\sigma}\gamma_5$$

and obtain

$$\begin{aligned} 4\psi\bar\chi &= -(\bar\chi\psi) - \gamma_\mu(\bar\chi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\chi\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar\chi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\chi\gamma_5\psi) \\ 4\psi\bar\chi - 4\chi\bar\psi &= -2\gamma_\mu(\bar\chi\gamma^\mu\psi) + \gamma_{\mu\nu}(\bar\chi\gamma^{\mu\nu}\psi). \end{aligned} \tag{3.66}$$

In particular, we find

$$\begin{aligned} 4\psi\bar\psi &= -(\bar\psi\psi) - \gamma_\mu(\bar\psi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar\psi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\psi\gamma_5\psi) \\ 4\gamma_5\psi\bar\psi\gamma_5 &= -(\bar\psi\psi) + \gamma_\mu(\bar\psi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) - \gamma_5\gamma_\mu(\bar\psi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\psi\gamma_5\psi) \\ \psi\bar\psi + \gamma_5\psi\bar\psi\gamma_5 &= -\tfrac{1}{2}(\bar\psi\psi) + \tfrac{1}{4}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) - \tfrac{1}{2}\gamma_5(\bar\psi\gamma_5\psi). \end{aligned}$$

For *Majorana spinors* the second last term vanishes because of (3.54), and multiplying with ψ from the right, we find that

$$\psi(\bar\psi\psi) + \gamma_5\psi(\bar\psi\gamma_5\psi) = 0. \tag{3.67}$$

More useful Fierz identities can be found in [22].

3.7.3 Raising and lowering spinor indices

We raise and lower spinor indices in a way that is independent of the dimension of spacetime. First of all, spinors get a lower spinor index. So for a spinor ψ, we shall write ψ_α. Gamma matrices act on these spinors, and therefore the expression $\gamma_\mu\psi$ becomes $(\gamma_\mu)_\alpha{}^\beta\psi_\beta$. Note the position of the spinor indices on a usual Gamma matrix. This is then the same for products of Gamma matrices. Now we wish to raise and lower indices. To discuss symmetries of Gamma matrices, the $(\gamma_\mu)_\alpha{}^\beta$ can not be used, as we can not interchange upper and lower indices. To that purpose, we introduce matrices $C^{\alpha\beta}$ and $C_{\alpha\beta}$, which will be related to the charge conjugation matrix below. We raise and lower indices always in the Northwest–Southeast (NW-SE) convention. That means that the contraction indices should appear in that relative position, i.e.

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta\,, \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha}\,. \tag{3.68}$$

In order for these two equations to be consistent, we should have

$$C^{\alpha\beta}C_{\gamma\beta} = \delta^{\alpha}_{\gamma}\,, \qquad C_{\beta\alpha}C^{\beta\gamma} = \delta^{\gamma}_{\alpha}\,. \tag{3.69}$$

We choose the identifications such that the Majorana conjugate $\bar{\psi}_C = \psi^T\mathcal{C}^*$ is written as ψ^α. Comparing with (3.68), we conclude that $(C^{\alpha\beta})$ is $\mathcal{C}^{*T} = \mathcal{C}^{-1}$, and $(C_{\alpha\beta})$ is then $\mathcal{C}^T$. One may check that the symmetries of $\mathcal{C}^{-1}\gamma^{(n)}$ or $\gamma^{(n)}\mathcal{C}$, which were discussed before, is now the symmetry of $(C^{-1}\gamma_\mu)^{\alpha\beta} \equiv (\gamma_\mu)^{\alpha\beta}$ or $(\gamma_\mu)_{\alpha\beta}$.

3.8 Appendix A: $SL(2,\mathbb{C})$ representations

Every finite-dimensional representation of $SL(2,\mathbb{C})$ can be gotten by tensoring two fundamental 2-dimensional representations. The knowledge of the representation theory of $SL(2,\mathbb{C})$ is necessary to construct Lorentz-invariant field equations.
The components ζ_A and $\zeta_{\dot{A}}$ (with lower indices) are called covariant. The two representations of $A \in SL(2,\mathbb{C})$ are

$$\zeta_A \to A_A^{\ B}\zeta_B \quad \text{and} \quad \zeta_{\dot{A}} \to \bar{A}_{\dot{A}}^{\ \dot{B}}\zeta_{\dot{B}}. \tag{3.70}$$

The representations spaces carry natural symplectic structures:

$$(\chi,\zeta) = -(\zeta,\chi) = \chi_A\varepsilon^{AB}\zeta_B = \chi^T\varepsilon\zeta, \qquad \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \varepsilon \tag{3.71}$$

and similarly for the dotted spinors. For every 2×2 matrix

$$A^T\varepsilon A = \det(A)\cdot\varepsilon,$$

and if $A \in SL(2,\mathbb{C})$, we have

$$(A\chi, A\zeta) = (A\chi)^T\varepsilon(A\zeta) = \chi^T A^T\varepsilon A\zeta = \chi^T\varepsilon\zeta = (\chi,\zeta).$$

Hence, the skew-symmetric products are invariant under $SL(2,\mathbb{C})$ transformations:

$$(A\chi, A\zeta) = (\chi,\zeta) \quad \text{and} \quad (\bar{A}\chi, \bar{A}\zeta) = (\chi,\zeta). \tag{3.72}$$

Indices are raised and lowered with ε:

$$\zeta^A = \varepsilon^{AB}\zeta_B \quad \text{or} \quad \zeta_A = \zeta^B\varepsilon_{BA}.$$

Let us see, how $\zeta^{\dot{A}}$ transforms under $SL(2,\mathbb{C})$:

$$\tilde{\zeta}^{\dot{C}}\varepsilon_{\dot{C}\dot{A}} = \tilde{\zeta}_{\dot{A}} = \bar{A}_{\dot{A}}^{\ \dot{B}}\zeta_{\dot{B}} = \bar{A}_{\dot{A}}^{\ \dot{B}}\chi^{\dot{D}}\varepsilon_{\dot{D}\dot{B}}.$$

Hence, the contravariant indices transform with $-\varepsilon\bar{A}\varepsilon$. Since

$$\bar{A}^T\varepsilon\bar{A} = \varepsilon \Rightarrow \varepsilon\bar{A}\varepsilon = (\bar{A}^T)^{-1} = (A^\dagger)^{-1}.$$

