# Kapitel 5

# Symmetries

Supersymmetry is an extension of POINCARÉ symmetry and hence is intimately related to the geometry of our spacetime. A supersymmetric *field theory* in flat space time is a particular theory compatible with the principles of special relativity. Thus it is useful to review the implications of internal and spacetime symmetries for relativistic field theories.
All fundamental relativistic field equations are EULER-LAGRANGE equation of an POINCARÉ invariant action integral $S$. In a local field theory the action is the space-time integral of a local LAGRANGEan density $\mathcal{L}(x)$,

$$S = \int d^d x\ \mathcal{L}(x) \equiv \int dt d\boldsymbol{x} \mathcal{L}(t, \boldsymbol{x}), \tag{5.1}$$

where $\mathcal{L}$ is a function of the fields and their first space-time derivatives. Again the space-time dimension is left open. The volume element of space is denoted by $d\boldsymbol{x}$.
The action should be invariant under POINCARÉ *transformations*,

$$\tilde{x} = \Lambda(A)x + a \tag{5.2}$$

under which the field transforms as

$$\tilde{\phi}(\tilde{x}) = S(A)\phi(x) \quad \text{or} \quad \phi(x) \longrightarrow S(A)\phi\left(\Lambda^{-1}(A)(x-a)\right), \tag{5.3}$$

where $A \to S(A)$ is a representations of the spin group and $\Lambda(A)$ it the LORENTZ transformation corresponding to $A$.
In addition the action may be invariant under *global gauge transformations*

$$\phi \longrightarrow U\phi \sim \phi + iX\phi = \phi + \delta_X \phi, \qquad X^\dagger = X. \tag{5.4}$$

They are called global since the transformation matrix $U$ it the same for all space-time points.

## 5.1 Noether theorem and conserved charges

According to a theorem of EMMY NOETHER, to each parameter of the symmetry group there corresponds a conserved current. The global gauge transformations (5.4) leave the LAGRANGEan density invariant so that

$$0 = \delta_X \mathcal{L} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\, \partial_\mu \left(\delta_X \phi\right) + \frac{\delta \mathcal{L}}{\delta \phi}\, \delta_X \phi = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta_X \phi\right), \tag{5.5}$$

where we used the EULER-LAGRANGE equation (field equation, equation of motion)

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)-\frac{\delta\mathcal{L}}{\delta\phi}=0 \tag{5.6}$$

in the last step. Thus the conserved NOETHER *current* for an internal symmetry takes the form

$$J^\mu_X=\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_X\phi,\qquad \partial_\mu J^\mu_X=0. \tag{5.7}$$

Integrating the last equation over the space-time region $[t_0,t]\times\mathbb{R}^{d-1}$ and converting the volume- into a surface integral, shows that the NOETHER *charges*

$$Q_X=\int_{x^0}d\boldsymbol{x}J^0_X=\int_{x^0}d\boldsymbol{x}\,\pi(x)\delta_X\phi(x), \tag{5.8}$$

are time-independent. Here we introduced the momentum density conjugate to the field $\phi$,

$$\pi(x)=\frac{\partial\mathcal{L}}{\partial\dot\phi(x)}. \tag{5.9}$$

To every internal symmetry there is always one conserved NOETHER charge. The dimension of the symmetry group equals the number of independent vector fields $X$ in (5.8) and hence equals the number of independent NOETHER charges.
The fundamental POISSON bracket between field and conjugated momentum density has the form

$$\{\phi(x),\pi(y)\}_{x^0=y^0}=\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{5.10}$$

and can be used to calculate the POISSON brackets between the conserved charges and the field,

$$\{\phi(x),Q_X\}=\int d\boldsymbol{y}\,\{\phi(x),\pi(y)\,\delta_X\phi(y)\}=\delta_X\phi(x), \tag{5.11}$$

where we assumed, that $\delta_X\phi$ contains no time-derivatives of the field. We see that the conserved NOETHER charge generates the symmetry from which it has been derived.

