Kapitel 4

Spinors

In this chapter we study CLIFFORD algebras, spinors and spin transformation in d dimensions. Most of the time we shall assume the metric to have LORENTZian signature,

$$(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, \dots, -1). \tag{4.1}$$

Sometimes we consider spaces with EUCLIDean metrics, $\eta_{\mu\nu} = \delta_{\mu\nu}$.

4.1 Gamma matrices in d dimensions

The CLIFFORD algebra is the free algebra generated by the d elements $\gamma_0, \dots, \gamma_{d-1}$, modulo the quadratic relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \equiv \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\,. \tag{4.2}$$

We give explicit representations of these algebras for spaces with LORENTZian signature in terms of the PAULI-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

The relevant properties of these *hermitean* matrices are

$$\sigma_i\sigma_j = \delta_{ij}\sigma_0 + \mathrm{i}\epsilon_{ijk}\sigma_k, \quad \text{where} \quad \sigma_0 = \mathbb{1}_2.$$

We claim, that the following matrices furnish a representation,

$$\begin{aligned} \gamma_0 &= \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots \quad \gamma_2 = \mathrm{i}\,\sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \dots \quad \gamma_4 = \mathrm{i}\,\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \dots \\ \gamma_1 &= \mathrm{i}\,\sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots \quad \gamma_3 = \mathrm{i}\,\sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \dots \quad \dots\,. \end{aligned} \tag{4.4}$$

To prove this assertion one uses

$$\{A_1 \otimes \dots \otimes A_p, B_1 \otimes \dots \otimes B_p\} = A_1B_1 \otimes \dots \otimes A_pB_p + B_1A_1 \otimes \dots \otimes B_pA_p$$

and that if $[A_i, B_i] = 0$ for all i but $i = q$, then

$$\{A_1 \otimes \dots \otimes A_p, B_1 \otimes \dots \otimes B_p\} = A_1B_1 \otimes \dots \otimes \{A_q, B_q\} \otimes \dots \otimes A_pB_p. \tag{4.5}$$

For even dimensions this yields a $2^{d/2}$-dimensional representation of the CLIFFORD algebra. For odd dimensions we may choose the representation in $d-1$ dimensions, supplemented by the matrix $\gamma_{d-1} = \mathrm{i}\,\sigma_3 \otimes \cdots \otimes \sigma_3$ ($d/2-1$ factors). This way we have found *explicit* representations in all spacetime dimensions d and

$$\gamma^\mu \in \mathrm{GL}\big(2^{[d/2]}, \mathbb{C}\big), \tag{4.6}$$

where $[a]$ is the biggest integer less or equal to a. We have the following hermiticity property of the γ-matrices,

$$\gamma_0^\dagger = \gamma_0 \quad \text{and} \quad \gamma_i^\dagger = -\gamma_i. \tag{4.7}$$

Of course the above representation is not unique. Equivalent representations preserving (4.2) are

$$\gamma'_\mu = U\gamma_\mu U^{-1}\,. \tag{4.8}$$

To preserve (4.7) the transformation U should be *unitary*. It is clear, that $\pm\gamma^\mu$ can also be used as representations for the CLIFFORD algebra. For even dimensions these are unitary equivalent to the original one. In odd dimensions one finds *two unitary inequivalent representations*. For example,

$$\gamma_0 = \sigma_1, \quad \gamma_1 = \mathrm{i}\sigma_2 \quad \text{and} \quad \gamma_2 = \pm\mathrm{i}\sigma_3$$

are two inequivalent representation of the CLIFFORD algebra in 3 dimensions.
For applications the following observations are relevant:

- In *even dimensions* a complete set of $2^{d/2}\times 2^{d/2}$ matrices is provided by the antisymmetrized products $\{\gamma^{(n)}\}$ of $n = 0, 1, \dots d$ gamma-matrices,

$$\gamma^{(n)} = \gamma_{\mu_1\dots\mu_n} \equiv \gamma_{[\mu_1}\gamma_{\mu_2}\dots\gamma_{\mu_n]} \qquad \binom{d}{n} \quad \text{matrices.} \tag{4.9}$$

  The last matrix $\gamma^{(d)}$ is proportional to

$$\gamma_* = \mathrm{i}^{1+d/2}\gamma_0\dots\gamma_{d-1} = \underbrace{\sigma_3\otimes\sigma_3\otimes\dots\otimes\sigma_3}_{d/2} \quad \text{with} \quad \gamma_*\gamma_* = 1. \tag{4.10}$$

  In addition, we have

$$\gamma_{\mu_1\dots\mu_n} = \mathrm{i}^{1+d/2}\frac{1}{(d-n)!}\,\varepsilon_{\mu_1\dots\mu_d}\,\gamma_*\gamma^{\mu_d\dots\mu_{n+1}}\,. \tag{4.11}$$

  Note that γ_* anti-commutes with all γ_μ and thus can be viewed as γ_d in $d+1$ dimensions.

- In *odd dimensions* the product of all γ-matrices is a multiple of the identity. A basis for the $2^{(d-1)/2}$-dimensional matrices is formed by $\gamma^{(n)}$ with $n = 0, 1, \dots, (d-1)/2$.

4.2 Spin transformations

In this section we study the transformation property of spinors under 'LORENTZ'-transformations. For that aim we introduce the $d(d-1)/2$ matrices

$$\Sigma^{\mu\nu} = -\Sigma^{\nu\mu} = \frac{1}{2\mathrm{i}}\gamma^{\mu\nu}. \tag{4.12}$$

They possess the following commutators with the γ-matrices and themselves:

$$\begin{aligned}[\Sigma^{\mu\nu},\gamma^\rho] &= \mathrm{i}\,(\eta^{\mu\rho}\gamma^\nu-\eta^{\nu\rho}\gamma^\mu)\\ [\Sigma_{\mu\nu},\Sigma_{\rho\sigma}] &= \mathrm{i}\,(\eta_{\mu\rho}\Sigma_{\nu\sigma}+\eta_{\nu\sigma}\Sigma_{\mu\rho}-\eta_{\mu\sigma}\Sigma_{\nu\rho}-\eta_{\nu\rho}\Sigma_{\mu\sigma})\,.\end{aligned}\tag{4.13}$$

Hence, the Σ furnish a $2^{[d/2]}$-dimensional *representation* of the LORENTZ algebra. It is the socalled *spin-representation* to be analysed below.
Let $S(s)$ be the following one-parameter family of transformations

$$\Gamma^\rho(s)=S^{-1}(s)\gamma^\rho S(s)\quad\text{with}\quad S(s)=e^{i\frac{s}{2}(\omega,\Sigma)},\qquad(\omega,\Sigma)=\omega_{\mu\nu}\Sigma^{\mu\nu}$$

with 'initial value' $S(0)=\mathbb{1}$. Making use of (4.13) one obtains the differential equation

$$\frac{d}{ds}\Gamma^\rho(s)=-\frac{i}{2}\omega_{\mu\nu}S^{-1}(s)\,[\Sigma^{\mu\nu},\gamma^\rho]\,S(s)=\omega^\rho_{\ \sigma}\Gamma^\sigma(s)$$

the solution of which reads

$$\Gamma^\rho(s)=S^{-1}(s)\gamma^\rho S(s)=\left(e^{s\omega}\right)^\rho_{\ \sigma}\gamma^\sigma.\tag{4.14}$$

Recall that for antisymmetric $(\omega_{\mu\nu})$ the matrix $(e^\omega)^\rho_{\ \sigma}$ is a LORENTZ transformation. Thus, setting $s=1$ we conclude

$$S^{-1}\gamma^\rho S=\Lambda^\rho_{\ \sigma}\gamma^\sigma,\quad\text{where}\quad S=e^{\frac{i}{2}(\omega,\Sigma)},\quad\Lambda=e^{\frac{i}{2}(\omega,M)}=e^\omega.\tag{4.15}$$

To find the action of infinitesimal LORENTZ transformation on spinors, we recall that a DIRAC spinor transforms as

$$\psi(x)\longrightarrow S\psi\big(\Lambda^{-1}x\big)\equiv e^{\frac{i}{2}(\omega,\Sigma)}\psi\left(e^{-\omega}x\right).\tag{4.16}$$

This should be compared with the transformation of scalar and vector fields,

$$\phi(x)\longrightarrow\phi\big(\Lambda^{-1}x\big)\quad\text{and}\quad A^\mu(x)\longrightarrow\Lambda^\mu_{\ \nu}A^\nu\big(\Lambda^{-1}x\big).\tag{4.17}$$