We see, that $\zeta^{\dot{A}}$ transform as a *2 anti-spinor field*. To summarize:

spinor	ζ_A	$\zeta^A = \epsilon^{AB}\zeta_B$	$\zeta_{\dot{A}}$	$\zeta^{\dot{A}} = \zeta^{\dot{B}}\epsilon^{\dot{B}\dot{A}}$
transforms with	A	$\varepsilon A \varepsilon^{-1} = (A^T)^{-1}$	$\bar{A}$	$\varepsilon \bar{A} \varepsilon^{-1} = (A^\dagger)^{-1}$

Of course, the representations $(A^\dagger)^{-1}$ and $\bar{A}$ are equivalent via ε. Notation:

$$\text{representation } D^{(\frac{1}{2},0)}(A): \quad \zeta \to A\zeta$$
$$\text{representation } D^{(0,\frac{1}{2})}(A): \quad \zeta \to \bar{A}\zeta.$$

The tensor product of n un-dotted and m dotted spinors yield spinors of rank $n+m$ with n un-dotted and m dotted indices. Such a spinor transforms as follows under $SL(2,\mathbb{C})$ transformations:

$$\zeta_{A_1\ldots A_n\dot{A}_1\ldots\dot{A}_m} \longrightarrow A_{A_1}^{\ B_1}\cdots A_{A_n}^{\ B_n}\bar{A}_{\dot{A}_1}^{\ \dot{B}_1}\cdots A_{\dot{A}_m}^{\ \dot{B}_m}\,\zeta_{B_1\ldots B_n\dot{B}_1\ldots\dot{B}_m}.$$

These representations are reducible if $(n,m) \neq (\frac{1}{2},0)$ or $\neq (0,\frac{1}{2})$.

3.9 Appendix B: Representations of spin and Lorentz groups

The fields furnish a representation of the homogeneous Lorentz groups $SO(1,d-1)$ or, strictly speaking, of their universal covering groups spin$(1,d-1)$. Assuming that they transform according to a finite dimensional representation of spin$(1,d-1)$, all these representation can be obtained from the finite-dimensional unitary representation of the corresponding orthogonal group spin(d) by setting $\tau = ix^0$.

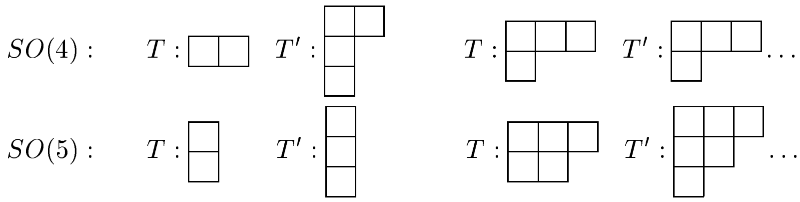
Young-tableau: Every irreducible representation of $O(n)$ can be obtained from the tensor product of the defining representation

$$\begin{array}{cclcl} T_\mu & \longrightarrow & R_\mu^{\ \nu}T_\nu, & R \in O(n) & \text{defining repr.} \\ T_{\mu_1\ldots\mu_r} & \longrightarrow & R_{\mu_1}^{\ \nu_1}\ldots R_{\mu_r}^{\ \nu_r}T_{\nu_1\ldots\nu_r} & & \text{tensor product repr.} \end{array}$$

by suitable symmetrization and anti-symmetrization in groups of indices. They are in one-to-one corresponding to admissible Young tableaus. Each tableaux T contains at most $n/2$ rows for even n and $(n-1)/2$ rows for odd n. If l_i is the number of boxes of the i'th row, then

$$l_1 \geq l_2 \geq \cdots \geq l_{[d/2]}.$$

To get all irreducible representations of $O(n)$ we associate a diagram T' to T as follows: The length a of the first column in T is less or equal to $n/2$, $a \leq n/2$, the length of the first column in T' is $n-a$, and all other columns in T and T' have the same length. Examples



When n is even and $a = n/2$ then the diagrams T and T' coincide (they transform differently under improper transformations) and T is said to be self-associate. If we restrict ourselves to the proper subgroup $SO(n)$ with determinant one, the representations corresponding to the associate diagrams T and T' become equivalent. For even n the representation corresponding to a selfassociate pattern T splits into two non-equivalent irreducible representations: the selfdual and antiselfdual antisymmetric tensors of rank $n/2$,

$$T_{\mu_1\dots\mu_{n/2}} = \pm\frac{1}{(n/2)!}\epsilon_{\mu_1\dots\mu_{n/2}\nu_1\dots\nu_{n/2}}T_{\nu_1\dots\nu_{n/2}}, \qquad T \quad \text{totally skew symmetric.}$$

Roots, weights and all that

Our Lie algebra conventions are as follows: Let $H_k,\ k=1,\dots,r$ be an orthogonal basis of the Cartan subalgebra $\mathcal{H}$, which are diagonal in a given representation

$$H_k|\mu\rangle = \mu_k|\mu\rangle \quad \text{and} \quad [H_k, E_\alpha] = \alpha_k E_\alpha. \tag{3.73}$$

Let

$$\alpha_{(i)}\ , \quad \mu_{(i)}\ , \quad \alpha^\vee_{(i)} = \frac{2\alpha_{(i)}}{\big(\alpha_{(i)},\alpha_{(i)}\big)} \quad \text{and} \quad \mu^\vee_{(i)} = \frac{2\mu_{(i)}}{\big(\alpha_{(i)},\alpha_{(i)}\big)}\ , \qquad i=1,\dots,r \tag{3.74}$$

be the simple roots, fundamental weights, co-roots and co-weights, respectively:

$$\big(\alpha_{(i)},\alpha^\vee_{(j)}\big) = K_{ij}, \quad \big(\alpha^\vee_{(i)},\mu_{(j)}\big) = \big(\alpha_{(i)},\mu^\vee_{(j)}\big) = \delta_{ij}, \quad \big(\mu_{(i)},\mu^\vee_{(j)}\big) = (K^{-1})_{ij}. \tag{3.75}$$

We used that the simple roots and fundamental weights are related by the Cartan matrix,

$$\alpha_{(i)} = \sum_{j=1}^r K_{ij}\ \mu_{(j)}. \tag{3.76}$$

The fundamental weight-states (which are the highest weight states of the r fundamental representations) and states in the adjoint representation obey

$$\alpha^\vee_{(i)}\cdot H\ |\mu_{(j)}\rangle = \delta_{ij}|\mu_{(j)}\rangle \quad \text{and} \quad \mu^\vee_{(i)}\cdot H\ |\alpha_{(j)}\rangle = \delta_{ij}|\alpha_{(j)}\rangle. \tag{3.77}$$