### 5.1.1 $U(1)$ gauge transformations

Gauge transformations in non-ABELian gauge theories are generalisations of phase transformations in electrodynamics. As an example consider the LAGRANGEan of a charged scalar field,

$$\mathcal{L}=\partial_\mu\phi^\dagger\partial^\mu\phi-m^2\phi^\dagger\phi-\lambda(\phi^\dagger\phi)^2. \tag{5.12}$$

It is invariant under the global phase transformations

$$\phi\longrightarrow e^{i\lambda}\phi\sim\phi+\delta_\lambda\phi\,,\quad\text{with}\quad\delta_\lambda\phi=i\lambda\phi,\quad\delta_\lambda\phi^\dagger=-i\lambda\phi^\dagger, \tag{5.13}$$

and the corresponding NOETHER current has the form

$$J^\mu_\lambda=\lambda j^\mu,\qquad j_\mu=i\big(\partial_\mu\phi^\dagger\phi-\phi^\dagger\partial_\mu\phi\big). \tag{5.14}$$

Using the equation of motion $\Box\phi+m^2\phi+2\lambda(\phi^\dagger\phi)\phi$ one can check that $j^\mu$ is conserved. The conserved charge is to be identified with the electric charge carried by the field,

$$Q=i\int_{x^0}d\boldsymbol{x}\big(\dot\phi^\dagger\phi-\phi^\dagger\dot\phi\big)=i\int_{x^0}d\boldsymbol{x}\,\left(\pi_\phi\phi-\pi_{\phi^\dagger}\phi^\dagger\right). \tag{5.15}$$

Again the charge generates the symmetry from which it has been derived,

$$\{\phi,Q\}=i\phi\quad\text{and}\quad\{\phi^\dagger,Q\}=-i\phi^\dagger. \tag{5.16}$$

### 5.1.2 The energy-momentum as Noether current

Under space-time translations the field transforms as

$$\phi(x+a) \sim \phi(x) + a^\mu \partial_\mu \phi(x) = \phi(x) + \delta_a \phi(x). \tag{5.17}$$

Since the space-time point is changing the LAGRANGEan is not invariant. But translations are symmetries and the *action* must be invariant. It is invariant since (off-shell) the LAGRANGEan only changes by total divergence,

$$\delta_a \mathcal{L} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\mu \big(\delta_a \phi\big) + \frac{\delta \mathcal{L}}{\delta \phi} \delta_a \phi = a^\nu \partial_\nu \mathcal{L}. \tag{5.18}$$

Using the field equation (5.6) the on-shell identity (5.5) is now modified to

$$\delta_a \mathcal{L} = a^\nu \partial_\nu \mathcal{L} = \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} a^\nu \partial_\nu \phi \right), \quad \text{or} \quad \partial_\mu J^\mu_a = 0 \tag{5.19}$$

with covariantly conserved *canonical energy-momentum* tensor as NOETHER *current*

$$J^\mu_a = a^\nu T^\mu_{\ \nu}, \qquad T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L}. \tag{5.20}$$

The conserved charges are the total energy-momentum of the field,

$$P^\mu = \int_{x^0} d\boldsymbol{x}\, T^{0\mu}, \qquad \dot{P}^\mu = 0, \tag{5.21}$$

and for the scalar field with LAGRANGEan (5.12) the *energy* and *momentum* have the form

$$P^0 \equiv H = \int_{x^0} d\boldsymbol{x}\, \big(\pi \dot{\phi} - \mathcal{L}\big) \quad \text{and} \quad P^i = \int_{x^0} d\boldsymbol{x}\, \pi\, \partial^i \phi. \tag{5.22}$$

These NOETHER charges generate the infinitesimal spacetime translations,

$$\{\phi, P_\mu\} = \partial_\mu \phi. \tag{5.23}$$

For the scalar field theory with LAGRANGEan (5.12) we find

$$T_{\mu\nu} = \partial_\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial_\mu \phi - \eta_{\mu\nu} \mathcal{L} \tag{5.24}$$

which leads to the conserved energy

$$H = \int d\boldsymbol{x}\, \big(\pi^\dagger \pi + \nabla \phi^\dagger \nabla \phi + m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2\big) \tag{5.25}$$

and conserved momentum

$$P_i = \int d\boldsymbol{x}\, \big(\pi \partial_i \phi + \partial_i \phi^\dagger \pi^\dagger\big). \tag{5.26}$$

In *electrodynamics* we introduce the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is derived from the 4-potential $A^\mu = (A^0, \boldsymbol{A})$. The entries of the field strength are the components of the electric and magnetic fields,

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} = (\boldsymbol{E}, \boldsymbol{B}). \tag{5.27}$$