The components of a scalar field (it could be a doublet as in the electroweak theory) do not transform at all, the components of a vector field transform with the LORENTZ transformation Λ and the components of a spinor field with the spin-transformation S. The mapping $S\to\Lambda(S)$ defined in (4.15) is a group homomorphism (a representation) from the simply connected spin group into the non-simply connected LORENTZ group.
The infinitesimal form of the transformation (4.16) reads

$$\delta\psi(x)=\tfrac{i}{2}\omega_{\mu\nu}\left(L^{\mu\nu}+\Sigma^{\mu\nu}\right)\psi(x),\tag{4.18}$$

where the $\Sigma_{\mu\nu}$ generates spin rotations and

$$L_{\mu\nu}=\frac{1}{i}\big(x_\mu\partial_\nu-x_\nu\partial_\mu\big)$$

orbital transformations. The $M_{\mu\nu}$ and $\Sigma_{\mu\nu}$ both satisfy the commutation relations of the LORENTZ algebra and so does their sum

$$J_{\mu\nu}=L_{\mu\nu}+\Sigma_{\mu\nu}.\tag{4.19}$$

The $J_{\mu\nu}$ generate the spin-transformations of Dirac spinors,

$$S\psi\big(\Lambda^{-1}x\big) = \Big(e^{\frac{i}{2}(\omega,J)}\psi\Big)(x) \tag{4.20}$$

and hence generalise the total angular momentum in non-relativistic quantum mechanics.
To construct *tensor fields* which are bilinear in the spinor fields we use that $\gamma^0 = (\gamma^0)^{-1}$ conjugates the γ and Σ-matrices into their adjoints,

$$\gamma^0\gamma_\mu\gamma^0 = \gamma_\mu^\dagger \quad , \quad \gamma^0\Sigma_{\mu\nu}\gamma^0 = \Sigma_{\mu\nu}^\dagger. \tag{4.21}$$

It follows at once that γ^0 conjugates the adjoint of S into the inverse of S,

$$\gamma^0 S^\dagger\gamma^0 = \gamma^0 e^{-\frac{i}{2}(\omega,\Sigma)^\dagger}\gamma^0 = e^{-\frac{i}{2}(\omega,\Sigma)} = S^{-1} \tag{4.22}$$

Now we are ready to define the Dirac *conjugate spinor*,

$$\bar\psi \equiv \psi^\dagger\gamma^0 \Longrightarrow \bar\psi^T = \gamma^{0T}\psi^* = \gamma^{0*}\psi^* \tag{4.23}$$

which enter the expression for the fermionic bilinears. Under spin-transformations it transforms with the inverse spin rotation,

$$\bar\psi \longrightarrow (S\psi)^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0 \stackrel{(4.22)}{=} \psi^\dagger\gamma^0 S^{-1} = \bar\psi S^{-1}. \tag{4.24}$$

With the help of (4.15) it is now easy to prove that the bilinear objects

$$A^{\mu_1\cdots\mu_n} = \bar\psi\gamma^{\mu_1\cdots\mu_n}\psi \tag{4.25}$$

are *antisymmetric tensor fields*. The transformation of these objects follow from that of ψ and $\bar\psi$,

$$A^{\mu_1\cdots\mu_n} \longrightarrow \bar\psi\, S^{-1}\gamma^{\mu_1\cdots\mu_n}S\,\psi = \Lambda^{\mu_1}_{\ \ \nu_1}\cdots\Lambda^{\mu_n}_{\ \ \nu_n}\,\bar\psi\gamma^{\nu_1\cdots\nu_n}\psi = \Lambda^{\mu_1}_{\ \ \nu_1}\cdots\Lambda^{\mu_n}_{\ \ \nu_n}\,A^{\nu_1\cdots\nu_n}.$$

In particular in 4 dimensions there are 5 tensor fields

$$\bar\psi\psi \quad , \quad \bar\psi\gamma_*\psi \quad , \quad \bar\psi\gamma^\mu\psi, \quad , \quad \bar\psi\gamma_*\gamma^\mu\psi \quad , \quad \bar\psi\gamma^{\mu\nu}\psi,$$

that is a scalar, pseudo-scalar, vector, pseudo-vector and antisymmetric 2-tensor field.

4.3 Charge conjugation

The best way to see how charge conjugation emerges is to consider the Dirac equation

$$i\gamma^\mu D_\mu(e)\psi - m\psi = 0, \qquad D_\mu(e) = \partial_\mu - ieA_\mu \tag{4.26}$$

in d dimensions. We multiply this equation from the left with γ^0

$$i\gamma^0\gamma^\mu\gamma^0 D_\mu(e)\gamma^0\psi - m\gamma^0\psi = i\gamma^{\mu\dagger}D_\mu(e)\gamma^0\psi - m\gamma^0\psi = 0.$$

With (4.23) the complex conjugate of this equation takes the form

$$-i\gamma^{\mu T}D_\mu(-e)\bar\psi^T - m^*\bar\psi^T = 0. \tag{4.27}$$

Let us now assume that there exists a *charge conjugation matrix* $\mathcal{C}$ which fulfils

$$\mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = \eta\,\gamma^\mu, \quad \text{with} \quad \eta = \pm 1, \tag{4.28}$$

in which case we define the *charge conjugated spinor*

$$\psi_{\rm C} = \mathcal{C}\bar\psi^T = \mathcal{C}\gamma_0^T\psi^*. \tag{4.29}$$

Now we multiply (4.27) with $\mathcal{C}$ from the left and obtain

$$i\gamma^\mu D_\mu(-e)\psi_{\rm C} + \eta m^*\psi_{\rm C} = 0. \tag{4.30}$$

We see that for a vanishing mass $\psi_{\rm C}$ fulfils the DIRAC equation with reversed electric charge and this justifies its name charge conjugates spinor[1]. Since γ_0 is hermitean and the γ_i are antihermitean we have

$$\gamma_0^* = \eta\,\mathcal{C}^{-1}\gamma_0\mathcal{C} \quad\text{and}\quad \gamma_i^* = -\eta\,\mathcal{C}^{-1}\gamma_i\mathcal{C}. \tag{4.31}$$

If there is a representation with *real* or *imaginary* γ's then we may choose $\mathcal{C}$ proportional to γ^0,

$$\begin{array}{llll} \gamma^\mu \text{ real:} & \mathcal{C} = \alpha\gamma^0 = \mathcal{C}^T, & \eta = 1, & \psi_{\rm C} = \alpha\psi^* \\ \gamma^\mu \text{ imaginary:} & \mathcal{C} = \alpha\gamma^0 = -\mathcal{C}^T, & \eta = -1, & \psi_{\rm C} = -\alpha\psi^* \end{array} \tag{4.32}$$

Such representations are called MAJORANA representations. For a MAJORANA representation and for $\alpha = \eta$ charge conjugation becomes complex conjugation. A *spinor* invariant under charge conjugations is called MAJORANA spinor. In a MAJORANA representation a MAJORANA spinor is real.
Now we are going to prove that there always exist a *symmetric* or *antisymmetric* matrix $\mathcal{C}$ with

$$\gamma_\mu^T = \eta\,\mathcal{C}^{-1}\gamma_\mu\mathcal{C}, \qquad \eta = \pm 1. \tag{4.33}$$

A formal proof (for arbitrary dimensions and signatures) can be found in [20, 21], but we may as well give explicit solutions in the representation (4.4). We use

$$\sigma_1\sigma_1\sigma_1 = \sigma_1^T \quad , \quad \sigma_1\sigma_2\sigma_1 = \sigma_2^T \quad , \quad \sigma_1\sigma_3\sigma_1 = -\sigma_3^T \quad\text{and}\quad \sigma_2\sigma_i\sigma_2 = -\sigma_i^T \tag{4.34}$$

to show that the hermitean and unitary matrices

$$\begin{array}{lcll} \mathcal{C}_+ & = & \sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\dots & \eta = +1 \\ \mathcal{C}_- & = & \sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\dots \propto \mathcal{C}_+\gamma_* & \eta = -1 \end{array} \tag{4.35}$$

are solutions of (4.33) in even dimensions. Being hermitean and unitary they square to $\mathbb{1}$. This then proves that there is a charge conjugation matrix in even dimensions.
Let $\gamma_0, \dots, \gamma_{d-1}$ be the γ-matrices for a space with even dimension d. As γ-matrices for the space with odd dimension $d+1$ we may take these γ-matrices, supplemented by $\gamma_d = \alpha\gamma_0\cdot\gamma_{d-1}$. The phase α must be chosen such that γ_d squares to $-\mathbb{1}$. This matrix must fulfil (4.33),

$$\mathcal{C}^{-1}\gamma_d\,\mathcal{C} = \alpha\eta^d\gamma_0^T\cdots\gamma_{d-1}^T \overset{d\text{ even}}{=} \alpha\,(\gamma_{d-1}\cdots\gamma_0)^T = (-)^{d/2}\gamma_d^T \overset{!}{=} \eta\,\gamma_d^T, \tag{4.36}$$

which implies $\eta = (-)^{d/2}$. Hence $\mathcal{C}_+$ is a charge conjugation matrix in $1+4n$ dimensions and $\mathcal{C}_-$ in $3+4n$ dimensions. In the following table we have summarised our findings. The symbols S and A denote symmetric and antisymmetric matrices. For example, there is an antisymmetric $\mathcal{C}_-$ solution but no $\mathcal{C}_+$-solution in 3 dimensions. Since the results are identical in d and $d+8n$ dimensions, it is sufficient to give the results for $d = 1, \dots, 8$:

d	1	2	3	4	5	6	7	8
$\mathcal{C}_+$	S	S		A	A	A		S
$\mathcal{C}_-$		A	A	A		S	S	S

(4.37)

[1] In the presence of a real mass we would want $\eta = -1$.

