Kapitel 3

Poincaré groups and algebras

A cornerstone of modern physics is that the fundamental laws are the same in all inertial frames. The transition between two inertial frames is given by a Poincaré transformation, depending on the relative spacetime displacement of the two systems, their relative velocity and orientation. In 4 spacetime dimensions it depends on $4+3+3=10$ real parameters. The set of all transformations form a non-compact Lie group – the Poincaré group containing the Lorentz group as subgroup.

In these lectures we shall consider supersymmetric field theories in various dimensions. There are good reasons for doing that. For example, superstring theories can consistently be formulated in 10 spacetime dimensions only. Our 'real world' emerges after a dimensional reduction. Or one may easily construct theories with *extended supersymmetry* by a dimensional reduction of theories with simple supersymmetry.

In these lectures M denotes the d-dimensional Minkowski spacetime with metric coefficients $\eta_{\mu\nu}=\text{diag}(1,-1,\dots,-1)$. Points in M are events characterised by their coordinates with respect to some inertial system,

$$x=(x^\mu),\quad \mu=0,\dots,d-1. \tag{3.1}$$

The bilinear scalar product between two tangential vectors is

$$(\xi,\eta)=\sum_{\mu\nu}\eta_{\mu\nu}\xi^\mu\eta^\nu\equiv\xi^\mu\eta_\mu.$$

Poincaré *transformations* are linear transformations

$$x^\mu\longrightarrow\tilde{x}^\mu=\Lambda^\mu{}_\nu x^\nu+a^\mu\quad\text{or}\quad\tilde{x}=\Lambda x+a,\qquad d\tilde{x}=\Lambda dx$$

with $ds^2=(dx,dx)=(d\tilde{x},d\tilde{x})$. They form the Poincaré group,

$$iL=\Big\{(\Lambda,a)\big|\,a\in\mathbb{R}^d,\Lambda\in L(\mathbb{R}^d),\Lambda^T\eta\Lambda=\eta\Big\}, \tag{3.2}$$

also called inhomogeneous Lorentz group, with group multiplication

$$(\Lambda_2,a_2)(\Lambda_1,a_1)=(\Lambda_2\Lambda_1,\Lambda_2a_1+a_2). \tag{3.3}$$

The unit element is $(\mathbb{1},0)$ and the inverse of (Λ,a) reads

$$(\Lambda,a)^{-1}=\big(\Lambda^{-1},-\Lambda^{-1}a\big). \tag{3.4}$$

The POINCARÉ group has dimension

$$\tfrac{1}{2}d(d-1)+d=\tfrac{1}{2}d(d+1),$$

which is just the number of independent generators. For example, in 4 dimensions the POINCARÉ group has 10 generators and in 5 dimensions it has 15. 10 is also the number of generators of the anti-DESITTER (AdS) group in 4 dimensions and of the conformal group in 3 dimensions. More generally, the AdS group in d dimensions is isomorphic to the *conformal group* in $d-1$ dimensions. This equivalence is at the heart of the so-called AdS-CFT-correspondence.
Note, that iL is the semi-direct product of space-time *translations* and LORENTZ *transformations*. The normal subgroup of translations is ABELian and the subgroup of LORENTZ transformations

$$L=\Big\{\Lambda\in L(\mathbb{R}^d)\big|\,\Lambda^T\eta\Lambda=\eta\Big\} \tag{3.5}$$

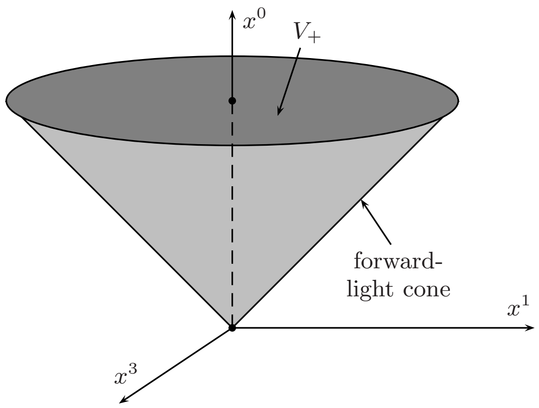
form the (non-compact) simple LIE group $O(1,d-1)$. From (3.5) it follows at once, that

$$\det\Lambda=\pm 1.$$

The transformations Λ with determinant one form the subgroup $SO(1,d-1)$ of *proper* LORENTZ *transformations*. Every Λ maps the forward light cone

$$V_+=\{\xi^0>0,(\xi,\xi)>0\}$$

into itself or into the past light cone V_-. In the second case the time direction is reversed.