Raising both indexes amount to changing the sign of the electric field, $(F^{\mu\nu}) = (-\boldsymbol{E}, \boldsymbol{B})$. The LAGRANGEan density in electromagnetism is

$$\mathcal{L} = \tfrac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\tfrac{1}{4}F^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \tag{5.28}$$

From the last representation it follows at once that

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\rho)} = -F^{\mu\rho} \tag{5.29}$$

holds true, and this result is used when calculating the energy-momentum tensor,

$$T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\rho)}\,\partial_\nu A_\rho - \eta_{\mu\nu}\mathcal{L} = -F_\mu^{\ \rho}\partial_\nu A_\rho + \tfrac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \tag{5.30}$$

This then leads to the following formula for the conserved energy-momentum in electromagnetism,

$$P_\nu = \int_{x^0} d\boldsymbol{x}\, T^0_{\ \nu} = \int_{x^0} d\boldsymbol{x}\,\left(F^{\rho 0}\,\partial_\nu A_\rho - \delta^0_{\ \nu}\mathcal{L}\right)$$

or written in components

$$H = \int_{x^0} d\boldsymbol{x}\,\left(E_i\dot{A}_i - \mathcal{L}\right) \quad \text{and} \quad P_i = \int_{x^0} d\boldsymbol{x}\,\left(E_j\partial_i A_j\right). \tag{5.31}$$

Although we succeeded in constructing a conserved second rank tensor which is to be interpreted as energy-momentum tensor, there is still some improvement necessary. The main reason for our unhappiness with the tensor (5.30) is its lack of symmetry and invariance. It is not gauge invariant and hence not observable. Also we would like the energy-momentum to be a symmetric tensor for several reasons. The most convincing one is that it appears on the right hand side of EINSTEIN's field equation and then it must be both symmetric and conserved [28]

### 5.1.3 Improving Noether currents

Only for scalar fields is the canonical energy-momentum tensor arising from NOETHER's theorem symmetric. It is possible to correct the non-symmetric ones arising in non-scalar theories through the BELINFANTE symmetrisation procedure [27]. This way one may arrive at the socalled metric energy-momentum tensor which is automatically symmetric and conserved.
Let the LAGRANGEan be invariant under a symmetry transformation $\phi \to \phi + \delta\phi$, up to a total divergence,

$$\delta\mathcal{L} = \partial_\mu V^\mu. \tag{5.32}$$

As we have seen, the corresponding conserved NOETHER current has the form

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi - V^\mu. \tag{5.33}$$

It is important to note that the vector field $V^\mu$ in (5.32) is determined only up to terms $\partial_\nu A^{\mu\nu}$ with antisymmetric $A_{\mu\nu}$ and hence the NOETHER current $J^\mu$ is not unique,

$$J^\mu \longrightarrow J^\mu - \partial_\nu A^{\mu\nu}. \tag{5.34}$$

For localised fields the conserved charges are not affected by this ambiguity since

$$J^0 \to J^0 - \partial_i A^{0i} \quad \text{and hence} \quad Q = \int_{x^0} d\boldsymbol{x}\, J^0 \longrightarrow Q.$$

The possible improvement terms have been studied widely in the literature. For example, in electrodynamics the conserved momentum $P^\mu$ is gauge invariant, since $\partial_\rho F^{0\rho} = 0$, and we expect that there exists an improved gauge invariant energy momentum tensor. Indeed, we can add an improvement term $-\partial_\rho(F^{\mu\rho}A^\nu)$ to $T^{\mu\nu}$ in (5.30) to obtain a gauge invariant and symmetric tensor. The improved energy-momentum tensor takes the well-known form

$$T_{\mu\nu} = \tfrac{1}{2} F_\mu^{\ \rho} F_{\rho\nu} + \tfrac{1}{4}\eta_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \tag{5.35}$$

giving rise to

$$H = \tfrac{1}{2}\int_{x^0} d\boldsymbol{x}\,\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \quad \text{and} \quad \boldsymbol{P} = \int_{x^0} d\boldsymbol{x}\; \boldsymbol{E} \wedge \boldsymbol{B}. \tag{5.36}$$

The conserved tensor (5.35) may alternatively be gotten by coupling the MAXWELL field minimally to gravity and varying the resulting LAGRANGEan with respect to the metric.

