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Will be discussed: **15th week of year**

Problems in Supersymmetry

Sheet 1

Problem 1: Susy-Basics: Harmonic Oscillator

Consider a (bosonic) harmonic oscillator. For simplicity assume $\hbar = c = \omega = \ldots = 1$. There are the well-known relations

$$[q,p] = i, \quad a = \frac{1}{\sqrt{2}}(q+ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(q-ip), \quad [a,a^\dagger] = 1.$$

For the eigenstates $|n\rangle$ we have: $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Up to now everything is bosonic. Number and Hamilton operator are

$$N_B = a^\dagger a, \quad H_B = \frac{1}{2}\left(p^2 + q^2\right).$$

Express H_B in terms of N_B. What do we get for

$$[N_B, a], \quad [N_B, a^\dagger], \quad N_B|n\rangle, \quad H_B|n\rangle.$$

Add a 2-state system (analogous to Spin-1/2 states $|\vec{S}^2, S_3\rangle$):

$$|\frac{1}{2}, \frac{1}{2}\rangle = |+\rangle \quad \text{and} \quad |\frac{1}{2}, -\frac{1}{2}\rangle = |-\rangle.$$

Use $S_\pm = S_1 \pm iS_2$ to define fermionic annihilation and creation operators:

$$b^\dagger := S_+, \quad b := S_-$$

What are the anti-commutation relations of $b, b^\dagger$?
Analogous to a spin in a magnetic field define fermionic number and Hamilton operator:

$$N_F = b^\dagger b, \quad H_F = S_z = ?$$

How do $b^\dagger$, b, N_F act on the states $|+\rangle$ and $|-\rangle$? Assume that the bosonic operators $a, a^\dagger$ commute with the fermionic operators $b, b^\dagger$. Calculate the eigenvalues and their degeneracies for the total Hamiltonian $H = H_B + H_F$.

Problem 2: Grassmann numbers

A Grassmann number θ is an anticommuting object, $\{\theta, \theta\} = 0$.

- What follows for the Taylor series of the function $\phi(\theta)$?
- If we impose translational invariance for the integrals

  $$\int_{-\infty}^{\infty} dx\, \phi(x) = \int_{-\infty}^{\infty} dx\, \phi(x+c)$$

  for Grassmann variables, what integration rules do we get? Calculate

  $$\int d\theta, \quad \int d\theta\, \theta, \quad \frac{\partial}{\partial\theta} \quad .$$

  Use the most simple relation for $\int d\theta\, \theta$.

- What are the changes in the rules if there is a set of (anticommuting) Grassmann variables $\theta = \{\theta_1, \dots, \theta_n\}$.

Problem 3: Tensors

Show that the subspaces of tensors for which all pair traces are zero form an invariant subspace under Lorentz-transformations.
Show that if T is (anti)symmetric in two indexes, then the transformed tensor is also (anti)symmetric in the same indexes.

Problem 4: Clifford algebra I

Show that the sixteen matrices of the $4d$ Clifford-algebra

$$M_i = \mathbb{1}, \quad \gamma^\mu, \quad \gamma^{\mu\nu} = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \quad \gamma_5\gamma^\mu, \quad \gamma_5$$

are orthogonal with respect to the scalar product

$$(M_i, M_j) = \mathrm{Tr}\,(M_i M_j).$$

Problem 5: Clifford algebra II

Prove, that the matrices

$$\begin{aligned}
\gamma_0 &= \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots \quad & \gamma_2 &= i\,\sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \dots \quad & \gamma_4 &= i\,\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \dots \\
\gamma_1 &= i\,\sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots \quad & \gamma_3 &= i\,\sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \dots \quad & & \dots
\end{aligned}$$

generate a Clifford-algebra.

Problem 6: Clifford algebra III

Let d be odd and let $\gamma_0, \dots, \gamma_{d-2}$ be a representation of the CLIFFORD algebra in $d-1$ dimension. Prove that in d dimensions the two representations

$$\begin{aligned}
\gamma_0, \dots, \gamma_{d-2}, \gamma_{d-1} &= +\alpha\gamma_0 \cdots \gamma_{d-2} \\
\gamma_0, \dots, \gamma_{d-2}, \gamma_{d-1} &= -\alpha\gamma_0 \cdots \gamma_{d-2}
\end{aligned}$$

are never equivalent. The phase α is chosen such that γ_{d-1} squares to $-\mathbb{1}$.