To determine the dimension of the representation with highest weight μ we may use Weyl's formula

$$\dim V_\mu = \prod_{\alpha>0}\frac{\langle\mu+\delta,\alpha\rangle}{\langle\delta,\alpha\rangle}, \tag{3.78}$$

where the Weyl vector δ is half the sum over all positive roots or the sum over all fundamental weights,

$$\delta = \tfrac{1}{2}\sum_{\alpha>0}\alpha = \sum\mu_{(i)} \tag{3.79}$$

A positive root and highest weight is a linear combination of the simple roots and fundamental weights with non-negative integer coefficients, respectively,

$$\alpha = \sum_i m_i \alpha_{(i)} \quad \text{and} \quad \mu = \sum_i k_i \mu_{(i)}, \qquad m_i, k_i \in \mathbb{N}_0. \tag{3.80}$$

Inserting this into Weyl's formula and using $(\mu_{(i)}, \alpha_{(j)}) = \delta_{ij} {\alpha_{(j)}}^2/2$ yields

$$\dim V_\mu = \prod_{\alpha>0} \frac{\sum_i (1+k_i) m_i {\alpha_{(i)}}^2}{\sum_i m_i {\alpha_{(i)}}^2}. \tag{3.81}$$

To find the center elements we find conditions on $\rho \cdot H \in \mathcal{H}$ such that $\exp(2\pi i \rho \cdot H)$ is in the center $\mathcal{Z}$. Center elements are the identity in the adjoint representation. Because of the second set of equations in (3.77) they must be powers of

$$z_i = \exp\left(2\pi i \mu^\vee_{(i)} \cdot H\right) \Longrightarrow z_i|\mu\rangle = \exp\left(2\pi i (kK^{-1})_i\right)|\mu\rangle, \qquad \mu = \sum_i k_i \mu_{(i)}. \tag{3.82}$$

In an irreducible representation a center element acts the same way on all states. Hence, a necessary and sufficient condition for $z_i \neq \mathbb{1}$ is that

$$z_i\, |\mu_{(j)}\rangle = \exp\left(2\pi i K^{-1}_{ji}\right)|\mu_{(j)}\rangle \neq |\mu_{(j)}\rangle, \quad \text{or that} \quad K^{-1}_{ji} \notin \mathbb{Z} \tag{3.83}$$

for at least one fundamental weight $\mu_{(j)}$. Here we have used that the inner products of the weights with the co-weights yield the inverse Cartan matrix, see (3.75). The order of the center group is just $\det(K)$.

The spin groups

Let us now turn to the spin groups, i.e. the universal (double) covering groups of the $SO(n)$-groups[3]. In the Cartan classification spin$(2n) \sim D_n$ and spin$(2n+1) \sim B_n$:

group	dim	fundamental reps
spin(4)= $D_2 \sim A_1 \times A_1$	6	$2, 2$
spin(5)= $B_2 \sim C_2$	10	$5, 4$
spin(6)= $D_3 \sim A_3$	15	$6, 4, 4$
spin(7)= B_3	21	$7, 21, 8$
spin(8)= D_4	28	$8, 28, 8, 8$
spin(9)= B_4	36	$9, 36, 84, 16$
spin(10)= D_5	45	$10, 45, 120, 16, 16$

The Dynkin diagrams of the spin groups are depicted in the following figure: The representation with highest weight $\mu_{(1)}$ is always the defining representation of $SO(n)$. The representation with

[3] we use the conventions as implemented by LIE 2.1

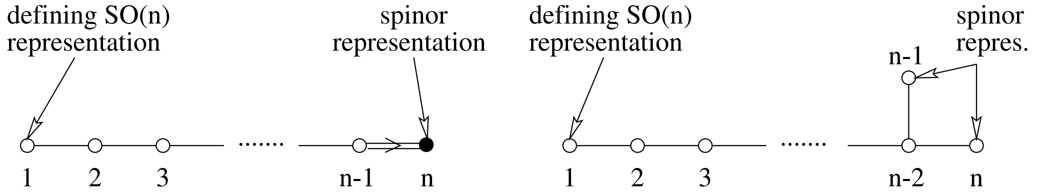


Figure 3.1: *Dynkin diagrams of the B_n series (left) and D_n series (right). $\bullet$ is a short root.*

highest weight $\mu_{(n)}$ is a spinor representation for all groups. For the D_n the second last representation belonging to $\mu_{(n-1)}$ is the second spinor representation. The Cartan matrices of the B_n and D_n groups have the form

$$K_B = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}, \quad K_D = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

The centers and their generators are

group	center	generators	action of center element(s)
D_n, n even	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$z_1 = e^{2\pi i \mu^\vee_{(1)}}$ $z_2 = e^{2\pi i \mu^\vee_{(n)}}$	$z_1 \vert\mu_{(j)}\rangle = +\vert\mu_{(j)}\rangle, \quad j \le n-2$ $z_1 \vert\mu_{(j)}\rangle = -\vert\mu_{(j)}\rangle, \quad j \ge n-1$ $z_2 \vert\mu_{(j)}\rangle = (-)^j \vert\mu_{(j)}\rangle, \quad j \le n-2$ $z_2 \vert\mu_{(n-1)}\rangle = -(-)^{n/2} \vert\mu_{(n-1)}\rangle$ $z_2 \vert\mu_{(n)}\rangle = (-)^{n/2} \vert\mu_{(n)}\rangle$
D_n, n odd	$\mathbb{Z}_4$	$z = e^{2\pi i \mu^\vee_{(n)}}$	$z \vert\mu_{(j)}\rangle = (-)^j \vert\mu_{(j)}\rangle, \quad j \le n-2$ $z \vert\mu_{(n-1)}\rangle = -i^n \vert\mu_{(n-1)}\rangle$ $z \vert\mu_{(n)}\rangle = i^n \vert\mu_{(n)}\rangle$
B_n	$\mathbb{Z}_2$	$z = e^{2\pi i \mu^\vee_{(1)}}$	$z \vert\mu_{(j)}\rangle = \vert\mu_{(j)}\rangle, \quad j \le n-1$ $z \vert\mu_{(n)}\rangle = -\vert\mu_{(n)}\rangle$

The group spin$(5) \sim$ B$_2$

This group is related to the isometry groups of the 5-dimensional de Sitter and anti-de Sitter spaces and to the conformal group of the 4-dimensional flat spaces (with different signatures). Therefore we shall discuss the corresponding representations in some detail. The discussion for the higher spin groups is very similar. The Cartan matrix of spin(5) and its inverse are

$$K = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad K^{-1} = \tfrac{1}{2} \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

