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Will be discussed: **18th week of year**

Problems in Supersymmetry

Sheet 4

Problem 13: Spin- vs. Lorentz transformations

The earlier introduced matrices

$$\Sigma^{\mu\nu} = -\Sigma^{\nu\mu} = \frac{1}{2\mathrm{i}}\gamma^{\mu\nu}$$

possess the following commutators with the γ-matrices and themselves:

$$\begin{aligned}[\Sigma^{\mu\nu},\gamma^\rho] &= \mathrm{i}\,(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu)\\ [\Sigma_{\mu\nu},\Sigma_{\rho\sigma}] &= \mathrm{i}\,(\eta_{\mu\rho}\Sigma_{\nu\sigma} + \eta_{\nu\sigma}\Sigma_{\mu\rho} - \eta_{\mu\sigma}\Sigma_{\nu\rho} - \eta_{\nu\rho}\Sigma_{\mu\sigma})\,.\end{aligned}$$

Let $S(s)$ be the following one-parameter family of transformations

$$\Gamma^\rho(s) = S^{-1}(s)\gamma^\rho S(s) \quad \text{with} \quad S(s) = e^{\frac{\mathrm{i}s}{2}(\omega,\Sigma)}$$

with 'initial value' $S(0) = \mathbb{1}$. Prove that

$$\Gamma^\rho(s) = S^{-1}(s)\gamma^\rho S(s) = (e^{s\omega})^\rho{}_\sigma\,\gamma^\sigma.$$

Set $s = 1$ and discuss the resulting relation $S^{-1}\gamma^\rho S = \Lambda^\rho{}_\sigma\gamma^\sigma$ between the matrices

$$S = e^{\frac{\mathrm{i}}{2}(\omega,\Sigma)} \quad \text{and} \quad \Lambda = e^\omega.$$

What type of matrix is Λ? Prove that $S \to \Lambda$ is a representation.

Problem 14: Super-Liealgebras and Jacobi-Identities

A super-Liealgebra (graduated algebra) uses the brackets

$$[A,B\} = C \quad \text{with} \quad [A,B\} = AB - (-1)^{ba}BA,$$

and

$$a = g(A) = \begin{cases} 0 & \text{if } A \text{ bosonic}\\ 1 & \text{if } A \text{ fermionic,}\end{cases}$$

and similarly for $b = g(B)$. The grade of C is $g(C) := (a+b) \bmod 2$ ($\mathbb{Z}_2$ grading).

1. What is the relation between $[A,B\}$ and $[B,A\}$?
2. Prove the super-Jacobi identity

$$[[A,B\},C\} + (-1)^{(b+c)a}[[B,C\},A\} + (-1)^{c(a+b)}[[C,A\},B\} = 0$$

by considering the four cases $\mathcal{BBB}$, $\mathcal{FBB}$, $\mathcal{FFB}$ and $\mathcal{FFF}$ ($\mathcal{B}$ for a bosonic and $\mathcal{F}$ for a fermionic Operator).

3. Check whether the super-Jacobi identities are fulfilled for the simple superalgebra

$$\{Q, Q^\dagger\} = 2H, \quad \{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\}, \quad [H, Q] = 0 = [H, Q^\dagger].$$

Problem 15: Equations of motion for super-particle

The action of a supersymmetric particle in flat space has the form

$$S[q, \psi, \bar{\psi}] = \int dt L\left(q, \dot{q}, \psi, \dot{\psi}, \bar{\psi}, \dot{\bar{\psi}}\right).$$

1. Find the corresponding equations of motion.
2. Apply the general result for a system with

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\left(W'(q)\right)^2 + \frac{\mathrm{i}}{2}\left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi\right) - W''(q)\bar{\psi}\psi,$$

   where W does not depend on $\dot{q}$.

Problem 16: On shell susy transformations

The supersymmetry transformations for the degrees of freedom of a superparticle read

$$\delta q = \varepsilon\psi + \bar{\psi}\bar{\varepsilon}, \quad \delta\psi = -\bar{\varepsilon}\left(\mathrm{i}\dot{q} + W'(q)\right), \quad \delta\bar{\psi} = \left(\mathrm{i}\dot{q} - W'(q)\right)\varepsilon.$$

1. Prove that the action is invariant.
2. Calculate the commutator of two supersymmetry transformations, $[\delta_1, \delta_2]$ on every field q, ψ and $\bar{\psi}$. You may use the equations of motion derived earlier.