### 5.1.4 The conserved angular momentum

Deriving the conserved quantities belonging to the LORENTZ transformations is a bit more elaborate. Under LORENTZ transformation a field transforms as follows,

$$\phi(x) \longrightarrow e^{\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}}\phi\left(e^{-\omega}x\right) \sim \phi(x) + \tfrac{i}{2}\,\omega^{\mu\nu}J_{\mu\nu}\phi(x) = \phi(x) + \delta_\omega\phi(x), \tag{5.37}$$

where the infinitesimal generators of LORENTZ transformations $J_{\mu\nu}$ are the sum of orbital and spin terms,

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} = \frac{1}{i}\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) + S_{\mu\nu}. \tag{5.38}$$

The $L_{\mu\nu}$ commute with the $S_{\mu\nu}$ and both fulfil the LORENTZ algebra (3.10). Again the LAGRANGEan is off-shell invariant up to a total divergence,

$$\delta_\omega\mathcal{L} = \tfrac{i}{2}\,(\omega, M)\,\mathcal{L} = \partial_\mu V^\mu \quad \text{with} \quad V^\mu = -\omega^{\mu\rho}x_\rho\mathcal{L}. \tag{5.39}$$

The first term in (5.33) reads

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_\omega\phi = \frac{i}{2}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\omega^{\rho\sigma}\left(L_{\rho\sigma} + S_{\rho\sigma}\right)\phi$$

and subtracting $V^\mu$ in (5.39) yields the NOETHER current

$$J^\mu_\omega = \frac{\omega^{\rho\sigma}}{2}\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\phi - (\delta^\mu_\sigma x_\rho - \delta^\mu_\rho x_\sigma)\mathcal{L} + i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,S_{\rho\sigma}\phi\right)$$

The spin-independent terms between the brackets are $x_\rho T^\mu_{\ \sigma}$ and $-x_\sigma T^\mu_{\ \rho}$ and thus we obtain

$$J^\mu_\omega = \omega_{\rho\sigma}M^{\mu\rho\sigma}, \quad M^{\mu\rho\sigma} = \tfrac{1}{2}x^\rho T^{\mu\sigma} - \tfrac{1}{2}x^\sigma T^{\mu\rho} + \tfrac{i}{2}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,S^{\rho\sigma}\phi = -M^{\mu\sigma\rho} \tag{5.40}$$

and the corresponding conserved charges

$$J^{\rho\sigma} = -J^{\sigma\rho} = \int_{x^0} d\boldsymbol{x}\, M^{0\rho\sigma}. \tag{5.41}$$

The space-time and space-space components

$$\begin{aligned} J^{i0} &= \tfrac{1}{2}\int_{x^0} d\boldsymbol{x}\, \left(x^i\mathcal{H} - \pi x^0\partial^i\phi - i\pi S^{0i}\phi\right) \\ J^{ij} &= \tfrac{1}{2}\int_{x^0} d\boldsymbol{x}\, \pi\left(x^i\partial^j\phi - x^j\partial^i\phi + iS^{ij}\phi\right). \end{aligned} \tag{5.42}$$

A simple calculation reveals that

$$\{\phi, J^{\mu\nu}\} = \tfrac{i}{2}\big(L^{\mu\nu} + S^{\mu\nu}\big)\phi, \tag{5.43}$$

or that the NOETHER charges $J^{0i}$ generate boosts and the charges $J^{ij}$ rotations. For example, for electrodynamics

$$M^{\mu\rho\sigma} = \big(\tfrac{1}{4}\eta^{\mu\rho}x^\sigma F^2 - F^{\mu\xi}x^\sigma\partial^\rho A_\xi - F^{\mu\rho}A^\sigma\big) - \big(\rho\leftrightarrow\sigma\big). \tag{5.44}$$

In passing we note, that for theories with spinor fields things may become tricky. The canonical energy-momentum tensor is generically not symmetric and must be improved. For bosonic fields the most efficient way to do this is to couple the fields to gravity and vary the resulting LAGRANGEan with respect to the metric. However, when coupling fermions to gravity one needs to introduce the vielbein. When one varies the action with respect to the vielbein one gets a conserved but not necessarily symmetric energy momentum tensor.

