Problems in Supersymmetry: sheet 3

The celebrated COLEMAN-MANDULA theorem states that in a theory with non-trivial scattering in more than $1 + 1$ dimensions, the only possible conserved quantities that transform as tensors under the LORENTZ group (i.e. without spinor indexes) are the usual energy momentum vector P_μ, the generators of LORENTZ transformations $J_{\mu\nu}$, as well as possible 'internal' symmetry charges Z_i which commute with P_μ and $J_{\mu\nu}$. (There is an extension of this result for massless particles which allows the generators of conformal transformations.)

Problem 5.1: To illustrate the COLEMAN-MANDULA theorem we consider the example of theory of 2 free real bose fields,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 \tag{2}$$

Such a theory has infinitely many conserved currents. For example, it follows immediately from the equations of motion $\partial_\mu\partial^\mu\phi_i = 0$, that the series of currents

$$\begin{aligned} J_\mu &= (\partial_\mu\phi_2)\phi_1 - \phi_2\partial_\mu\phi_1, \\ J_{\mu\rho} &= (\partial_\mu\phi_2)\partial_\rho\phi_1 - \phi_2\partial_\mu\partial_\rho\phi_1, \\ J_{\mu\rho\sigma} &= (\partial_\mu\phi_2)\partial_\rho\partial_\sigma\phi_1 - \phi_2\partial_\mu\partial_\rho\partial_\sigma\phi_1, \quad \textit{etc.} \end{aligned}$$

are conserved, e.g. $\partial_\mu J^{\mu\rho} = 0$. Prove this. Construct further conserved tensorial currents. Show that the charges

$$Q_{\rho\sigma} = \int d^{d-1}x\; J_{0\rho\sigma} \qquad \textit{etc.}$$

are time-independent (conserved) for fields which vanish at spatial infinity.

Problem 5.1: The COLEMAN-MANDULA theorem does not apply to spinor charges, though. Consider a free theory of two real massless scalars and a MAJORANA fermion in 4 dimensions with LAGRANGEan

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi$$

Why is there an i multiplying the fermion term? Derive the equation of motion. Again, there is an infinite number of currents with spinor indexes, e.g.

$$\begin{aligned} S_{\mu\alpha} &= \partial_\rho(\phi_1 - i\phi_2)(\gamma^\rho\gamma_\mu\psi)_\alpha \\ S_{\mu\nu\alpha} &= \partial_\rho(\phi_1 - i\phi_2)(\gamma^\rho\gamma_\mu\partial_\nu\psi)_\alpha, \quad \textit{etc.} \end{aligned}$$

Show that these currents are conserved. Construct further conserved currents. Adding the interaction

$$-g\bar{\psi}(\phi_1 - i\gamma_5\phi_2)\psi - \tfrac{1}{2}g^2(\phi_1^2 + \phi_2^2)^2$$

to the LANGREAN $\mathcal{L}_0$ of the free model, $S_{\mu\alpha}$ (with corrections proportional to g) remains conserved. Try to find these corrections. $Q_{\nu\alpha}$ can never be deformed to yield a conserved quantity of the interacting theory. Why would this contradict the above theorem (consider $\{Q_{\nu\alpha}, Q^\dagger_{\nu\alpha}\}$).