The center element $z = \exp(\pi i \mu_1^\vee)$ is the identity on the representation with highest weight μ_1 which is the defining representation of $SO(5)$. It is $-\mathbb{1}$ on the spinor representation with highest weight μ_2. Hence, a representation with highest weight $\mu = k_1\mu_1 + k_2\mu_2$ is a faithful representation of

$$SO(5) = \mathrm{spin}(5)/\mathbb{Z}_2 = B_2/\mathbb{Z}_2$$

if and only if

$$z|\mu\rangle = |\mu\rangle \Longleftrightarrow k_2 \in 2\mathbb{N}_0.$$

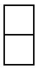
The positive roots are

$$\text{short roots:}\quad m = (0,1),(1,1) \qquad \text{long roots:}\quad m = (1,0),(1,2)$$

Using Weyl's formula we find that the dimension of the representation with highest weight μ is

$$\dim V_\mu = \frac{1}{6}(1+k_1)(1+k_2)(3+2k_1+k_2)(2+k_1+k_2).$$

The quantum numbers can easily be related to the length (l_1, l_2) of the rows in the Young tableau. We need

 : 5 and  : 10 .

The 5 and 10-dimensional representation have

$$5: \quad l = (1,0), k = (1,0) \quad \text{and} \quad 10: \quad l = (1,1), k = (0,2).$$

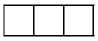
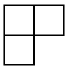
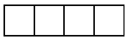
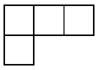
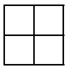
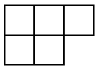
From this follows that

$$l_1 = k_1 + k_2/2 \quad \text{and} \quad l_2 = k_2/2$$

In terms of the lengths l_1, l_2 of the Young tableaux we obtain the following formula for the dimensions for the faithful $SO(5)$-representations

$$\mathrm{dim} V_l = (l_1 - l_2 + 1)(2l_2 + 1)(l_1 + l_2 + 2)(2l_1 + 3)/6.$$

Examples:

 : 14  : 35  : 55  : 91  : 35  : 105.

The group spin(6) $\sim$ D$_3$
The simple laced group D_3 has Cartan matrix

$$K = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad K^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

The center $\mathbb{Z}_4$ is generated by

$$z = \exp\left(2\pi i \mu_{(3)}^\vee\right) \quad \text{with} \quad z|\mu\rangle = \exp\left(\frac{i\pi}{2}(2k_1 + k_2 + 3k_3)\right)|\mu\rangle.$$

Since on the defining representation of $SO(6)$ it is $-\mathbb{1}$, a representation of spin(6) is a faithful representation of

$$SO(6) = \mathrm{spin}(6)/\mathbb{Z}_2 = D_3/\mathbb{Z}_2$$

if $z = \pm\mathbb{1}$ on this representation. This is equivalent to

$$2k_1 + k_2 + 3k_3 \in 2\mathbb{N}_0 \Longleftrightarrow k_2 + k_3 \in 2\mathbb{N}_0.$$

The positive roots are

$$m = (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (1,1,1)$$

and Weyl's formula yields

$$\dim V_\mu = \frac{1}{6}(1+k_1)(1+k_2)(1+k_2)(2+k_1+k_2)(2+k_1+k_3)(3+k_1+k_2+k_3).$$

The lowest representations of $SO(6)$ are

$$6 : \begin{array}{|c|}\hline \\ \hline \end{array} \quad 15 : \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \quad \text{and} \quad 20 : \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}.$$

Using LIE it easy to relate the highest weights to these representations (recall, that the 20 splits into $10_S \oplus 10_A$).

dim.	l	k
5	(1,0,0)	(1,0,0)
15	(1,1,0)	(0,1,1)
20	l=(1,1,1)	(0,2,0),(0,0,2)

The group spin(7) $\sim$ $\mathbf{B}_3$
A representation of spin(7) is a faithful representation of $SO(7)$ if k_3 is an even integer. Since B_3 has no selfassociate diagrams it is easy to relate the lengths of the rows in a Young-tableau to a highest weight:

$$k_1 = l_1 - l_2 \quad k_2 = l_2 - l_3 \quad \text{and} \quad k_3 = 2l_3.$$

The dimensions of the various representations and the corresponding results for the simples spin-groups are listed in the following table

group	dimension
spin(4)	$(k_1+1)(k_2+1)$
SO(5)	$(l_1-l_2+1)(2l_2+1)(l_1+l_2+2)(2l_1+3)/6$
spin(6)	$(1+k_1)(1+k_2)(1+k_2)$
	$\cdot(2+k_1+k_2)(2+k_1+k_3)(3+k_1+k_2+k_3)/6$
SO(7)	$(2l_1+5)(2l_2+3)(2l_3+1)(l_1+l_2+4)(l_1-l_2+1)$
	$\cdot(l_1-l_3+2)(l_1+l_3+3)(l_2+l_3+2)(l_2-l_3+1)/720$

4 Symmetries

Supersymmetry is related to the structure of spacetime. Hence I will first review the well-known spacetime symmetries. A dynamical system is identified by its action integral. In a d-dimensional local field theory the action S is the space-time integral of a local Lagrangian density $\mathcal{L}(x)$,

$$S = \int d^d x \; \mathcal{L}(x),$$

where $\mathcal{L}$ is a function of the fields and their first space-time derivatives. The action will be invariant under *Poincaré transformations*,

$$\tilde{x} = \Lambda(A)x + a, \qquad \Lambda^T \eta \Lambda = \eta, \quad \eta = (\eta_{\mu\nu}) = \mathrm{diag}(1, -1, \dots, -1), \qquad a \in \mathbb{R}^d$$

under which the field transforms as

$$\phi(\tilde{x}) = S(A)\phi(x) \quad \text{or} \quad \phi(x) \longrightarrow S(A)\phi\big(\Lambda(A)^{-1}(x-a)\big),$$

where $S(A), \Lambda(A)$ are representations of the spin group, i.e. the universal covering of $SO(1, d-1)$. The action will also be invariant under *global gauge transformations*. The field transforms as

$$\phi \longrightarrow U\phi \sim \phi + iX\phi = \phi + \delta_X \phi, \qquad X^\dagger = X.$$

4.1 Noether theorem and conserved charges

According to a theorem of Emmy Noether, to each parameter of the symmetry group there corresponds a conserved current. For the global gauge transformations the Lagrangian density itself is invariant and

$$\delta_X \mathcal{L} = 0 = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \, \partial_\mu \big(\delta_X \phi\big) + \frac{\delta \mathcal{L}}{\delta \phi} \delta_X \phi = \partial_\mu \Big(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta_X \phi \Big), \tag{4.1}$$

where we used the equation of motion in the last step. Hence, the conserved Noether current reads

$$J^\mu_X = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta_X \phi, \qquad \partial_\mu J^\mu_X = 0.$$

This way we find the conserved charges

$$\dot{Q}_X = 0 \quad \text{with} \quad Q_X = \int\limits_{x^0} d^{d-1}x \; J^0_X = \int\limits_{x^0} d^{d-1}x \; \pi(x)\, \delta_X \phi(x),$$

where we introduced the momentum density conjugate to the field,

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}.$$

The fundamental Poisson bracket between the momentum density and field ϕ is

$$\{\phi(x), \pi(y)\}_{x^0=y^0} = \delta(\vec{x} - \vec{y}).$$

It follows, that

$$\{\phi(x), Q_X\} = \int d^{d-1}y \{\phi(x), \pi(y)\delta_X\phi(y)\} = \delta_X\phi(x),$$

where we assumed, that $\delta_X\phi$ does not contain time-derivatives of ϕ.