## 5.2 Conformal symmetry

Under conformal transformations

$$x^\mu \to y^\mu = y^\mu(x^\nu) \tag{5.45}$$

the metric tensor is unchanged up to a factor $e^\varphi$,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\sigma}dy^\rho dy^\sigma = e^{\varphi(y)}\eta_{\rho\sigma}dy^\rho dy^\sigma. \tag{5.46}$$

These transformation leave the light cone invariant and a spacelike vector can be mapped into a timelike vector. The apparent problem with causality has been solved by LÜSCHER and MACK [24].

### 5.2.1 The conformal algebra

Lets look at the infinitesimal conformal transformations,

$$y^\mu \sim x^\mu + X^\mu(x) \quad\text{and}\quad e^\varphi \sim 1 + \delta\varphi \tag{5.47}$$

for which

$$\big(\eta_{\mu\nu} + X_{\mu,\nu} + X_{\nu,\mu}\big)\big(1+\delta\varphi\big)dx^\mu dx^\nu \sim \eta_{\mu\nu}dx^\mu dx^\nu$$

holds true. The linearision of this equation leads to

$$\delta\varphi\,\eta_{\mu\nu} + X_{\mu,\nu} + X_{\nu,\mu} = 0.$$

Taking the trace implies that $\delta\varphi$ is proportional to the covariant divergence of the vector field $X$, so that $X$ satisfies the conformal Killing *equation*

$$X_{\mu,\nu} + X_{\nu,\mu} = \frac{2}{d}\eta_{\mu\nu}\partial_\rho X^\rho. \tag{5.48}$$

Hence the conformal coordinate transformations are generated by conformal Killing vector fields.

In 3 and more dimensions there are $\frac{1}{2}(d+1)(d+2)$ conformal Killing fields. These are

$$\begin{array}{rlcl} d\ \text{translations} & y^\mu = x^\mu + a^\mu & \Longrightarrow & X^\mu = a^\mu \\ \frac{1}{2}d(d-1)\ \text{Lorentztransf.} & y^\mu = \Lambda^\mu_{\ \nu} x^\nu & \Longrightarrow & X^\mu = \omega^\mu_{\ \nu} x^\nu \\ 1\ \text{dilatation} & y^\mu = \lambda x^\mu & \Longrightarrow & X^\mu = \lambda x^\mu \\ d\ \text{special conf. transf.} & y^\mu = \dfrac{x^\mu - x^2 c^\mu}{1 - 2c\cdot x + c^2 x^2} & \Longrightarrow & X^\mu = 2(c\cdot x)x^\mu - x^2 c^\mu. \end{array} \tag{5.49}$$

The corresponding hermitian generators are

$$iX^\mu\partial_\mu = \{a^\mu P_\mu, 2\omega^{\mu\nu}L_{\mu\nu}, \lambda D, c^\mu K_\mu\} \tag{5.50}$$

where

$$P_\mu = i\partial_\mu, \quad L_{\mu\nu} = \frac{1}{i}(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad D = ix^\mu\partial_\mu \quad \text{and} \quad K_\mu = 2ix_\mu x^\rho\partial_\rho - ix^2\partial_\mu \tag{5.51}$$

generate translations, Lorentz transformations, dilatations and special conformal transformations. The conformal algebra is

$$\begin{array}{rcl} [P_\mu, P_\nu] &=& [K_\mu, K_\nu] = [D, D] = 0 \quad , \quad [P_\mu, D] = iP_\mu \quad , \quad [K_\mu, D] = -iK_\mu \\ [P_\mu, K_\nu] &=& 2iL_{\mu\nu} + 2i\eta_{\mu\nu}D \quad , \quad [L_{\mu\nu}, D] = 0 \\ [P_\rho, L_{\mu\nu}] &=& -i\big(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\big) \quad , \quad [K_\rho, L_{\mu\nu}] = -i\big(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\big) \\ [L_{\mu\nu}, L_{\rho\sigma}] &=& i\big(\eta_{\mu\rho}L_{\nu\sigma} + \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\rho}L_{\mu\sigma}\big). \end{array} \tag{5.52}$$

These are just the commutation relations of the generators of $SO(2,d)$. To see that, one introduces the coordinates

$$\xi_m = (\xi_{-1}, x_\mu, \xi_d)$$

in a $d+2$-dimensional space with signature $(2,d)$:

$$d\xi^2 = d\xi_{-1}^2 + \eta_{\mu\nu}dx^\mu dx^\nu - d\xi_d^2 = \eta_{mn}d\xi^m d\xi^n. \tag{5.53}$$