The LORENTZ groups are not (simply) connected, e.g.

$$\begin{aligned}\pi_1\big(SO_e(1,2)\big)=\mathbb{Z}\quad &,\quad \pi_0\big(O(1,3)\big)=\mathbb{Z}_2\times\mathbb{Z}_2\\ \pi_0\big(SO_e(1,3)\big)=\mathbb{Z}_2\quad &,\quad \pi_1\big(SO_e(1,3)\big)=\mathbb{Z}_2.\end{aligned}$$

3.1 Poincaré Algebras

The LORENTZ *group* in d dimensions is generated by $d(d-1)/2$ elements $M_{\mu\nu}=-M_{\nu\mu}$,

$$(\Lambda,0)=\exp\big(\tfrac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\big)=\exp\big(\tfrac{i}{2}(\omega,M)\big)=\mathbb{1}+\tfrac{i}{2}(\omega,M)+O(\omega^2). \tag{3.6}$$

In the defining representation each $M_{\mu\nu}$ is a d-dimensional imaginary matrix. The condition (3.5) on the LORENTZ transformation implies

$$(\omega, M)^T\eta + \eta(\omega, M) = 0 \Longleftrightarrow (\omega, M)_{\rho\sigma} = -(\omega, M)_{\sigma\rho}$$

for all ω. Hence each $M_{\mu\nu}$ must be antisymmetric if both its indexes ρ and ρ are *lower* ones:

$$(M_{\mu\nu})_{\rho\sigma} = -(M_{\mu\nu})_{\sigma\rho}. \tag{3.7}$$

There are $\frac{1}{2}d(d-1)$ independent antisymmetric matrices and as independent ones we choose

$$\big(M_{\mu\nu}\big)_{\rho\sigma} = -\big(M_{\nu\mu}\big)_{\rho\sigma} = -i\big(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\nu\rho}\eta_{\mu\sigma}\big) \Longrightarrow \tfrac{i}{2}(\omega, M)^\rho_{\ \sigma} = \omega^\rho_{\ \sigma}. \tag{3.8}$$

These generators fulfil the commutation relations

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\sigma}]_{\lambda\xi} &= (M_{\mu\nu})_{\lambda\tau}(M_{\rho\sigma})^\tau_{\ \xi} - (M_{\rho\sigma})_{\lambda\tau}(M_{\mu\nu})^\tau_{\ \xi}\\ &= i\big(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\big)_{\lambda\xi}.\end{aligned}$$

It is the relativistic generalisation of the well-known angular momentum algebra and is generated by the infinitesimal rotations M_{ij} and boosts M_{0i}.
The subgroup of spacetime *translations*

$$(\mathbb{1}, a) = \exp(ia^\mu P_\mu) = \exp(i(a, P)) = \mathbb{1} + i(a, P) + O(a^2) \tag{3.9}$$

form a normal ABELian subgroup of the POINCARÉ group with commuting generators

$$[P_\mu, P_\nu] = 0, \qquad \mu, \nu = 0, \dots, d-1.$$

The commutation relations of infinitesimal LORENTZ transformations and translations follow from

$$(\Lambda, 0)(\mathbb{1}, a)(\Lambda, 0)^{-1} = (\mathbb{1}, \Lambda a)$$

after inserting the infinitesimal transformations (3.6,3.9). The term linear in ω and in a reads

$$[(\omega, M), (a, P)] = \omega^{\mu\nu}\big(M_{\mu\nu}a, P\big).$$

This holds true for any a, ω and with (3.8) we conclude

$$[M_{\mu\nu}, P_\rho] = (M_{\mu\nu})^\sigma_{\ \rho}P_\sigma = i\big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\big).$$

To summarise, the important POINCARÉ *algebra* is given by

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\sigma}] &= i\big(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\big)\\ [M_{\mu\nu}, P_\rho] &= i\big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\big)\\ [P_\mu, P_\nu] &= 0\end{aligned} \tag{3.10}$$

One of the main task will be to extend this LIE algebra to supersymmetric algebras containing commutators and anticommutators. Now we turn to the particular and most important cases.

3.2 Lorentz-algebras in low dimensions

In the following we study the LORENTZ algebras and groups in two, three and four spacetime dimensions in more detail. Particular emphasis is put on the four-dimensional case.