4.1.1 $U(1)$ gauge transformations

As an example consider the Lagrangian of a charged scalar field,

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{4.2}$$

It is invariant under the simple $U(1)$ phase transformations

$$\phi \longrightarrow e^{i\lambda}\phi \sim \phi + i\lambda\phi \Longrightarrow \delta_\lambda\phi = i\lambda\phi, \quad \delta_\lambda\phi^\dagger = -i\lambda\phi^\dagger.$$

The Noether current is

$$J_\mu(\lambda) = \lambda j_\mu, \qquad j_\mu = i\big(\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi\big).$$

The conserved charge is the electric charge,

$$Q = i\int_{x^0} \big(\dot{\phi}^\dagger\phi - \phi^\dagger\dot{\phi}\big) = i\int_{x^0} \big(\pi_\phi\phi - \pi_{\phi^\dagger}\phi^\dagger\big). \tag{4.3}$$

It generates the gauge transformations,

$$\{\phi, Q\} = i\phi \quad \text{and} \quad \{\phi^\dagger, Q\} = -i\phi^\dagger.$$

4.1.2 The energy-momentum as Noether current of translations

For the d space-time translations

$$\phi(x+a) \sim \phi(x) + a^\mu\partial_\mu\phi(x) = \phi(x) + \delta_a\phi(x),$$

the action is invariant but the Lagrangian not. It is (off-shell) invariant only up to a total divergence,

$$\delta_a\mathcal{L} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\big(\delta_a\phi\big) + \frac{\delta\mathcal{L}}{\delta\phi}\delta_a\phi = a^\nu\partial_\nu\mathcal{L},$$

and the on-shell identity (4.1) is modified to

$$\delta_a \mathcal{L} = a^\nu \partial_\nu \mathcal{L} = \partial_\mu \Big(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} a^\nu \partial_\nu \phi \Big), \quad \text{or} \quad \partial_\mu J^\mu_a = 0 \tag{4.4}$$

with Noether current

$$J^\mu_a = a^\nu T^\mu_{\ \nu}, \qquad T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L}.$$

The conserved quantities are the energy and momentum of the field,

$$P^\mu = \int_{x^0} T^{0\mu}, \qquad \dot{P}^\mu = 0.$$

In particular,

$$P^0 \equiv H = \int_{x^0} \big(\pi \dot{\phi} - \mathcal{L}\big), \quad P^i = \int_{x^0} \pi \, \partial^i \phi,$$

such that

$$\{\phi, P_\mu\} = \partial_\mu \phi.$$

For the scalar field theory with Lagrangian (4.2) we find

$$T_{\mu\nu} = \partial_\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial_\mu \phi - \eta_{\mu\nu} \mathcal{L}$$

which leads to the conserved energy

$$H = \int \Big(\dot{\phi}^\dagger \dot{\phi} + \nabla \phi^\dagger \nabla \phi + m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \Big)$$

and conserved momentum

$$P_i = \int \Big(\dot{\phi}^\dagger \partial_i \phi + \partial_i \phi^\dagger \dot{\phi} \Big).$$

In electrodynamics we introduce the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A^\mu = (A^0, \vec{A})$, whose entries are the components of the electric and magnetic fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

The Lagrangian density reads

$$\mathcal{L} = \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\tfrac{1}{4} F^{\mu\nu} \big(\partial_\mu A_\nu - \partial_\nu A_\mu \big) \Longrightarrow \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)} = -F^{\mu\rho}$$

and we find

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)} \partial_\nu A_\rho - \eta_{\mu\nu} \mathcal{L} = -F_\mu^{\ \rho} \partial_\nu A_\rho + \tfrac{1}{4} \eta_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}.$$

This leads to the conserved energy-momentum

$$P_\nu = \int T^0_{\ \nu} = \int \Big(- F^{0\rho}\partial_\nu A_\rho - \tfrac{1}{2}\delta^0_{\ \nu}(\vec{E}^2 - \vec{B}^2)\Big)$$

or in components

$$H = \frac{1}{2}\int \Big(E_i(2\dot{A}_i - E_i) + \vec{B}^2\Big)$$

and

$$P_i = \int \big(E_j\partial_i A_j\big).$$

4.1.3 Improving Noether currents

Let $\mathcal{L}$ be invariant under $\phi \to \phi + \delta\phi$, up to a total divergence,

$$\delta\mathcal{L} = \partial_\mu V^\mu.$$

Then

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - V^\mu$$

is conserved. Note that the vector field V^μ is determined only up to terms $\partial_\nu A^{\mu\nu}$ with $A_{\mu\nu} = -A_{\nu\mu}$ and hence J is not unique,

$$J^\mu \longrightarrow J^\mu - \partial_\nu A^{\mu\nu}. \tag{4.5}$$

For localized fields the conserved charges are not affected by this ambiguity for the Noether current since

$$J^0 \to J^0 - \partial_i A^{0i} \quad \text{and hence} \quad Q = \int_{x^0} J^0 \to Q.$$

The possible improvement terms have been studied widely in the literature. For example, the energy-momentum tensor $T^{\mu\nu}$ in electrodynamics in not gauge invariant. The conserved energy momentum P^μ is gauge invariant, since $\partial_\rho F^{0\rho} = 0$. One can improve the Noether current $T^{\mu\nu}$ with $-\partial_\rho(F^{\mu\rho}A^\nu)$ such that it becomes gauge invariant. The improved energy-momentum tensor takes the well-known gauge invariant form

$$T_{\mu\nu} = \tfrac{1}{2}F_\mu^{\ \rho}F_{\rho\nu} + \tfrac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \Longrightarrow H = \tfrac{1}{2}\int(\vec{E}^2 + \vec{B}^2), \quad \vec{P} = \int \vec{E}\wedge\vec{B}.$$