If one changes the representation as in (4.8), then, to preserve (4.33), the charge conjugation matrix must transform as

$$\mathcal{C}' = U\mathcal{C}U^T\,. \tag{4.38}$$

Hence $\mathcal{C}$ in unitary and (anti)symmetric in *any representation*, but need not always square to $\mathbb{1}$.

4.3.1 Explicit Majorana representations

Sometimes it is useful to have an explicit MAJORANA representation for the γ-matrices at hand. As will be demonstrated below, such representations only exist in dimensions d for which

$$d \bmod 8 \in \{0,1,2,3,4\}.$$

We give MAJORANA representations in $2,3$ and 4 spacetime dimensions.

- In 2 dimensions we may choose imaginary γ-matrices

$$\gamma^0 = \sigma_2,\quad \gamma^1 = i\sigma_1 \qquad \text{with}\quad \gamma_* = -\gamma_0\gamma_1 = \sigma_3,\quad \mathcal{C} = -\gamma^0, \tag{4.39}$$

  or real γ-matrices

$$\gamma^0 = \sigma_3,\quad \gamma^1 = i\sigma_2 \qquad \text{with}\quad \gamma_* = -\gamma_0\gamma_1 = \sigma_1,\quad \mathcal{C} = \gamma^0 \tag{4.40}$$

- In 3 dimensions there exists the imaginary representation

$$\gamma^0 = \sigma_2,\quad \gamma^1 = i\sigma_1,\quad \gamma^2 = i\sigma_3 \quad \text{with}\quad \mathcal{C} = -\gamma^0 \tag{4.41}$$

- In 4 dimensions there is again an imaginary representation

$$\begin{gathered}\gamma^0 = \sigma_0\otimes\sigma_2,\quad \gamma^1 = i\sigma_0\otimes\sigma_3,\quad \gamma^2 = i\sigma_1\otimes\sigma_1,\quad \gamma^3 = i\sigma_3\otimes\sigma_1\\ \text{with}\quad \gamma_* = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_2\otimes\sigma_1 \quad\text{and}\quad \mathcal{C} = -\gamma^0.\end{gathered} \tag{4.42}$$

4.4 Irreducible spinors

We do not know whether the spinor representations (4.16) are *irreducible*. In general they are not and there are two types of projections onto invariant subspaces that one can envisage:
• The first exist in even dimensions, where one has a γ_* anti-commuting with all γ-matrices. This element can be used to define *left and right spinors*,

$$\psi_L = \tfrac{1}{2}\left(1-\gamma_*\right)\psi \equiv P_L\psi\,,\qquad \psi_R = \tfrac{1}{2}\left(1+\gamma_*\right)\psi \equiv P_R\psi\,. \tag{4.43}$$

Since γ_* commutes with the LORENTZ generators $\Sigma_{\mu\nu}$ the left- and right-handed spinors form invariant subspaces with respect to spin rotations,

$$\delta(\psi_L) = \big(\delta\psi\big)_L,\qquad \delta\psi = \tfrac{i}{2}(\omega,\Sigma)\psi. \tag{4.44}$$

• The second possible projection is a *reality condition*. Consider first the reality properties of the γ matrices. Combining (4.21) and (4.28), we have

$$\mathcal{C}\gamma_0^T\gamma_\mu^*\gamma_0^T\mathcal{C}^{-1} = \mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = \eta\gamma_\mu \Longrightarrow \mathcal{C}\gamma_0^T\Sigma_{\mu\nu}^*\gamma_0^T\mathcal{C}^{-1} = -\Sigma_{\mu\nu}. \tag{4.45}$$

Now we try to impose the MAJORANA *reality condition*

$$\psi = \psi_C = \mathcal{C}\gamma_0^T\psi^*. \tag{4.46}$$

Consistency with LORENTZ transformations demands

$$\mathcal{C}\gamma_0^T(\delta\psi)^* = \delta\psi,$$

which for infinitesimal LORENTZ transformations (4.44) is just the condition (4.45). If a spinor is MAJORANA in one inertial frame then it is MAJORANA in any other inertial frame. Since $\psi^{**} = \psi$ we must demand

$$(\mathcal{C}\gamma_0^T)^*(\mathcal{C}\gamma_0^T) = \mathcal{C}^*\gamma_0\mathcal{C}\gamma_0^T \overset{(4.28)}{=} \eta\,\mathcal{C}^*\mathcal{C} \overset{!}{=} \mathbb{1} \quad \text{or} \quad \mathcal{C}^* = \eta\,\mathcal{C}^{-1} = \eta\,\mathcal{C}^\dagger. \tag{4.47}$$

Since $\mathcal{C}$ is unitary this condition is equivalent to $\mathcal{C}^T = \eta\,\mathcal{C}$. Comparing with the above table leads to the solutions given in the following table:

d	1	2	3	4	5	6	7	8
$\mathcal{C}_+$	S	S						S
$\mathcal{C}_-$		A	A	A				
		MW				SM		

(4.48)

In cases there are no MAJORANA spinors, there is still another possibility for theories with extended supersymmetry. One can define *symplectic* MAJORANA *spinors*

$$\psi_i = \mathcal{C}\gamma_0^T\,\Omega_{ij}\psi_j^*\,, \tag{4.49}$$

where Ω is some antisymmetric matrix, with $\Omega\Omega^* = -1$. Symplectic MAJORANA spinors exist in $6, 14, \dots$ dimensions. The existence of symplectic MAJORANA spinors is marked with SM in the above table.

Having two projections, to chiral spinors and to MAJORANA spinors, one may ask whether one can define a reality condition respecting the chiral projection. For the left handed spinors this means

$$P_L\mathcal{C}\gamma_0^T = \mathcal{C}\gamma_0^T P_L^* \quad \text{or} \quad \gamma_*\mathcal{C}\gamma_0^T = \mathcal{C}\gamma_0^T\gamma_*^*. \tag{4.50}$$

On the other hand, the definition (4.10) of γ_* together with (4.45) imply

$$\mathcal{C}\gamma_0^T\gamma_*^* = -\mathrm{i}^d\gamma_*\mathcal{C}\gamma_0^T$$

and comparing with (4.50) leads to

$$d \bmod 8 \in \{2, 6\}$$

Since there are no MAJORANA fermions in 6 dimensions the second solutions does not exist. Hence there are only MAJORANA-WEYL spinors in $2, 10, 18, \dots$ dimensions.

4.5 Chiral representations

When dealing with chiral fermions, which exist in all even dimensions, it is convenient to use a chiral representation for then γ-matrices. These are representations for which γ_* is diagonal,

$\gamma_* = -\sigma_3 \otimes \mathbb{1}$. Then the chiral projectors onto the left- and right handed spinors (often called *chiral* and *anti-chiral fermions*) are diagonal as well.

$$P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{4.51}$$

Left-handed spinors have only upper and right-handed spinors only lower components in these chiral representations. The left-handed object λ_α with lower components is called *chiral spinor* while the right-handed object $\bar{\lambda}^{\dot{\alpha}}$ with upper components is called *anti-chiral spinor*. Sometimes one calls them left-handed and right-handed WEYL-spinors.
The following representations in even dimensions are chiral:

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma_j = \begin{pmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{pmatrix}, \quad j = 1, \dots, d-1, \tag{4.52}$$

with hermitean α_j generating an EUCLIDean CLIFFORD algebra in one dimension less,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}. \tag{4.53}$$

Without loss of generality we may assume that the hermitean matrix $\mathrm{i}^{1-d/2}\alpha_1 \cdots \alpha_{d-1}$ is the identity matrix. Then we find

$$\gamma_* = \mathrm{i}^{1+d/2}\gamma_0 \cdots \gamma_{d-1} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = -\sigma_3 \otimes \mathbb{1}. \tag{4.54}$$