We consider infinitesimal transformations in this space and decompose the generators as follows,

$$2L_{mn} = -2L_{nm} = \begin{pmatrix} 0 & P_\mu & D \\ -P_\mu & L_{\mu\nu} & P_\mu \\ -D & -P_\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & K_\mu & D \\ -K_\mu & L_{\mu\nu} & -K_\mu \\ -D & K_\mu & 0 \end{pmatrix}. \tag{5.54}$$

The $SO(2,d)$ commutation relations

$$[L_{mn}, L_{pq}] = i\big(\eta_{mp}L_{nq} + \eta_{nq}L_{mp} - \eta_{mq}L_{np} - \eta_{np}L_{mq}\big)$$

are equivalent to commutation relations (5.52) of the conformal algebra. The number of generators equals the dimension of $SO(2,d)$ which is $\frac{1}{2}(d+1)(d+2)$.

### 5.2.2 The conformal group

Th relation between conformal transformations in $\mathbb{R}^d$ and $SO(2,d)$-transformations in $(\mathbb{R}^{d+2},\eta_{mn})$ exists not only on the algebraic level. To see that one introduces the *projective* coordinates

$$\xi^m=(\psi-\chi,\xi^\mu,\psi+\chi),\quad \text{where}\quad \xi^\mu=2\psi x^\mu,\qquad \chi=\psi x^2\quad\text{and}\quad \xi^m\xi_m=0. \tag{5.55}$$

The nonlinear conformal transformations are then represented by linear $SO(2,d)$ transformations in $\mathbb{R}^{d+2}$ as follows:

$$\xi'^m=\zeta\tilde\Lambda^m_{\ n}\xi^n\sim\tilde\Lambda^m_{\ n}\xi^n\quad\text{with}\quad \tilde\Lambda^m=e^\Omega\in SO(2,d).$$

The infinitesimal transformations have the form

$$\Omega_T=\begin{pmatrix}0&-a_\mu&0\\ a^\mu&0&a^\mu\\ a&a_\mu&0\end{pmatrix}\quad,\quad \Omega_L=\begin{pmatrix}0&0&0\\0&\omega^\mu_{\ \nu}&0\\0&0&0\end{pmatrix}$$
$$\Omega_D=-\begin{pmatrix}0&0&\log\lambda\\0&0&0\\ \log\lambda&0&0\end{pmatrix}\quad,\quad \Omega_S=\begin{pmatrix}0&-c_\mu&0\\ c^\mu&0&-c^\mu\\0&-c_\mu&0\end{pmatrix}.$$

For completeness we note that

$$\zeta_T=\xi_L=\frac{\psi'}{\psi},\quad \zeta_D=\lambda\frac{\psi'}{\psi}\quad\text{and}\quad \zeta_S=\frac{\psi'/\psi}{1-2c\cdot x+c^2x^2}.$$

### 5.2.3 Transformation of fields and conformal weights

Conformal transformations are particular coordinate transformations and tensor fields transform under small transformations as

$$\delta_X T_{\rho\sigma\dots}=\frac{d}{d\epsilon}\left(\frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\sigma}\dots T_{\mu\nu\dots}(x(y))\right)|_{\epsilon=0}\equiv L_X T_{\rho\sigma\dots}, \tag{5.56}$$

The infinitesimal transformations of a matter field is given by the LIE derivative $L_X$.
For example, for a vector field $V_\mu$ it is

$$L_XV_\mu=X^\rho\partial_\rho V_\mu+X^{\ \rho}_{;\mu}V_\rho. \tag{5.57}$$

Inserting the corresponding KILLING field one finds

$$\begin{aligned}
&\text{transl.} && iL_aV_\mu=a^\rho P_\rho V_\mu\\
&\text{Lorentztrf.} && iL_\omega V_\mu=\tfrac12\omega^{\rho\sigma}L_{\rho\sigma}V_\mu+i\omega^\rho{}_\mu V_\rho\\
&\text{dilatation} && iL_\lambda V_\mu=\lambda DV_\mu+i\lambda V_\mu\\
&\text{spez. Trf.} && iL_cV_\mu=c^\rho K_\rho V_\mu+2i\big(c_\mu x^\rho-x_\mu c^\rho+c\cdot x\,\delta^\rho_\mu\big)V_\rho,
\end{aligned} \tag{5.58}$$