4.1.4 The conserved angular momentum

Deriving the conserved quantities for the Lorentz transformations is a bit trickier. A field transforms as (see (3.15))

$$\phi(x) \longrightarrow e^{\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}}\phi(e^{-\omega}x) \sim \phi(x) + \tfrac{i}{2}\omega^{\mu\nu}J_{\mu\nu}\phi(x) = \phi(x) + \delta_\omega\phi(x),$$

where

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} = \frac{1}{i}\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big) + S_{\mu\nu}.$$

The $M_{\mu\nu}$ commute with the spin operators $S_{\mu\nu}$ and both sets satisfy the Lorentz algebra, so that

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\big(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}\big). \tag{4.6}$$

Again the Lagrangian is off-shell only invariant up to a total divergence,

$$\delta_\omega\mathcal{L} = \tfrac{i}{2}\,(\omega, M)\mathcal{L} = \tfrac{1}{2}\omega^{\rho\sigma}\partial_\mu\Big(\big(\delta^\mu_{\ \sigma}x_\rho - \delta^\mu_{\ \rho}x_\sigma\big)\mathcal{L}\Big).$$

Since

$$\partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_\omega\phi\Big) = \tfrac{1}{2}\omega^{\rho\sigma}\partial_\mu\Big\{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}(x_\rho\partial_\sigma - x_\sigma\partial_\rho + iS_{\rho\sigma})\phi\Big\}$$

the Noether identity becomes

$$\partial_\mu\Big\{\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\sigma\phi - \delta^\mu_{\ \sigma}\phi\Big)x_\rho - \Big(\rho\leftrightarrow\rho\Big) + i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}S_{\rho\sigma}\phi\Big\} = 0.$$

The Noether current is

$$J^\mu_\omega = \omega_{\rho\sigma}M^{\mu\rho\sigma}, \qquad M^{\mu\rho\sigma} = \tfrac{1}{2}T^{\mu\sigma}x^\rho - \tfrac{1}{2}T^{\mu\rho}x^\sigma + \tfrac{i}{2}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}S^{\rho\sigma}\phi = -M^{\mu\sigma\rho} \tag{4.7}$$

and the conserved charges are

$$J^{\rho\sigma} = \int_{x^0} M^{0\rho\sigma}.$$

In components,

$$J^{i0} = \tfrac{1}{2}\int\big(\mathcal{H}x^i - \pi\partial^i\phi x^0 - i\pi S^{0i}\phi\big), \quad J^{ij} = \tfrac{1}{2}\int\pi\big(\partial^j\phi x^i - \partial^i\phi x^j + iS^{ij}\phi\big).$$

A simple calculation reveals, that

$$\{\phi, J^{\mu\nu}\} = \tfrac{i}{2}\big(L^{\mu\nu} + S^{\mu\nu}\big)\phi.$$

For example, for electrodynamics

$$M^{\mu\rho\sigma} = \big(\tfrac{1}{4}\eta^{\mu\rho}x^\sigma F^2 - F^{\mu\xi}x^\sigma\partial^\rho A_\xi - F^{\mu\rho}A^\sigma\big) - \big(\rho\leftrightarrow\sigma\big).$$

When one couples matter to gravity one needs a conserved, symmetric and gauge invariant energy-momentum tensor $T_{\mu\nu}$. The tensor gotten by the Noether procedure is not always symmetric and needs to be improved. To get a symmetric tensor one may couple matter to gravity and define $T^{\mu\nu}$ as response of the action to a variation of the metric. For diffeomorphism-invariant bosonic theories one obtains this was a symmetric and conserved energy momentum tensor. It is equal to the Noether current plus an improvement term, which does not alter the conserved quantities.

For theories containing fermions the situation is a bit trickier. Again the Noether current is not necessarily symmetric. To couple fermions to gravity one needs to introduce the d-bein. When one varies the action with respect to the vielbein one gets a conserved but not necessarily symmetric energy momentum tensor. Further improvements are necessary.

4.2 Conformal symmetry

Under conformal transformations

$$x^\mu \to y^\mu = y^\mu(x^\nu) \tag{4.8}$$

the metric tensor is unchanged up to a factor e^φ,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\sigma}dy^\rho dy^\sigma = e^{\varphi(y)}\eta_{\rho\sigma}dy^\rho dy^\sigma. \tag{4.9}$$

These transformation leave the light cone invariant and spacelike x maybe mapped into timelike x. The apparent problem with causality has been solved by Lüscher and Mack [23].

4.2.1 The conformal algebra

Lets look at the infinitesimal conformal transformations,

$$y^\mu \sim x^\mu + X^\mu(x) \quad \text{and} \quad e^\varphi \sim 1+\delta\varphi$$

such that

$$\big(\eta_{\mu\nu}+X_{\mu,\nu}+X_{\nu,\mu}\big)\big(1+\delta\varphi\big)dx^\mu dx^\nu \sim \eta_{\mu\nu}dx^\mu dx^\nu \quad \text{or} \quad \delta\varphi\eta_{\mu\nu}+X_{\mu,\nu}+X_{\nu,\mu}=0.$$

Taking the trace we find

$$\delta\varphi = -\frac{2}{d}\,\partial_\rho X^\rho \eta_{\mu\nu} \longrightarrow X_{\mu,\nu}+X_{\nu,\mu} = \frac{2}{d}\eta_{\mu\nu}\partial_\rho X^\rho. \tag{4.10}$$

The last equation is the conformal Killing equation so that conformal coordinate transformations are generated by *conformal Killing fields.*
In 3 and more dimensions there $\frac{1}{2}(d+1)(d+2)$ conformal Killing fields. These are

$$\begin{array}{rlcl}
d \text{ translations} & y^\mu = x^\mu + a^\mu & \Longrightarrow & X^\mu = a^\mu \\
\frac{1}{2}d(d-1) \text{ Lorentztransf.} & y^\mu = \Lambda^\mu{}_\nu x^\nu & \Longrightarrow & X^\mu = \omega^\mu{}_\nu x^\nu \\
1 \text{ dilatation} & y^\mu = \lambda x^\mu & \Longrightarrow & X^\mu = \lambda x^\mu \\
d \text{ special conf. transf.} & y^\mu = \dfrac{x^\mu - x^2 c^\mu}{1-2c\cdot x + c^2x^2} & \Longrightarrow & X^\mu = 2(c\cdot x)x^\mu - x^2c^\mu.
\end{array} \tag{4.11}$$