3.2.1 Two dimensions

In this simple case the LORENTZ group is ABELian and has one independent generator M_{01}, since

$$(M_{\mu\nu}) = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\sigma_1. \tag{3.11}$$

Thus a proper LORENTZ transformation has the form

$$\Lambda = \exp\left(\tfrac{i}{2}\omega_{\mu\nu}M_{\mu\nu}\right) = \exp\left(-\alpha\sigma_1\right) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}, \quad \text{where} \quad \alpha = \omega_{01}. \tag{3.12}$$

Since every irreducible representations of ABELian groups is 1-dimensional, this representation should decompose into two irreducible part. Indeed, introducing the light-cone components

$$\xi^+ = \xi^0 + \xi^1 \quad \text{and} \quad \xi^- = \xi^0 - \xi^1 \tag{3.13}$$

of a vector ξ we obtain the simple transformation law,

$$\xi^+ \longrightarrow e^{-\alpha}\xi^+ \quad , \quad \xi^- \longrightarrow e^{\alpha}\xi^-. \tag{3.14}$$

These components transform according to different 1-dimensional irreducible representations of the 2-dimensional proper LORENTZ group $SO(1,1)$. An important point which is worth making at this stage is that this finite dimensional representation is *non-unitary*, since the generator M_{01} is not hermitean. It is generally true that any finite-dimensional represenation of a non-compact (semi-simple) LIE group is non-unitary.

3.2.2 Three dimensions

In 3 spacetime dimensions the LORENTZ group in non-ABELian and has 3 independent generators,

$$(M_{\mu\nu}) = \begin{pmatrix} 0 & K_1 & K_2 \\ -K_1 & 0 & J \\ -K_2 & -J & 0 \end{pmatrix}. \tag{3.15}$$

K_1 and K_2 generate boosts in the two spatial directions and J generates rotations in 2-dimensional space. The LORENTZ algebra takes the form

$$[K_1, K_2] = iJ \quad , \quad [J, K_1] = -iK_2 \quad \text{and} \quad [J, K_2] = iK_1. \tag{3.16}$$

The explicit representation (3.8) leads to the following form for these generators,

$$K_1 = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}. \tag{3.17}$$

In order to find the LORENTZ transformations in (3.6) we need to exponentiate

$$\tfrac{i}{2}(\omega, M) = \omega = \begin{pmatrix} 0 & \omega^0_{\ 1} & \omega^0_{\ 2} \\ \omega^0_{\ 1} & 0 & \omega^1_{\ 2} \\ \omega^0_{\ 2} & -\omega^1_{\ 2} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & \theta \\ \alpha_2 & -\theta & 0 \end{pmatrix}. \tag{3.18}$$

Since ω^3 is proportional to ω we obtain the following form for the LORENTZ transformations

$$\Lambda(\boldsymbol{\alpha}, \theta) = e^{\omega} = \mathbb{1} + \frac{\omega}{\kappa}\sinh\kappa + \frac{\omega^2}{\kappa^2}\left(\cosh\kappa - 1\right), \qquad \kappa^2 = \boldsymbol{\alpha}^2 - \theta^2. \tag{3.19}$$

As expected, for vanishing $\boldsymbol{\alpha}$ they reduce to rotations in $SO(2) \subset SO(1,2)$ in space,

$$\Lambda(\boldsymbol{0},\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

On the other hand, for vanishing θ and α_2 we obtain a boost in x^1-direction,

$$\Lambda(\alpha_1,0,0) = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \cosh\alpha_1 = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}. \tag{3.20}$$

In passing we note, that the the complexified LORENTZ algebra is just A_2. To see that we set

$$K_1 = \mathrm{i}L_1, \quad K_2 = -\mathrm{i}L_2 \quad \text{and} \quad J = L_2,$$

and observe, that the hermitean operators L_i fulfil the wellknown commutation relations of the angular momentum operators,

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k. \tag{3.21}$$

It follows that the finite dimensional (and hence non-unitary) representations of the LORENTZ group $SO(1,2)$ are classified by an integer number which corresponds to the total angular momentum in quantum mechanics. If this number is half-integer, we would obtain a representation of the spin-group which is the double-cover of the LORENTZ group. We refrain from further study these representations and instead turn to 4 spacetime dimensions.

