Problems in Supersymmetry: sheet 2

Problem 4.1: Prove, that the matrices

$$
\begin{aligned}
\gamma_0 &= \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots & \gamma_2 &= i\,\sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \dots & \gamma_4 &= i\,\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \dots \\
\gamma_1 &= i\,\sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots & \gamma_3 &= i\,\sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \dots & & \dots
\end{aligned}
$$

generate a CLIFFORD algebra.

Problem 4.2: Let d be odd and let $\gamma_0, \dots, \gamma_{d-2}$ be a representation of the CLIFFORD algebra in $d-1$ dimension. Prove that in d dimensions the two representations

$$
\begin{aligned}
\gamma_0, \dots, \gamma_{d-2}, \gamma_{d-1} &= +\alpha \gamma_0 \cdots \gamma_{d-2} \\
\gamma_0, \dots, \gamma_{d-2}, \gamma_{d-1} &= -\alpha \gamma_0 \cdots \gamma_{d-2}
\end{aligned}
$$

are never equivalent. The phase α is chosen such that γ_{d-1} squares to $-\mathbb{1}$.

Problem 4.3: In a 'classical' treatment the components of a spinor field anticommute. Let ψ be a MAJORANA spinor in 4 space-time dimensions,

$$
\psi = \psi_{\mathrm{C}} = \mathcal{C}\gamma_0^T \psi^* = \mathcal{C}\bar{\psi}^T, \qquad \mathcal{C}^T = -\mathcal{C}, \quad \mathcal{C}^\dagger = \mathcal{C}^{-1}
$$

Use the properties of γ^0 and $\mathcal{C}$ to prove the following properties for the MAJORANA bilinears

$$
\begin{aligned}
(\bar{\psi} M \chi) &= \begin{cases} +(\bar{\chi} M \psi) & M = \mathbb{1}, \gamma_5\gamma_\mu, \gamma_5 \\ -(\bar{\chi} M \psi) & M = \gamma_\mu, \gamma_{\mu\nu} \end{cases} \\
(\bar{\psi} M \chi)^* &= \begin{cases} +(\bar{\psi} M \chi) & M = \mathbb{1}, \gamma_\mu, \gamma_{\mu\nu} \\ -(\bar{\psi} M \chi) & M = \gamma_5\gamma_\mu, \gamma_5 \end{cases} .
\end{aligned}
$$

What follows for $\bar{\psi} M \psi$ and can you interpret some of the results.

Problem 4.4: Let θ be a MAJORANA spinor. Try to prove the following identities:

$$
\theta_\alpha(\bar{\theta}\theta) = -(\gamma_5\theta)_\alpha(\bar{\theta}\gamma_5\theta), \quad (\bar{\theta}\theta)^2 = -(\bar{\theta}\gamma_5\theta)^2, \quad (\bar{\theta}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta) = -\eta_{\mu\nu}(\bar{\theta}\gamma_5\theta)^2.
$$