4.5.1 4 dimensions

In this most important case we choose the chiral representation

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad \sigma_\mu = (\sigma_0, -\sigma_i), \quad \tilde{\sigma}_\mu = (\sigma_0, \sigma_i) \tag{4.55}$$

for which the infinitesimal spinor-rotations have the form

$$\gamma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde{\sigma}_{\mu\nu} \end{pmatrix} \quad \text{with} \quad \sigma_{\mu\nu} = \tfrac{1}{2}(\sigma_\mu\tilde{\sigma}_\nu - \sigma_\nu\tilde{\sigma}_\mu), \quad \tilde{\sigma}_{\mu\nu} = \tfrac{1}{2}(\tilde{\sigma}_\mu\sigma_\nu - \tilde{\sigma}_\nu\sigma_\mu). \tag{4.56}$$

Note that the σ_{oi} are hermitean, whereas the σ_{ij} are antihermitian, $\sigma^\dagger_{\mu\nu} = -\tilde{\sigma}_{\mu\nu}$. It follows that the spin rotations have the form

$$S = \exp\left(\tfrac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\right) = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger-1} \end{pmatrix} \quad \text{with} \quad A = \exp\left(\tfrac{1}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right) \in SL(2,\mathbb{C}). \tag{4.57}$$

The complex 2-dimensional matrix A has determinant one since the $\sigma_{\mu\nu}$ are traceless. A Dirac spinor consists of a left-handed and a right-handed part,

$$\psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{4.58}$$

The left-handed spinors transform with A and the right-handed with $A^{\dagger-1}$

$$\lambda_\alpha \longrightarrow A_\alpha{}^\beta \lambda_\beta \quad , \quad \bar{\chi}^{\dot{\alpha}} \longrightarrow (A^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}} \tag{4.59}$$

Next we define the ε-tensors

$$(\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon^{\alpha\beta}) = (\varepsilon^{\dot\alpha\dot\beta}) \quad = \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.60}$$

which obey the relations,

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma \quad , \quad \varepsilon_{\alpha\beta}\varepsilon^{\delta\gamma} = \delta_\alpha^\gamma\delta_\beta^\delta - \delta_\alpha^\delta\delta_\beta^\gamma. \tag{4.61}$$

Because $A^T\varepsilon A = \varepsilon$ for any matrix A with determinant one, the following bilinears are Lorentz-invariant,

$$\lambda\chi = \lambda^\alpha\chi_\alpha \quad \text{and} \quad \bar\lambda\bar\chi = \bar\lambda_{\dot\alpha}\bar\chi^{\dot\alpha}, \tag{4.62}$$

where the raising and lowering of indexes are done with ε,

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta, \quad \lambda_\alpha = \varepsilon_{\alpha\beta}\lambda^\beta \quad \text{and} \quad \bar\lambda^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta}, \quad \bar\lambda_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\beta}. \tag{4.63}$$

The condition $A^T\varepsilon A = \varepsilon$ translates into

$$\left(A^{T-1}\right)^\alpha{}_\beta = \varepsilon^{\alpha\rho}A_\rho{}^\sigma\varepsilon_{\sigma\beta} \quad \text{and} \quad \left(A^{\dagger-1}\right)^{\dot\alpha}{}_{\dot\beta} = \varepsilon^{\dot\alpha\dot\rho}\bar A_{\dot\rho}{}^{\dot\sigma}\varepsilon_{\dot\sigma\dot\beta} \tag{4.64}$$

which means, that A can be conjugated into A^{T-1} and $\bar A$ into $A^{\dagger-1}$. Using these formulae one can show that the spin-transformation in (4.59) are equivalent to

$$\lambda^\alpha \longrightarrow \left(A^{T-1}\right)^\alpha{}_\beta\lambda^\beta \quad , \quad \bar\chi_{\dot\alpha} \longrightarrow \bar A_{\dot\alpha}{}^{\dot\beta}\bar\chi_{\dot\beta} \tag{4.65}$$

The components λ^α transform as the complex conjugate of $\bar\lambda^{\dot\alpha}$ and the components $\bar\chi_{\dot\alpha}$ as the complex conjugate of the χ_α. The index structure of the Dirac-conjugate spinor is

$$\bar\psi = (\chi^\alpha, \bar\lambda_{\dot\alpha}) \quad \text{such that} \quad \bar\psi\psi = \chi\lambda + \bar\lambda\bar\chi. \tag{4.66}$$

A mass term vanishes for left- or for right-handed spinors. Since σ_μ maps right- into left-handed spinors and $\tilde\sigma_\mu$ left- into right-handed ones they have the index structure

$$(\sigma_\mu)_{\alpha\dot\beta} \quad \text{and} \quad (\tilde\sigma_\mu)^{\dot\alpha\beta}. \tag{4.67}$$

The generator $\sigma_{\mu\nu}$ maps left- into left-handed spinors and $\tilde\sigma_{\mu\nu}$ right- into right-handed ones and they have the index structure

$$(\sigma_{\mu\nu})_\alpha{}^\beta \quad \text{and} \quad (\tilde\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}. \tag{4.68}$$

Since spinor components are Grassmann variables we find the following symmetry properties,

$$\begin{aligned} \lambda\chi &= \lambda^\alpha\chi_\alpha = -\chi_\alpha\lambda^\alpha = \chi^\alpha\lambda_\alpha = \chi\lambda \\ \bar\lambda\bar\chi &= \bar\lambda_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\lambda_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\lambda^{\dot\alpha} = \bar\chi\bar\lambda \end{aligned} \tag{4.69}$$

The vector current can be written as

$$\bar\psi\gamma^\mu\psi = \bar\lambda_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\beta}\lambda_\beta + \chi^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\chi^{\dot\beta} = \bar\lambda\,\tilde\sigma^\mu\lambda + \chi\sigma^\mu\bar\chi, \tag{4.70}$$

so that the 'kinetic term' for fermions reads

$$\bar\psi\partial\!\!\!/\psi = \bar\lambda\,\tilde\sigma^\mu\partial_\mu\lambda + \chi\sigma^\mu\partial_\mu\bar\chi. \tag{4.71}$$

4.5.2 SL(2,$\mathbb{C}$)-representations

Every finite-dimensional representation of $SL(2,\mathbb{C})$ can be gotten by tensoring two fundamental 2-dimensional representations. The knowledge of the representation theory of $SL(2,\mathbb{C})$ is necessary to construct Lorentz invariant field equations.
The components λ_α and $\bar\lambda_{\dot\alpha}$ (with lower indexes) are called covariant. The two inequivalent fundamental representations of $SL(2,\mathbb{C})$ are

$$\lambda_\alpha \to A_\alpha{}^\beta \lambda_\beta \quad \text{and} \quad \bar\lambda_{\dot A} \to \bar A_{\dot\alpha}{}^{\dot\beta}\bar\lambda_{\dot\beta}. \tag{4.72}$$

The transformation of the different components are summarised in the following table:

spinor	λ_α	$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta$	$\bar\lambda_{\dot\alpha}$	$\bar\lambda^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta}$
transforms with	A	$\varepsilon A\varepsilon^{-1} = (A^T)^{-1}$	$\bar A$	$\varepsilon\bar A\varepsilon^{-1} = (A^\dagger)^{-1}$

The representations A^{T-1} and $A^{\dagger-1}$ are equivalent to the representations A and $\bar A$. One uses the following notations

$$\begin{aligned} &\text{representation} \quad D^{(1/2,0)}: && \lambda \to A\lambda \\ &\text{representation} \quad D^{(0,1/2)}: && \bar\lambda \to \bar A\bar\lambda. \end{aligned} \tag{4.73}$$

The tensor product of n un-dotted and m dotted spinors yield a spinor of rank $n+m$ with n un-dotted and m dotted indexes. Such a spinor transforms as follows under $SL(2,\mathbb{C})$ transformations:

$$\lambda_{\alpha_1\dots\alpha_n\dot\alpha_1\dots\dot\alpha_m} \longrightarrow A_{\alpha_1}^{\beta_1}\cdots A_{\alpha_n}^{\beta_n}\bar A_{\dot\alpha_1}^{\dot\beta_1}\cdots\bar A_{\dot\alpha_m}^{\dot\beta_m}\,\lambda_{\beta_1\dots\beta_n\dot\beta_1\dots\dot\beta_m}.$$

These representations are *reducible* if $(n,m)\neq(\frac{1}{2},0)$ or $\neq(0,\frac{1}{2})$.

4.6 Composite fields

In this section we collect and partly prove some useful properties and formulae for fermionic bilinears and other composites. These technical results will later be of use in our study of supersymmetric field theories.