and similarly for tensor fields of higher rank [25].
The algebra of infinitesimal transformations of the fields is the same as that of the generators of the symmetry since

$$[L_X,L_Y]=L_{[X,Y]}.$$

The transformations

$$x^\mu\longrightarrow x^\mu+X^\mu\quad\text{and}\quad \phi\longrightarrow L_X\phi \tag{5.59}$$

is a symmetry of any diffeomorphism invariant theory if one also maps the metric $\eta_{\mu\nu}$ to $e^{\varphi}\eta_{\mu\nu}$. For a conformally invariant theory the transformation of the metric can be un-done by a compensating WEYL transformations [26] such that

$$\delta_X T_{\mu\nu\dots} = \left(L_X - \frac{2\alpha}{d}\partial_\rho X^\rho\right) T_{\mu\nu\dots}. \tag{5.60}$$

The real constant $\alpha$ is the WEYL-weight of $T_{\mu\nu\dots}$. It can be determined by the following recipe: If the metric transforms as $g_{\mu\nu} \to e^{\varphi} g_{\mu\nu}$ then the field must transform as $\phi \to e^{\alpha\varphi}\phi$ in order for the action to be invariant. One finds the following weights for scalar-, spinor- and vector fields in $d$ dimensions:

$$\text{scalar field: } -2\alpha = \frac{d-2}{2}, \quad \text{Dirac field: } -2\alpha = \frac{d-1}{2}, \quad \text{gauge potential in d=4}: \ \alpha = 0. \tag{5.61}$$

In particular, for dilatations

$$T_{\mu\nu\dots} \longrightarrow \lambda(x^\rho\partial_\rho + s - 2\alpha)T_{\mu\nu\dots}, \tag{5.62}$$

where $s$ is the number of covariant minus the number of contravariant indexes of $T$. The number $\Delta_\phi = s - 2\alpha$ is called *conformal weight* of $\phi$.

### 5.2.4 Noether currents

We assume that the action is invariant under conformal transformations of the fields and calculate the corresponding NOETHER currents. For the action to be invariant the LAGRANGEan density must have WEYL-weight $\alpha = -d/2$ and

$$\delta_X\mathcal{L} = \big(X^\mu\partial_\mu\mathcal{L} + \partial_\mu X^\mu\big)\mathcal{L} = \partial_\mu V^\mu, \qquad V^\mu = X^\mu\mathcal{L}. \tag{5.63}$$

The NOETHER current for the conformal symmetry has the form

$$J^\mu_X = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\left(L_X - \frac{2\alpha}{d}\partial_\rho X^\rho\right)\phi - X^\mu\mathcal{L}$$

Recalling the form of the canonical energy-momentum tensor (5.20) this can be rewritten as

$$J^\mu_X = T^\mu_{\ \nu}X^\nu + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\left(L_X - X^\rho\partial_\rho - \frac{2\alpha}{d}\partial_\rho X^\rho\right)\phi. \tag{5.64}$$

In particular for dilatations

$$J^\mu_D = T^\mu_{\ \nu}x^\nu + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\Delta_\phi\phi,$$

where $\Delta_\phi = s - 2\alpha$ is the conformal weight of $\phi$. Note that

$$\Delta_{\text{scalar}} = \frac{d-2}{2}, \quad \Delta_{\text{Dirac}} = \frac{d-1}{2} \quad \text{and} \quad \Delta_{\text{Photon}}(d=4) = 1.$$

The conserved *dilation charge* is

$$Q_D = \int_{x^0} d\boldsymbol{x}\, \big(T^0_{\ \nu}x^\nu + \Delta_\phi\phi\,\pi\big) = \int_{x^0} d\boldsymbol{x}\, \big(x^0\mathcal{H} + \pi(\boldsymbol{x}\nabla\phi) + \Delta_\phi\phi\,\pi\big)\,.$$

As expected, this charge generated the infinitesimal dilatations,

$$\{\phi, Q_D\} = x^\mu\partial_\mu\phi + \Delta_\phi\phi.$$

For example, for a free neutral scalar field with LAGRANGEan $\frac{1}{2}(\partial\phi)^2$ we find

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} \quad \text{and} \quad J^\nu_D = T^\mu_{\ \nu}x^\nu + \tfrac{1}{2}(d-2)\,\phi\,\partial^\mu\phi.$$

This current is conserved on shell, as it must be.