3.2.3 Four dimensions

In this physically most relevant case there are 6 independent LORENTZ generators $M_{\mu\nu}$. We denote the *infinitesimal boosts* by K_i and the infinitesimal rotations by J_i such that

$$(M_{\mu\nu}) = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{pmatrix}. \tag{3.22}$$

The explicit form of the antihermitean boost generators is

$$K_1 = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{3.23}$$

and that of the infinitesimal hermitean rotations

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.24}$$

These generators fulfil the commutation relations

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \quad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k \quad \text{and} \quad [J_i, K_j] = [K_i, J_j] = \mathrm{i}\epsilon_{ijk}K_k. \tag{3.25}$$

In a next step we introduce the following generators of the complexified Lorentz algebra,

$$S_i = \tfrac{1}{2}(J_i + \mathrm{i}K_i) \quad \text{and} \quad A_i = \tfrac{1}{2}(J_i - \mathrm{i}K_i). \tag{3.26}$$

They obey the simple commutation relations of the $A_1 \times A_1$ Lie algebra

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k, \quad [A_i, A_j] = \mathrm{i}\epsilon_{ijk} A_k \quad \text{and} \quad [S_i, A_j] = 0. \tag{3.27}$$

We conclude, that

$$so(4, \mathbb{C}) \sim sl(2, \mathbb{C}) \times sl(2, \mathbb{C}),$$

so that every irreducible representation of $so(1, 3)$ is uniquely determined by (n, m), where the half-integers n and m characterise the $sl(2, \mathbb{C})$ representations.
Note that the two $sl(2, \mathbb{C})$ subalgebras are not independent. They can be interchanged by the *parity operation*. Parity acts on the rotation - and boost - generators as follows:

$$J_i \to J_i \quad \text{and} \quad K_i \to -K_i, \tag{3.28}$$

and this shows that parity transforms S_i into A_i and A_i into S_i. If the generators J_i and K_i are represented by hermitean operators, then parity operation is equivalent to hermitean conjugation. As examples we consider the *smallest representations:*

- $(0, 0)$: this trivial one-dimensional scalar representation has total spin zero.
- $(\frac{1}{2}, 0)$: this two-dimensional spinor representation is called the left-handed representation. *left-handed spinors* transform trivial under one subgroup $sl(2)$ and according to the spin $\frac{1}{2}$-representation with respect to the other subgroup $sl(2)$.
- $(0, \frac{1}{2})$: this two-dimensional spinor representation is called the right-handed representation and spinors transforming according to this representation are called *right handed*.
- Since parity exchanges S_i and A_i, the *handed spinors* have no fixed parity: parity transforms the representation $(\frac{1}{2}, 0)$ into $(0, \frac{1}{2})$ and vice-versa. To obtain a representation such that parity acts as a linear transformation, one needs to combine the two spinor representation to

  $$(\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}) \tag{3.29}$$

  which yields the Dirac *spinor representation.*
- Any representation can be generated from the left- and right-handed spinor representations. For example, the tensor product of the right- and left-handed representations is

  $$(\tfrac{1}{2}, 0) \otimes (0, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2}) \tag{3.30}$$

  and yields the 4-dimensional vector-representation, giving the transformations of a 4-vector. Also,

  $$(\tfrac{1}{2}, 0) \otimes (\tfrac{1}{2}, 0) = (0, 0) \oplus (1, 0) \tag{3.31}$$

  and yields the scalar and spin-one representation, the second given by self-dual antisymmetric second rank tensors.

- More generally, for tensors of rank two we find

$$\begin{aligned} T^{\mu\nu} &\in (\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2}) = (\tfrac{1}{2},0)\otimes(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})\otimes(\tfrac{1}{2},0) \\ &= \big[(1,0)\oplus(0,0)\big]\otimes\big[(0,1)\oplus(0,0)\big] \\ &= (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0). \end{aligned}$$

  Every second rank tensor contains a scalar part $(0,0)$, a spin-1 part $(1,0)\oplus(0,1)$ and a spin-2 part $(1,1)$. This decomposition corresponds to

$$T^{\mu\nu} = a\eta^{\mu\nu} + A^{\mu\nu} + S^{\mu\nu} \quad \text{with} \quad a = \tfrac{1}{4}T^{\rho}_{\ \rho}, \quad A^{\mu\nu} = -A^{\nu\mu}, \quad S^{\mu\nu} = S^{\nu\mu}, \quad S^{\rho}_{\ \rho} = 0.$$

  The antisymmetric part can be further decomposed into its selfdual and anti-selfdual pieces.

3.3 Representations of the 4-dimensional Poincaré algebra

There are two cases to be distinguished:

1. There are massive particles. Then the symmetry consists of spacetime translations, Lorentz transformations and internal symmetries commuting with the spacetime symmetries.
2. There are only massless particles. Then there is the extra possibility of having conformal symmetry.