4.6.1 Bilinears for charge-conjugated fields

The Dirac-conjugate of the charge conjugated spinor $\psi_{\rm C} = \mathcal{C}\bar\psi^T$ has the form

$$\psi_{\rm C} = \mathcal{C}\bar\psi^T, \quad \bar\psi_C = \psi_{\rm C}^\dagger\gamma_0 = \psi^T\gamma_0^T\mathcal{C}^{-1}\gamma_0 = \eta\psi^T\mathcal{C}^{-1} \tag{4.74}$$

and is used to compute the bilinears for the charge conjugated fields,

$$\begin{aligned} \bar\psi_{\rm C}\gamma^{(n)}\chi_{\rm C} &= \eta\,\psi^T\mathcal{C}^{-1}\gamma^{(n)}\mathcal{C}\bar\chi^T = \eta^{1+n}(-1)^{n(n-1)/2}\psi^T\gamma^{(n)T}\bar\chi^T \\ &= -\eta^{1+n}(-1)^{n(n-1)/2}\bar\chi\gamma^{(n)}\psi. \end{aligned}$$

In the second last step we used, that the components of fermion fields *anticommute.* In particular, for Majorana spinors we find

$$\bar\psi\gamma^{(n)}\chi = -\eta^{1+n}(-1)^{n(n-1)/2}\,\bar\chi\gamma^{(n)}\psi. \tag{4.75}$$

In 4 dimensions and $\eta=-1$ we get for Majorana spinors

$$\begin{aligned} \bar\psi\chi = \bar\chi\psi, \quad \bar\psi\gamma_5\chi = \bar\chi\gamma_5\psi, \quad \bar\psi\gamma_5\gamma_\mu\chi = \bar\chi\gamma_5\gamma_\mu\psi \\ \bar\psi\gamma_\mu\chi = -\bar\chi\gamma_\mu\psi \quad \text{and} \quad \bar\psi\gamma_{\mu\nu}\chi = -\bar\chi\gamma_{\mu\nu}\psi. \end{aligned} \tag{4.76}$$

4.6.2 Hermitean conjugation

We *define* the hermitean conjugate as if the spinor-components are operators in a HILBERT-space (which they are in quantum field theory). For example

$$(\bar\psi\chi)^\dagger \equiv \chi^\dagger\bar\psi^\dagger \tag{4.77}$$

and the GRASSMANN-property has not been used. Now it follows, that

$$(\bar\psi M\chi)^\dagger = \chi^\dagger M^\dagger\bar\psi^\dagger = -\bar\psi^* M^*\chi^* = -\eta\,\bar\psi_{\rm C} M_C\chi_{\rm C}, \qquad M_{\rm C} = \mathcal{C}\gamma_0^T M^*\gamma_0^T\mathcal{C}^{-1}. \tag{4.78}$$

This formula shows that we can hermitean conjugate by using the $\mathcal{C}$ operation. In a MAJORANA representation we choose $\mathcal{C}\gamma_0^T = \mathbb{1}$ (such that a MAJORANA spinor is real) and obtain

$$(\bar\psi M\chi)^\dagger = -\eta\,\bar\psi_C M^*\chi_C \qquad \text{in MAJORANA representation.} \tag{4.79}$$

For imaginary γ-matrices we have $\eta = -1$ and for real γ-matrices $\eta = 1$. For MAJORANA spinors the bilinear tensor fields are hermitean or antihermitean, properties,

$$\begin{array}{rl} \bar\psi\chi,\ \bar\psi\gamma^{\mu\nu}\chi,\ \dots & \text{hermitean} \\ \bar\psi\gamma^{\mu}\chi,\ \bar\psi\gamma^{\mu\nu\rho}\chi,\ \dots & \text{antihermitean.} \end{array} \tag{4.80}$$

In 4 dimensions and with $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ we have

$$\begin{array}{rl} \bar\psi\chi,\ \bar\psi\gamma^{\mu\nu}\chi,\ \bar\psi\gamma_5\gamma^{\mu}\chi & \text{hermitean} \\ \bar\psi\gamma^{\mu}\chi,\ \bar\psi\gamma_5\chi & \text{antihermitean.} \end{array} \tag{4.81}$$

4.6.3 Fierz identities for Dirac spinors

In d dimensions the γ^μ are Δ-dimensional matrices, where $\Delta = 2^{[d/2]}$. As *orthonormal basis* in the linear space of $\mathrm{Mat}(\Delta,\mathbb{C})$ with scalar product $(M,N) = \mathrm{Tr}\,M^\dagger N$ we may choose

$$\frac{1}{\sqrt{\Delta}}\big\{\mathbb{1},\gamma_\mu,\gamma_{\mu_1\mu_2},\dots,\gamma_{\mu_1\dots\mu_D}\big\}, \quad \text{where} \quad \begin{cases} D = d & \text{for even } d \\ D = \frac{1}{2}(d-1) & \text{for odd } d\,, \end{cases} \tag{4.82}$$

where we assume $\mu_1 < \mu_2 < \dots$. With $\gamma_\mu^\dagger = \gamma^\mu$ we have

$$\gamma^\dagger_{\mu_1\mu_2\dots\mu_n} = \big(\gamma_{\mu_1}\gamma_{\mu_2}\dots\gamma_{\mu_n}\big)^\dagger = \gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1} = (-)^{n(n-1)/2}\gamma^{\mu_1\mu_2\dots\mu_n}. \tag{4.83}$$

Therefore every $\Delta\times\Delta$-matrix M can be expanded as

$$M = \frac{1}{\Delta}\sum_{n=0}^{D}\frac{1}{n!}\,(-)^{n(n-1)/2}\gamma_{\mu_1\dots\mu_n}\,\mathrm{Tr}\,\left(\gamma^{\mu_1\dots\mu_n}M\right)\,. \tag{4.84}$$

Since the sums extend over all indexes $\mu_1,\dots,\mu_n$ and not only the ordered ones we needed corrected this over counting by the factor $n!$ in the denominator. Now we take two spinors ψ and χ whose components anticommute and choose

$$M = \psi\bar\chi \quad \text{or} \quad M_\alpha^{\ \beta} = \psi_\alpha\bar\chi^\beta \quad \text{such that} \quad \mathrm{Tr}\big(\gamma^{\mu_1\dots\mu_n}M\big) = -\bar\chi\gamma^{\mu_1\dots\mu_n}\psi.$$

The minus sign originate from the anticommuting nature of the spinor components. The identity (4.84) implies the following general FIERZ-identity

$$\psi\bar\chi = -\frac{1}{\Delta}\sum_n\frac{1}{n!}\,(-)^{n(n-1)/2}\,\gamma_{\mu_1\dots\mu_n}\,\big(\bar\chi\gamma^{\mu_1\dots\mu_n}\psi\big). \tag{4.85}$$

These general identity allows for the expansion of the matrix $\psi\bar\chi$ as linear combination of the basis elements $\gamma^{(n)}$. The expansion coefficients are the antisymmetric tensor fields $\bar\chi\gamma^{(n)}\psi$. For example, in 4 dimensions

$$4\psi\bar\chi = -(\bar\chi\psi) - \gamma_\mu(\bar\chi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\chi\gamma^{\mu\nu}\psi) + \tfrac{1}{3!}\gamma_{\mu\nu\rho}(\bar\chi\gamma^{\mu\nu\rho}\psi) - \tfrac{1}{4!}\gamma_{\mu\nu\rho\sigma}(\bar\chi\gamma^{\mu\nu\rho\sigma}\psi). \tag{4.86}$$

Now we may use (4.11)

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_{\mu\nu\rho} = i\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma \quad \text{and} \quad \gamma_{\mu\nu\rho\sigma} = i\epsilon_{\mu\nu\rho\sigma}\gamma_5$$

to rewrite the Fierz-identity as follows

$$4\psi\bar\chi = -(\bar\chi\psi) - \gamma_\mu(\bar\chi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\chi\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar\chi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\chi\gamma_5\psi). \tag{4.87}$$

In particular, we find

$$\begin{aligned}
4\psi\bar\psi &= -(\bar\psi\psi) - \gamma_\mu(\bar\psi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar\psi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\psi\gamma_5\psi) && (4.88)\\
4\gamma_5\psi\bar\psi\gamma_5 &= -(\bar\psi\psi) + \gamma_\mu(\bar\psi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) - \gamma_5\gamma_\mu(\bar\psi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\psi\gamma_5\psi) && (4.89)\\
\psi\bar\psi + \gamma_5\psi\bar\psi\gamma_5 &= -\tfrac{1}{2}(\bar\psi\psi) + \tfrac{1}{4}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) - \tfrac{1}{2}\gamma_5(\bar\psi\gamma_5\psi). && (4.90)
\end{aligned}$$

For Majorana *spinors* the second last term vanishes because of (4.76). Multiplying with ψ from the right, we find

$$\psi(\bar\psi\psi) + \gamma_5\psi(\bar\psi\gamma_5\psi) = 0, \qquad (\psi \text{ Majorana}). \tag{4.91}$$

More useful Fierz identities for Dirac- and Majorana spinors can be found in [23].