The corresponding hermitean generators are

$$iX^\mu\partial_\mu = \big\{a^\mu P_\mu, 2\omega^{\mu\nu}M_{\mu\nu}, \lambda D, c^\mu K_\mu\big\} \tag{4.12}$$

where

$$P_\mu = i\partial_\mu, \quad M_{\mu\nu} = \frac{1}{i}(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad D = ix^\mu\partial_\mu \quad \text{and} \quad K_\mu = 2ix_\mu x^\rho\partial_\rho - ix^2\partial_\mu. \tag{4.13}$$

The conformal algebra is

$$
\begin{array}{rcl}
[P_\mu, P_\nu] &=& [K_\mu, K_\nu] = [D, D] = 0 \quad , \quad [P_\mu, D] = iP_\mu \quad , \quad [K_\mu, D] = -iK_\mu \\
[P_\mu, K_\nu] &=& 2iM_{\mu\nu} + 2i\eta_{\mu\nu}D \quad , \quad [M_{\mu\nu}, D] = 0 \\
[P_\rho, M_{\mu\nu}] &=& -i\big(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\big) \quad , \quad [K_\rho, M_{\mu\nu}] = -i\big(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\big) \\
[M_{\mu\nu}, M_{\rho\sigma}] &=& i\big(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\big).
\end{array}
\tag{4.14}
$$

These are just the commutation relations of the generators of $SO(t+1, s+1)$. To see that, one introduces the coordinates

$$\xi_m = (\xi_{-1}, x_\mu, \xi_d)$$

in a $d+2$-dimensional space with signature $(t+1, s+1)$:

$$d\xi^2 = d\xi_{-1}^2 + \eta_{\mu\nu}dx^\mu dx^\nu - d\xi_d^2 = \eta_{mn}d\xi^m d\xi^n, \qquad \eta_{\mu\nu} = \mathrm{diag}(1, 1, \ldots, -1, -1).$$

We consider small 'Lorentz-transformations' in this space

$$2M_{mn} = -2M_{nm} = \begin{pmatrix} 0 & P_\mu & D \\ -P_\mu & M_{\mu\nu} & P_\mu \\ -D & -P_\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & K_\mu & D \\ -K_\mu & M_{\mu\nu} & -K_\mu \\ -D & K_\mu & 0 \end{pmatrix}. \tag{4.15}$$

The $SO(t+1, s+1)$ commutation relations

$$[M_{mn}, M_{pq}] \;=\; i\big(\eta_{mp}M_{nq} + \eta_{nq}M_{mp} - \eta_{mq}M_{np} - \eta_{np}M_{mq}\big)$$

imply the conformal algebra (4.14). The number of generators equals the dimension of $SO(t+1, s+1)$ which is $\frac{1}{2}(d+1)(d+2)$.

4.2.2 The conformal group

This relation between conformal transformations in $(\mathbb{R}^d, \eta_{\mu\nu})$ and 'Lorentz-transformations' in $(\mathbb{R}^{d+2}, \eta_{mn})$ exists not only on the algebraic level. To see that one introduces the *projective* coordinates

$$\xi^m = (\psi - \chi, \xi^\mu, \psi + \chi), \quad \text{where} \quad \xi^\mu = 2\psi x^\mu, \qquad \chi = \psi x^2 \quad \text{and} \quad \xi^m \xi_m = 0.$$

The nonlinear conformal transformations are then represented by linear $SO(t+1, s+1)$ transformations in $\mathbb{R}^{d+2}$ as follows:

$$\xi'^m = \zeta \tilde{\Lambda}^m_{\ n}\xi^n \sim \tilde{\Lambda}^m_{\ n}\xi^n \quad \text{with} \quad \tilde{\Lambda}^m = e^\Omega \in SO(t+1, s+1).$$

The infinitesimal transformations are

$$
\begin{array}{ll}
\Omega_T = \begin{pmatrix} 0 & -a_\mu & 0 \\ a^\mu & 0 & a^\mu \\ a & a_\mu & 0 \end{pmatrix} \quad , & \Omega_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega^\mu_{\ \nu} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\Omega_D = -\begin{pmatrix} 0 & 0 & \log\lambda \\ 0 & 0 & 0 \\ \log\lambda & 0 & 0 \end{pmatrix} \quad , & \Omega_S = \begin{pmatrix} 0 & -c_\mu & 0 \\ c^\mu & 0 & -c^\mu \\ 0 & -c_\mu & 0 \end{pmatrix}.
\end{array}
$$

For completeness we note that

$$\zeta_T = \xi_L = \frac{\psi'}{\psi}, \quad \zeta_D = \lambda\frac{\psi'}{\psi} \quad \text{and} \quad \zeta_S = \frac{\psi'/\psi}{1-2c\cdot x + c^2x^2}.$$

4.2.3 Transformation of fields and conformal weights

Conformal transformations are very particular coordinate transformations and tensor fields transform under small transformations as

$$\delta_X T_{\rho\sigma\ldots} = \frac{d}{d\epsilon}\Big(\frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\sigma}\ldots T_{\mu\nu\ldots}(x(y))\Big)|_{\epsilon=0} \equiv L_X T_{\rho\sigma\ldots},$$

The infinitesimal transformations of a matter field is given by the Lie derivative L_X.
For example, for a vector field V_μ it is

$$L_X V_\mu = X^\rho\partial_\rho V_\mu + X,_\mu^\rho V_\rho. \tag{4.16}$$

Inserting the corresponding Killing field one finds

$$\begin{array}{rl} \text{transl.} & iL_a V_\mu = a^\rho P_\rho V_\mu \\ \text{Lorentztrf.} & iL_\omega V_\mu = \frac{1}{2}\omega^{\rho\sigma}M_{\rho\sigma}V_\mu + i\omega^\rho{}_\mu V_\rho \\ \text{dilatation} & iL_\lambda V_\mu = \lambda D V_\mu + i\lambda V_\mu \\ \text{spez. Trf.} & iL_c V_\mu = c^\rho K_\rho V_\mu + 2i\big(c_\mu x^\rho - x_\mu c^\rho + c\cdot x\,\delta^\rho_\mu\big)V_\rho, \end{array} \tag{4.17}$$

and similarly for tensor fields of higher rank [24].
The algebra of infinitesimal transformations of the fields is the same as that of the generators of the symmetry since

$$[L_X, L_Y] = L_{[X,Y]}.$$

The transformations

$$x^\mu \longrightarrow x^\mu + X^\mu \quad \text{and} \quad \phi \longrightarrow L_X\phi$$

is a symmetry of any diffeomorphism invariant theory if one also maps the metric $\eta_{\mu\nu}$ to $e^\varphi\eta_{\mu\nu}$. For a conformally invariant theory the transformation of the metric can be un-done by a compensating Weyl transformations [25] such that

$$\delta_X T_{\mu\nu\ldots} = \big(L_X - \frac{2\alpha}{d}\partial_\rho X^\rho\big)T_{\mu\nu\ldots}. \tag{4.18}$$