First we find two Casimir *operators* of the Poincaré algebra. It is not difficult to prove

Lemma 1 *$P^2 = P_\mu P^\mu$ is a Casimir operator for the* Poincaré *algebra.*

Proof: Clearly, $[P^2, P_\mu] = 0$. Using the Leibniz rule and Poincaré algebra (3.10) one has

$$[P^2, M_{\rho\sigma}] = P_\mu[P^\mu, M_{\rho\sigma}] + [P_\mu, M_{\rho\sigma}]P^\mu = 0$$

and this proves the lemma. The last equation just states that for any 4-vector V^μ the object $V^2 = V^\mu V_\mu$ is a Lorentz scalar.
The second Casimir-operator can be constructed from the Pauli-Ljubanski polarisation vector, defined by

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \qquad \epsilon_{0123} = 1. \tag{3.32}$$

This 4-vector is orthogonal to the 4-momentum,

$$W_\mu P^\mu = 0. \tag{3.33}$$

In the $1+3$ split this vector takes the form

$$W_0 = -P^i J_i, \quad W_i = P^0 J_i - \epsilon_{ijk}P^j K_k. \tag{3.34}$$

For a particle at rest $P = (m, 0)$ and $W = (0, m\boldsymbol{J})$. Now we shall prove the

Lemma 2 *$W^2 = W_\mu W^\mu$ is a Casimir operator for the* Poincaré *algebra.*

Proof: Since

$$[W_\mu, P_\xi] = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu[M^{\rho\sigma}, P_\xi] = \mathrm{i}\epsilon_{\mu\nu\rho\xi}P^\nu P^\rho = 0 \tag{3.35}$$

holds true, W^2 commutes with the 4-momentum P_σ. Since W_μ is a 4-vector, its square W^2 is a LORENTZ scalar and hence commutes with the infinitesimal LORENTZ transformations $M_{\mu\nu}$. This then proves the lemma.
We conclude that P^2 and W^2 are CASIMIR operators of the POINCARÉ algebra. Since the LORENTZ algebra has rank 2 these are all CASIMIR operators. Thus we have constructed a maximal set of operators commuting with translations and LORENTZ transformations[1]. Hence all physical states (fields, particles) in a quantum field theory may be classified by the eigenvalues of these two CASIMIR operators.
The infinitesimal LORENTZ transformations consists of two parts, namely the external orbital and the internal spin part[2]:

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}, \qquad L_{\mu\nu} = -x_\mu P_\nu + x_\nu P_\mu, \quad [P_\mu, x^\nu] = i\delta_\mu^{\ \nu}. \tag{3.36}$$

Since the cyclic sum of $P^\nu L^{\rho\sigma}$ vanishes, the PAULI-LJUBANSKI vector simplifies to

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu \Sigma^{\rho\sigma}. \tag{3.37}$$

3.3.1 Massive particles

A massive particle has a non-vanishing rest mass m,

$$P^2 = m^2 > 0 \tag{3.38}$$

and we may transform to its rest frame, in which

$$P^\mu = (m, \boldsymbol{0}). \tag{3.39}$$

In this frame we have

$$W_0 = 0 \quad \text{and} \quad W_i = -\frac{m}{2}\epsilon_{ijk}\Sigma^{jk} = mS_i,$$

where the operators

$$S_i = -\tfrac{1}{2}\epsilon_{ijk}\Sigma^{jk}, \qquad [S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k \tag{3.40}$$

generate the spin rotations. Hence W^μ is the relativistic generalisation of the spin. Since

$$W^2 = -\boldsymbol{W}^2 = -m^2 \boldsymbol{S}^2,$$

we see that the massive representations are labelled by the mass m and the spin s:

$$P^2 = m^2 \quad \text{and} \quad W^2 = -m^2 s(s+1), \quad s = 0, \tfrac{1}{2}, 1, \dots. \tag{3.41}$$

The representations are characterised by the mass and spin of the particle. Massive particles fall into $(2s+1)$-dimensional irreducible multiplets the states of which are distinguished by the third component of the spin and the continuous eigenvalues of $\boldsymbol{P}$. Examples of massive states are the *electron* with spin $\frac{1}{2}$ and the *pion* with spin 0.

[1] and which are in the enveloping algebra of then POINCARÉ algebra
[2] See section (4.2).