4.6.4 Fierz identities for Weyl-spinors

The following table contains a rather useful list of Fierz-identities for chiral fermions.

$$\begin{aligned}
(\theta\lambda)(\theta\chi) &= -\tfrac{1}{2}(\theta\theta)(\lambda\chi) && (4.92)\\
(\bar\theta\bar\lambda)(\bar\theta\bar\chi) &= -\tfrac{1}{2}(\bar\lambda\bar\chi)(\bar\theta\bar\theta) && (4.93)\\
\lambda\sigma_\mu\bar\chi &= -\bar\chi\tilde\sigma_\mu\lambda && (4.94)\\
(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &= \tfrac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta) && (4.95)\\
(\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) &= \tfrac{1}{2}\eta^{\mu\nu}\theta_\alpha(\bar\theta\bar\theta) - i(\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta) && (4.96)\\
(\theta\lambda)(\bar\theta\bar\chi) &= \tfrac{1}{2}(\theta\sigma^\mu\bar\theta)(\lambda\sigma_\mu\bar\chi) && (4.97)\\
(\bar\theta\bar\chi)(\theta\lambda) &= \tfrac{1}{2}(\theta\sigma^\mu\bar\theta)(\lambda\sigma_\mu\bar\chi) = (\theta\lambda)(\bar\theta\bar\chi) && (4.98)\\
(\theta\lambda)(\bar\chi\tilde\sigma^\mu\theta) &= -\tfrac{1}{2}\theta^2(\bar\chi\tilde\sigma^\mu\lambda) && (4.99)\\
(\bar\theta\bar\lambda)(\chi\sigma^\mu\bar\theta) &= -\tfrac{1}{2}\bar\theta^2(\chi\sigma^\mu\bar\lambda) && (4.100)\\
(\theta\lambda)(\chi\sigma^\mu\bar\theta) &= -\tfrac{1}{2}(\theta\sigma^\nu\bar\theta)(\lambda\sigma_\nu\tilde\sigma^\mu\chi) && (4.101)\\
(\bar\theta\bar\lambda)(\bar\chi\tilde\sigma^\mu\theta) &= \tfrac{1}{2}(\theta\sigma^\nu\bar\theta)(\bar\lambda\tilde\sigma_\nu\sigma^\mu\bar\chi) && (4.102)\\
(\bar\lambda\tilde\sigma^\mu\theta)(\theta\sigma^\nu\bar\theta) &= -\tfrac{1}{2}\theta^2\bar\theta(\tilde\sigma^\nu\sigma^\mu)\bar\lambda && (4.103)\\
(\lambda\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &= \tfrac{1}{2}\bar\theta^2\theta(\sigma^\nu\tilde\sigma^\mu)\lambda && (4.104)
\end{aligned}$$

Most of these identities are easily be proven by using the explicit representation

$$\begin{aligned}
\lambda\chi &= \lambda^\alpha\chi_\alpha = -\lambda^1\chi^2 + \lambda^2\chi^1 = -\lambda_1\chi_2 + \lambda_2\chi_1, \quad \theta^1 = \theta_2,\ \theta^2 = -\theta_1\\
\bar\lambda\bar\chi &= \bar\lambda_{\dot\alpha}\bar\chi^{\dot\alpha} = +\bar\lambda^{\dot1}\bar\chi^{\dot2} - \bar\lambda^{\dot2}\bar\chi^{\dot1} = +\bar\lambda_{\dot1}\bar\chi_{\dot2} - \bar\lambda_{\dot2}\bar\chi_{\dot1}, \quad \bar\theta^{\dot1} = \bar\theta_{\dot2},\ \bar\theta^{\dot2} = -\bar\theta_{\dot1}.
\end{aligned} \tag{4.105}$$

together with the anticommutation relations.

4.7 Spinors in Euclidean spaces

The 'LORENTZ transformations' on a d-dimensional space with EUCLIDEAN metric are the rotations in $SO(d)$. The space symmetries form a compact LIE group. The γ^μ are hermitean and so are the generators $\Sigma_{\mu\nu}$ of the spin group. It follows that the spin transformation S in

$$\psi(x) \longrightarrow S\psi\big(\Lambda^{-1}x\big) \tag{4.106}$$

is a unitary matrix. As explicit representation for the EUCLIDean γ's we may take the matrices (4.4) without the factors i. It follows, that the charge conjugation matrices $\mathcal{C}_\pm$ in (4.35) are still solutions of

$$\gamma_\mu^T = \eta\,\mathcal{C}^{-1}\gamma_\mu\mathcal{C} \quad \text{with} \quad \eta = \pm 1. \tag{4.107}$$

Since the γ^μ are hermitean, the charge conjugation of a spinor in EUCLIDean spaces has a different form as in MINKOWSKI space times,

$$\psi_{\rm C} = \mathcal{C}\psi^*. \tag{4.108}$$

The MAJORANA condition $\psi_{\rm C} = \psi$ is again consistent with rotational invariance. Now the consistency condition $\psi^{**} = \psi$ implies the *symmetry* of the charge-conjugation matrix, $\mathcal{C} = \mathcal{C}^T$. Comparing with table (4.37) yields the following solutions

d	1	2	3	4	5	6	7	8
$\mathcal{C}_+$	S	S						S
$\mathcal{C}_-$						S	S	S
				SM				MW

(4.109)

The Existence of MAJORANA-WEYL fermions in $8, 16, 24, \dots$ dimensions can be proved similarly as in MINKOWSKI space times. Symplectic MAJORANA fermions exist only in $4, 12, 20, \dots$ dimensions. Since spin transformations are unitary, the antisymmetric tensor fields in EUCLIDean spaces are

$$A_{\mu_1\mu_2\dots\mu_n} = \psi^\dagger\gamma_{\mu_1\mu_2\dots\mu_n}\psi. \tag{4.110}$$

Since the hermiticity relations (4.83) hold true for all signatures, the general FIERZ-identity remain unaltered,

$$\psi\chi^\dagger = -\frac{1}{\Delta}\sum_n \frac{1}{n!}\,(-)^{n(n-1)/2}\,\gamma_{\mu_1\dots\mu_n}\,\big(\chi^\dagger\gamma^{\mu_1\dots\mu_n}\psi\big). \tag{4.111}$$

Note that for the metric $\delta_{\mu\nu}$ we have $\gamma^{\mu_1\mu_2\cdots} = \gamma_{\mu_1\mu_2\dots}$. The identity (4.86) in 4 dimensions still holds true (of course, after setting $\bar\chi = \chi^\dagger$). The definition of γ_5 is slightly modified in order to get a hermitean matrix which squares to $\mathbb{1}$,

$$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3 \quad \text{and} \quad \gamma_{\mu\nu\rho} = -\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma \quad \text{and} \quad \gamma_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}\gamma_5.$$

Nevertheless, the FIERZ-identity (4.87) remains unchanged,

$$4\psi\chi^\dagger = -(\chi^\dagger\psi) - \gamma_\mu(\chi^\dagger\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\chi^\dagger\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\chi^\dagger\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\chi\gamma_5\psi). \tag{4.112}$$

When one computes the fermionic bilinears for the charge conjugated fields one obtains

$$\psi^\dagger\gamma^{(n)}\chi = -\eta^n(-1)^{n(n-1)/2}\,\chi^\dagger\gamma^{(n)}\psi. \tag{4.113}$$

since in the relation $\psi_{\rm C}^\dagger = \psi^T \mathcal{C}^{-1}$ there is no η, contrary to the corresponding identity (4.74) in Minkowski space time.
Let us finally investigate the hermiticity properties of the bilinears. The relation (4.78) becomes

$$(\psi^\dagger M\chi)^\dagger = \chi^\dagger M^\dagger \psi = -\psi^T M^* \chi^* = -\psi_{\rm C}^\dagger M_{\rm C} \chi_{\rm C}, \qquad M_{\rm C} = \mathcal{C} M^* \mathcal{C}^{-1}. \tag{4.114}$$

In a Majorana representation with real or imaginary γ^μ all γ's are symmetric or antisymmetric. Then $\mathcal{C}$ is proportional to $\mathbb{1}$ and $M_{\rm C} = M^*$. For Majorana spinors in a Majorana representation we obtain the simple relation

$$(\psi^\dagger M\chi)^\dagger = -\psi^\dagger M^* \chi. \tag{4.115}$$

In passing we note that there are no Majorana spinors in 4-dimensional Euclidean space. Strictly speaking there is no Euclidean version of a theory with Majorana fermions in 4 dimensions. There have been suggestions to overcome this obstacle and there have been numerical simulations of such Euclidean models on lattices.