## 5.3 Implementing symmetries in quantised theories

In a quantum field theory for the field $\phi$ we assign to the classical field the operator $\hat{\phi}(x)$ acting on a suitable HILBERT space $\mathcal{H}$. In accordance with WIGNER's theorem we assume that the symmetries are represented by unitary representation on $\mathcal{H}$[1]. For the POINCARÉ group this means

$$U(a,A)\hat{\phi}(x)U^{-1}(a,A) = S\,\hat{\phi}\left((\Lambda^{-1}(x-a)\right) \tag{5.65}$$

with unitary $U(a,A)$ for $A$ from the spin group which is the quantum analog of the LORENTZ group. $\Lambda$ is the LORENTZ transformation belonging to $A$ and $S(A)$ is the representation according to which the field transforms. For translations we have

$$U(a) = \exp\left(ia^\mu \hat{P}_\mu\right), \tag{5.66}$$

such the infinitesimal form of (5.65) reads

$$[\hat{P}_\mu, \hat{\phi}] = P_\mu \hat{\phi} \quad \text{with} \quad P_\mu = i\partial_\mu. \tag{5.67}$$

For spin transformations we write

$$U(A) = \exp\left(\tfrac{i}{2}\omega^{\mu\nu}\hat{J}_{\mu\nu}\right) \tag{5.68}$$

such that

$$[\hat{J}_{\nu\nu}, \hat{\phi}] = J_{\mu\nu}\hat{\phi} \quad \text{with} \quad J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}. \tag{5.69}$$

Since $(a,A) \to U(a,A)$ is a representation of the quantum mechanical POINCARÉ group, the infinitesimal generators $(\hat{P}_\mu, \hat{J}_{\mu\nu})$ form a POINCARÉ algebra.
Let us assume, that the vacuum is left invariant by POINCARÉ transformations,

$$U(a,A)|0\rangle = |0\rangle. \tag{5.70}$$

It follows at once that a GREEN function is invariant under a simultaneous shift of its arguments. For example, the 2-point function

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|e^{-i(xP)}\phi(0)e^{i(x-y)P}\phi(0)e^{iyP}|0\rangle \\ &= \langle 0|\phi(x-y)\phi(0)|0\rangle = S(x-y)\,, \end{aligned} \tag{5.71}$$

is just a function of the vector connecting it two arguments. This expresses the homogeneity of space and time. We have skipped the hats. The GREEN functions also transform covariant under LORENTZ transformations. For example,

$$S(x) = \langle 0|\phi(x)\phi(0)|0\rangle = \langle 0|U^{-1}(A)S\phi(\Lambda^{-1}x)\,S\phi(0)U(A)|0\rangle = \langle 0|S\phi(\Lambda^{-1}x)\,S\phi(0)|0\rangle. \tag{5.72}$$

For the 2-point function of a spin-1 field this means

$$S_{\mu\nu}(x) \equiv \langle 0|A_\mu(x)A_\nu(0)|0\rangle = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma} S_{\rho\sigma}\left(\Lambda^{-1}x\right). \tag{5.73}$$

This condition heavily constrains the LORENTZ structure of the propagator in momentum space.
How does one (formally) construct the infinitesimal generators in the quantum theory? For that we recall that POINCARÉ brackets should be replaced by $-i$ times the commutator,

$$\{\phi,\pi\} = \delta \longrightarrow [\hat{\phi},\hat{\pi}] = i\delta \quad \text{and} \quad \{\phi, Q_X\} = \delta_X\phi \longrightarrow [\hat{\phi},\hat{Q}_X] = i\delta_X\hat{\phi}.$$

[1] Antilinear and antiunitary implementations would be the only other possibility.

The last relation states, that $\hat{Q}_X$ generates the infinitesimal symmetries,

$$\delta_X \hat{\phi} = i[\hat{Q}_X, \hat{\phi}], \tag{5.74}$$

and thus is the infinitesimal form of the transformation

$$\hat{\phi} \longrightarrow U \hat{\phi} U^{-1}, \qquad U = \exp\big(i\hat{Q}\big), \tag{5.75}$$

in (5.65). We conclude, that the symmetry generators $\hat{Q}$ in quantum field theory are (formally) constructed by replacing the classical field and momentum density in the expression for the classical NOETHER charges by the field operator and the momentum field operator.