The real constant α is the Weyl-weight of $T_{\mu\nu\ldots}$. It can be determined by the following recipe: If the metric transforms as $g_{\mu\nu}\to e^\varphi g_{\mu\nu}$ the the field transforms as $\phi\to e^{\alpha\varphi}\phi$ such that the action is invariant. One finds

$$\text{scalar field:} \ -2\alpha = \frac{d-2}{2}, \quad \text{Dirac field:} \ -2\alpha = \frac{d-1}{2}, \quad \text{gauge potential in d=4}: \ \alpha = 0.$$

In particular, for dilatations

$$T_{\mu\nu\dots} \longrightarrow \lambda(x^\rho\partial_\rho + s - 2\alpha)T_{\mu\nu\dots}, \tag{4.19}$$

where s is the number of covariant minus the number of contravariant indices of T. The number $\Delta = s - 2\alpha$ is called *conformal weight* of ϕ.

4.2.4 Noether currents

The Lagrangian density has Weyl weight $\alpha = -d/2$ and

$$\delta_X\mathcal{L} = \big(X^\rho\partial_\rho\mathcal{L} + \partial_\rho X^\rho\big)\mathcal{L} = \partial_\rho\big(X^\rho\mathcal{L}\big).$$

The Noether identity reads

$$\delta_X\mathcal{L} = \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_X\phi\Big) \Longleftrightarrow \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\big(L_X\phi - \frac{2\alpha}{d}\phi\big) - X^\mu\mathcal{L}\Big).$$

This leads to the conserved Noether current

$$J^\mu_X = T^\mu_{\ \nu}X^\nu + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\big(L_X - X^\rho\partial_\rho - \frac{2\alpha}{d}\big)\phi. \tag{4.20}$$

In particular for dilatations

$$J^\mu_D = T^\mu_{\ \nu}x^\nu + \Delta\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\phi,$$

where $\Delta = s - 2\alpha$ is the conformal weight of ϕ. Note that

$$\Delta_{\text{scalar}} = \frac{d-2}{2}, \quad \Delta_{\text{Dirac}} = \frac{d-1}{2} \quad \text{and} \quad \Delta_{\text{Photon}}(d=4) = 1.$$

The conserved dilation charge is

$$Q_D = \int_{x^0}\big(T^0_{\ \nu}x^\nu + \Delta\phi\pi\big) = \int_{x^0}\big(\mathcal{H}x^0 + \pi(\vec{x},\nabla)\phi + \Delta\phi\pi\big).$$

As expected, this charge generated the infinitesimal dilatations,

$$\{\phi, Q_D\} = x^\mu\partial_\mu\phi + \Delta\phi.$$

For example, for a free neutral scalar field

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi$$

we find

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} \quad \text{and} \quad J^\nu_D = T^\mu_{\ \nu}x^\nu + \tfrac{1}{2}(d-2)\,\phi\partial^\mu\phi.$$

This current is conserved on shell, as it must be.

4.3 Implementation of symmetries in quantized theories

We assign to a classical field $\phi(x)$ the operator $\hat{\phi}(x)$ on a Hilbert space $\mathcal{H}$. We assume that the symmetries are represented by unitary representation on $\mathcal{H}$. For the Poincaré group

$$U(a,\Lambda)\hat{\phi}(x)U^{-1}(a,\Lambda) = S\hat{\phi}\big((\Lambda^{-1}(x-a)\big) = e^{\frac{i}{2}(\omega,J)}\hat{\phi}(x). \tag{4.21}$$

In particular, for translations we write

$$U(a) = \exp\big(i(a,\hat{P})\big)$$

such that the infinitesimal form of (4.21) reads

$$[\hat{P}_\mu,\hat{\phi}] = P_\mu\hat{\phi}, \quad P_\mu = i\partial_\mu. \tag{4.22}$$

For Lorentz transformation we write

$$U(\Lambda) = \exp\big(\tfrac{i}{2}(\omega,\hat{J})\big)$$

such that

$$[\hat{J}_{\nu\nu},\hat{\phi}] = J_{\mu\nu}\hat{\phi}, \qquad J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}. \tag{4.23}$$

Since $(a,\Lambda)\to U(a,\Lambda)$ is a representation, the infinitesimal generators must obey the Poincaré algebra. Hence, the $(\hat{P}_\mu,\hat{J}_{\mu\nu})$ fulfill the Poincaré algebra (2.6).
Let us assume, that the vacuum in Poincaré invariant,

$$U(a,\Lambda)|0\rangle = |0\rangle. \tag{4.24}$$

Then[1]

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|e^{i(xP)}\phi(0)e^{i(y-x)P}\phi(0)e^{-iyP}|0\rangle\\ &= \langle 0|\phi(0)e^{-i(x-y)P}\phi(0)|0\rangle = \langle 0|\phi(x-y)\phi(0)|0\rangle = S(y-x).\end{aligned}$$

and

$$\langle 0|\phi(x)\phi(0)|0\rangle = \langle 0|U^{-1}(\Lambda)S\phi(\Lambda^{-1}x)S\phi(0)|0\rangle = \langle 0|S\phi(\Lambda^{-1}x)S\phi(0)|0\rangle.$$

For example, the propagator for a spin-1 field satisfies

$$\langle 0|A_\mu(x)A_\nu(0)|0\rangle = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}\langle 0|A_\rho(\Lambda^{-1}x)A_\sigma(0)|0\rangle.$$

This condition fixed almost uniquely the Lorentz structure of the propagator in momentum space. How does one (formally) construct the infinitesimal generators in the quantum theory? For that we recall that

$$\{\phi,\pi\} = \delta \longrightarrow [\phi,\pi] = i\delta \quad \text{and} \quad \{\phi,Q_X\} = \delta_X\phi \longrightarrow [\phi,Q_X] = i\delta_X\phi,$$

[1] we skip the hats

or that

$$i[Q_X, \phi] = \delta_X \phi.$$

The left hand side is the infinitesimal form of

$$\phi \to U\phi U^{-1}, \qquad U = e^{iQ}$$

and the right hand side of

$$S\phi\big(\Lambda^{-1}(x-a)\big).$$

We conclude, that the generators Q in the quantum theory are (formally) constructed by taking the classical Noether charges in which one replaces the classical fields by the quantum fields.