4.8 Appendix: Representations of spin and Lorentz groups

The fields furnish a representation of the homogeneous Lorentz groups $SO(1, d-1)$ or, strictly speaking, of their universal covering groups spin$(1, d-1)$. Assuming that they transform according to a finite dimensional representation of spin$(1, d-1)$, all these representation can be obtained from the finite-dimensional unitary representation of the corresponding orthogonal group spin(d) by setting $\tau = ix^0$.

4.8.1 Young-tableaux of O(n)

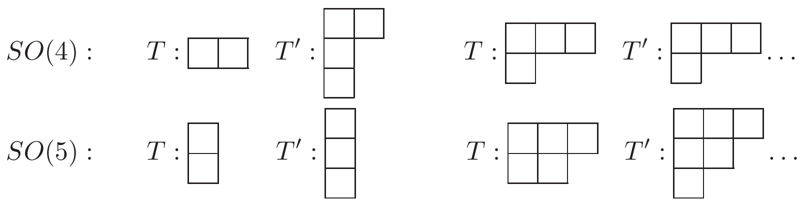
Every irreducible representation of $O(n)$ can be obtained from the tensor product of the defining representation

$$\begin{aligned} T_\mu &\longrightarrow R_\mu^{\ \nu} T_\nu, \qquad R \in O(n) \quad \text{defining repr.} \\ T_{\mu_1 \dots \mu_r} &\longrightarrow R_{\mu_1}^{\ \nu_1} \dots R_{\mu_r}^{\ \nu_r} T_{\nu_1 \dots \nu_r} \quad \text{tensor product repr.} \end{aligned} \tag{4.116}$$

by suitable symmetrisation and anti-symmetrisation in groups of indexes. They are in one-to-one corresponding to admissible Young tableaux. Each tableau T contains *at most* $[n/2]$ *rows*. If ℓ_i is the number of boxes of the i'th row, then

$$\ell_1 \geq \ell_2 \geq \dots \geq \ell_{[d/2]}.$$

To get all irreducible representations of $O(n)$ we associate a diagram T' to T as follows: The length a of the first column in T is less or equal to $n/2$, $a \leq n/2$, the length of the first column in T' is $n - a$, and all other columns in T and T' have the same length. Examples



When n is even and $a = n/2$, then the diagrams T and T' coincide (they transform differently under improper transformations) and T is said to be *self-associate*. If we restrict ourselves to the proper subgroup $SO(n)$ with determinant one, the representations corresponding to the associate diagrams T and T' become equivalent. For even n the representation corresponding to a selfassociate pattern T splits into two non-equivalent irreducible representations: the *selfdual* and *antiselfdual* antisymmetric tensors of rank $n/2$,

$$T_{\mu_1\dots\mu_{n/2}} = \pm\frac{1}{(n/2)!}\,\epsilon_{\mu_1\dots\mu_{n/2}\nu_1\dots\nu_{n/2}}T_{\nu_1\dots\nu_{n/2}}, \qquad T \quad \text{totally skew symmetric.} \tag{4.117}$$

4.8.2 Roots, weights and all that

Our Lie algebra conventions are as follows: Let $H_k,\ k=1,\dots,r$ be an orthogonal basis of the Cartan subalgebra $\mathcal{H}$, which we diagonalise in a given representation,

$$H_k|\mu\rangle = \mu_k|\mu\rangle \quad \text{and} \quad [H_k, E_\alpha] = \alpha_k E_\alpha. \tag{4.118}$$

Let

$$\alpha_{(i)}\ ,\quad \mu_{(i)}\ ,\quad \alpha^\vee_{(i)} = \frac{2\alpha_{(i)}}{\big(\alpha_{(i)},\alpha_{(i)}\big)} \quad \text{and} \quad \mu^\vee_{(i)} = \frac{2\mu_{(i)}}{\big(\alpha_{(i)},\alpha_{(i)}\big)}\ , \qquad i=1,\dots,r \tag{4.119}$$

be the *simple roots, fundamental weights, co-roots* and *co-weights*, respectively:

$$\big(\alpha_{(i)},\alpha^\vee_{(j)}\big) = K_{ij}, \quad \big(\alpha^\vee_{(i)},\mu_{(j)}\big) = \big(\alpha_{(i)},\mu^\vee_{(j)}\big) = \delta_{ij}, \quad \big(\mu_{(i)},\mu^\vee_{(j)}\big) = (K^{-1})_{ij}. \tag{4.120}$$

We used that the simple roots and fundamental weights are related by the Cartan matrix,

$$\alpha_{(i)} = \sum_{j=1}^r K_{ij}\ \mu_{(j)}. \tag{4.121}$$

The fundamental weight-states (which are the highest weight states of the r fundamental representations) and states in the adjoint representation obey

$$\alpha^\vee_{(i)}\cdot H\ |\mu_{(j)}\rangle = \delta_{ij}|\mu_{(j)}\rangle \quad \text{and} \quad \mu^\vee_{(i)}\cdot H\ |\alpha_{(j)}\rangle = \delta_{ij}|\alpha_{(j)}\rangle. \tag{4.122}$$

To determine the dimension of the representation with highest weight μ we may use Weyl's dimension formula

$$\dim V_\mu = \prod_{\alpha>0}\frac{\langle\mu+\delta,\alpha\rangle}{\langle\delta,\alpha\rangle}, \tag{4.123}$$

where the Weyl vector δ is half the sum over all positive roots or the sum over all fundamental weights,

$$\delta = \tfrac12\sum_{\alpha>0}\alpha = \sum\mu_{(i)} \tag{4.124}$$

A positive root and highest weight is a linear combination of the simple roots and fundamental weights with non-negative integer coefficients, respectively,

$$\alpha = \sum_i m_i\alpha_{(i)} \quad \text{and} \quad \mu = \sum_i k_i\mu_{(i)}, \qquad m_i, k_i\in\mathbb{N}_0. \tag{4.125}$$

Inserting this into Weyl's dimension formula and using $(\mu_{(i)},\alpha_{(j)}) = \delta_{ij}{\alpha_{(j)}}^2/2$ yields

$$\dim V_\mu = \prod_{\alpha>0}\frac{\sum_i(1+k_i)m_i{\alpha_{(i)}}^2}{\sum_i m_i{\alpha_{(i)}}^2}. \tag{4.126}$$

To find the *center elements* we find conditions on $\rho\cdot H\in\mathcal{H}$ such that $\exp(2\pi i\rho\cdot H)$ is in the center $\mathcal{Z}$. Center elements are the identity in the adjoint representation. Because of the second set of equations in (4.122) they must be powers of

$$z_i=\exp\left(2\pi i\mu^\vee_{(i)}\cdot H\right)\Longrightarrow z_i|\mu\rangle=\exp\left(2\pi i(kK^{-1})_i\right)|\mu\rangle,\qquad \mu=\sum_i k_i\mu_{(i)}.\tag{4.127}$$

In an irreducible representation a center element acts the same way on all states. Hence, a necessary and sufficient condition for $z_i\neq\mathbb{1}$ is that

$$z_i\,|\mu_{(j)}\rangle=\exp\left(2\pi i K^{-1}_{ji}\right)|\mu_{(j)}\rangle\neq|\mu_{(j)}\rangle,\quad\text{or that}\quad K^{-1}_{ji}\notin\mathbb{Z}\tag{4.128}$$

for at least one fundamental weight $\mu_{(j)}$. Here we have used that the inner products of the weights with the co-weights yield the inverse CARTAN matrix, see (4.120). The order of the center group is just $\det(K)$.

4.8.3 The spin groups

Let us now turn to the spin groups, i.e. the universal (double) covering groups of the $SO(n)$-groups[2]. In the CARTAN classification $\text{spin}(2n)\sim D_n$ and $\text{spin}(2n+1)\sim B_n$:

group	dim	dim. of fund. reps
$\text{spin}(4)=D_2\sim A_1\times A_1$	6	$2,2$
$\text{spin}(5)=B_2\sim C_2$	10	$5,4$
$\text{spin}(6)=D_3\sim A_3$	15	$6,4,4$
$\text{spin}(7)=B_3$	21	$7,21,8$
$\text{spin}(8)=D_4$	28	$8,28,8,8$
$\text{spin}(9)=B_4$	36	$9,36,84,16$
$\text{spin}(10)=D_5$	45	$10,45,120,16,16$

The DYNKIN diagrams of the spin groups are depicted in the following figure: The representation

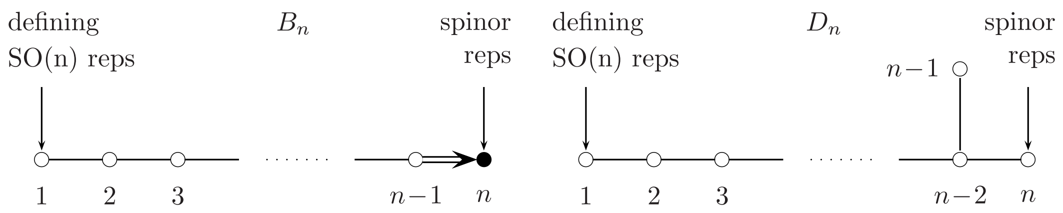


Abbildung 4.1: *Dynkin diagrams of the B_n and D_n series. $\bullet$ is a short root.*

with highest weight $\mu_{(1)}$ is always the defining representation of $SO(n)$ and has dimension n. The representation with highest weight $\mu_{(n)}$ is a spinor representation for all groups. For the D_n the second last representation belonging to $\mu_{(n-1)}$ is the second spinor representation.

[2]we use the conventions as implemented by LIE 2.1

The CARTAN matrices of the B_n and D_n groups have the form

$$K_B = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}, \quad K_D = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

The centers and their generators are

group	center	generators	action of center element(s)
D_n, n even	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$z_1 = e^{2\pi i \mu^\vee_{(1)}}$ $z_2 = e^{2\pi i \mu^\vee_{(n)}}$	$z_1\vert\mu_{(j)}\rangle = +\vert\mu_{(j)}\rangle, \quad j \le n-2$ $z_1\vert\mu_{(j)}\rangle = -\vert\mu_{(j)}\rangle, \quad j \ge n-1$ $z_2\vert\mu_{(j)}\rangle = (-)^j\vert\mu_{(j)}\rangle, \quad j \le n-2$ $z_2\vert\mu_{(n-1)}\rangle = -(-)^{n/2}\vert\mu_{(n-1)}\rangle$ $z_2\vert\mu_{(n)}\rangle = (-)^{n/2}\vert\mu_{(n)}\rangle$
D_n, n odd	$\mathbb{Z}_4$	$z = e^{2\pi i \mu^\vee_{(n)}}$	$z\vert\mu_{(j)}\rangle = (-)^j\vert\mu_{(j)}\rangle, \quad j \le n-2$ $z\vert\mu_{(n-1)}\rangle = -i^n\vert\mu_{(n-1)}\rangle$ $z\vert\mu_{(n)}\rangle = i^n\vert\mu_{(n)}\rangle$
B_n	$\mathbb{Z}_2$	$z = e^{2\pi i \mu^\vee_{(1)}}$	$z\vert\mu_{(j)}\rangle = \vert\mu_{(j)}\rangle, \quad j \le n-1$ $z\vert\mu_{(n)}\rangle = -\vert\mu_{(n)}\rangle$

The group spin(5) $\sim$ B$_2$

This group is related to the isometry groups of the 5-dimensional DESITTER and anti-DESITTER spaces and to the conformal group of the 4-dimensional flat spaces (with different signatures). Hence it is useful to discuss the corresponding representations in some detail. The discussion for the higher spin groups is very similar. The CARTAN matrix of spin(5) and its inverse are

$$K = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad K^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}. \tag{4.129}$$

The center element $z = \exp(\pi i \mu_1^\vee)$ is the identity on the defining $SO(5)$-representation with highest weight μ_1. It is $-\mathbb{1}$ on the spinor representation with highest weight μ_2. Hence, a representation with highest weight $\mu = k_1\mu_1 + k_2\mu_2$ is a faithful representation of

$$SO(5) = \text{spin}(5)/\mathbb{Z}_2 = B_2/\mathbb{Z}_2$$

if and only if

$$z|\mu\rangle = |\mu\rangle \Longleftrightarrow k_2 \in 2\mathbb{N}_0.$$

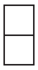
The positive roots are

$$\text{short roots:} \quad m = (0,1),(1,1) \qquad \text{long roots:} \quad m = (1,0),(1,2). \tag{4.130}$$

Using WEYL's formula we find that the dimension of the representation with highest weight μ is

$$\dim V_\mu = \frac{1}{6}(1+k_1)(1+k_2)(3+2k_1+k_2)(2+k_1+k_2). \tag{4.131}$$

The quantum numbers can easily be related to the length (ℓ_1, ℓ_2) of the rows in the YOUNG tableau. We need

 : 5 and  : 10 .

The 5 and 10-dimensional representation have

$$5: \quad \ell = (1,0),\ k = (1,0) \quad \text{and} \quad 10: \quad \ell = (1,1),\ k = (0,2).$$

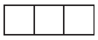
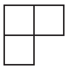
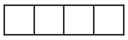
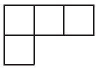
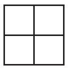
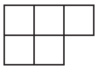
From this follows that

$$\ell_1 = k_1 + k_2/2 \quad \text{and} \quad \ell_2 = k_2/2$$

In terms of the lengths ℓ_1, ℓ_2 of the YOUNG tableau we obtain the following formula for the dimensions for the faithful $SO(5)$-representations

$$\dim V_\ell = (\ell_1 - \ell_2 + 1)(2\ell_2 + 1)(\ell_1 + \ell_2 + 2)(2\ell_1 + 3)/6. \tag{4.132}$$

Examples:

 : 14  : 35  : 55  : 91  : 35  : 105.

The group spin(6) $\sim$ $\mathbf{D}_3$

The simple laced group D_3 has CARTAN matrix

$$K = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad K^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

The center $\mathbb{Z}_4$ is generated by

$$z = \exp\left(2\pi i \mu^\vee_{(3)}\right) \quad \text{with} \quad z|\mu\rangle = \exp\left(\frac{i\pi}{2}(2k_1 + k_2 + 3k_3)\right)|\mu\rangle. \tag{4.133}$$

Since on the defining representation of $SO(6)$ it is $-\mathbb{1}$, a representation of spin(6) is a faithful representation of

$$SO(6) = \text{spin}(6)/\mathbb{Z}_2 = D_3/\mathbb{Z}_2 \tag{4.134}$$

if $z = \pm\mathbb{1}$ on this representation. This is equivalent to

$$2k_1 + k_2 + 3k_3 \in 2\mathbb{N}_0 \Longleftrightarrow k_2 + k_3 \in 2\mathbb{N}_0.$$

The positive roots are

$$m = (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (1,1,1), \tag{4.135}$$

and WEYL's formula yields

$$\dim V_\mu = \frac{1}{12}(1+k_1)(1+k_2)(1+k_3)(2+k_1+k_2)(2+k_1+k_3)(3+k_1+k_2+k_3). \tag{4.136}$$

The lowest representations of $SO(6)$ are

$$6:\begin{array}{|c|}\hline \\ \hline\end{array}\quad 15:\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\quad \text{and}\quad 20:\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\,.$$

Using LIE it easy to relate the highest weights to these representations (recall, that the 20 splits into $10_S \oplus 10_A$).

dim	ℓ	k
5	$(1,0,0)$	$(1,0,0)$
15	$(1,1,0)$	$(0,1,1)$
20	$(1,1,1)$	$(0,2,0),(0,0,2)$

(4.137)

The group spin$(7)\sim$ B$_3$

A representation of spin(7) is a faithful representation of $SO(7)$ if k_3 is an even integer. Since B_3 has no selfassociate diagrams it is easy to relate the lengths of the rows in a Young-tableau to a highest weight:

$$k_1=\ell_1-\ell_2 \quad k_2=\ell_2-\ell_3 \quad \text{and} \quad k_3=2\ell_3.$$

The dimensions of the various representations and the corresponding results for the smallest spin-groups are listed in the following table

group	dimension
spin(4)	$(k_1+1)(k_2+1)$
SO(5)	$(\ell_1-\ell_2+1)(2\ell_2+1)(\ell_1+\ell_2+2)(2\ell_1+3)/6$
spin(6)	$(1+k_1)(1+k_2)(1+k_3)$
	$\cdot(2+k_1+k_2)(2+k_1+k_3)(3+k_1+k_2+k_3)/12$
SO(7)	$(2\ell_1+5)(2\ell_2+3)(2\ell_3+1)(\ell_1+\ell_2+4)(\ell_1-\ell_2+1)$
	$\cdot(\ell_1-\ell_3+2)(\ell_1+\ell_3+3)(\ell_2+\ell_3+2)(\ell_2-\ell_3+1)/720$

(4.138)