3.3.2 Massless particles

A massless particle has a vanishing rest mass, $P^2 = 0$, and there is no system in which it is at rest. But we can always transform into a system where

$$P_\mu = (P_0, 0, 0, P_3) \quad \text{with} \quad P_0^2 - P_3^2 = 0 \tag{3.42}$$

holds true. Since $P_\mu W^\mu = 0$ we conclude that in this system

$$W_\mu = (\lambda P_0, W_1, W_2, \lambda P_3), \quad \text{such that} \quad W^2 = -W_1^2 - W_2^2 \leq 0.$$

For $W^2 = 0$ the 4-momentum and Pauli-Ljubanski vector become linearly dependent,

$$W_\mu = \lambda P_\mu.$$

This tensor identity holds in any inertial system. The factor λ is a Lorentz scalar and must commute with all generators of the Poincaré group,

$$[\lambda, P^\mu] = [\lambda, M^{\mu\nu}] = 0. \tag{3.43}$$

Thus λ is a Casimir operator in the massless case. Since $W_0 = -P^i J_i$ we conclude, that

$$\lambda = \frac{\boldsymbol{P} \cdot \boldsymbol{J}}{P_0} \tag{3.44}$$

which is the *helicity* of the massless particle. It must be half-integer[3]. Examples of particles falling into this category are the *photon* with helicities ± 1 and a massless *neutrino* with helicities $\pm\frac{1}{2}$. The remaining case $P^2 = 0$ and $W^2 < 0$ describes a particle with zero rest mass and an infinite number of polarisation states. These representations do not seem to be realised in nature.

3.4 Appendix: Anti-de Sitter algebras

In d-dimensional Minkowski spacetime the symmetry group has $\frac{1}{2}d(d+1)$ generators. There are only two other spaces with the same maximal number of symmetries, namely the deSitter and anti-deSitter spaces. Anti-deSitter spaces are consistent backgrounds for $\mathcal{N} = 1$ supergravity models, to be discussed next semester. They play a prominent role in recent duality conjectures and hence it maybe useful to introduce these maximally symmetric spaces. The main reference on *AdS* spaces is the book of Hawking and Ellis [14].

Anti-deSitter space maybe embedded as hyperboloid in a $d+1$-dimensional flat space with two time-like directions. The Cartesian coordinates of the embedding space $\mathbb{R}^{d+1}$ are denoted by ξ^m and the line element reads

$$ds^2 = \tilde{\eta}_{mn} d\xi^m d\xi^n, \qquad \tilde{\eta}_{mn} = (+, -, \dots, -, +). \tag{3.45}$$

The d-dimensional AdS_d-submanifold is defined by the $SO(2, d-1)$-invariant condition

$$\tilde{\eta}_{mn} \xi^m \xi^n = R^2. \tag{3.46}$$

Note that every point is invariant under rotations around the vector pointing from the origin to this point. Hence the little group of every point is $SO(1, d-1)$ and we conclude

$$AdS_d \sim \frac{SO(2, d-1)}{SO(1, d-1)}. \tag{3.47}$$

[3] See S. Weinberg, The Quantum Theory of Fields, I, p.90

Next we choose local coordinates $(x^\mu) = (y^\alpha, z)$ on AdS_d, where α is less or equal to $d-2$,

$$\xi^\alpha = \frac{Ry^\alpha}{z}, \qquad \xi^{d-1} = \frac{R^2 + y^\alpha y_\alpha - z^2}{2z} \quad \text{and} \quad \xi^d = \frac{R^2 - y^\alpha y_\alpha + z^2}{2z}, \tag{3.48}$$

where the α, β-indexes are lowered with the MINKOWSKI metric $\eta_{\alpha\beta} = \text{diag}(+1, -1, \dots, -1)$ in $d-1$ dimensions. The induced metric reads

$$ds^2 = \tilde{\eta}_{mn} d\xi^m d\xi^n = \frac{R^2}{z^2}\left(dy^\alpha dy_\alpha - dz^2\right) \equiv g_{\mu\nu} dx^\mu dx^\nu \tag{3.49}$$

and has signature $(1, d-1)$. Note that it is conformally flat,

$$g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}, \qquad e^\sigma = \frac{R}{z} \tag{3.50}$$

such that the CHRISTOFFEL-symbols have the form

$$\Gamma^\rho_{\mu\nu} = \left(\delta^\rho_\mu \sigma_{,\nu} + \delta^\rho_\nu \sigma_{,\mu} - \eta_{\mu\nu}\sigma^{,\rho}\right). \tag{3.51}$$

From these one can read off the RICCI tensor and RICCI scalar,

$$\begin{aligned} \mathcal{R}_{\mu\nu} &= (2-d)\left(\sigma_{;\mu\nu} - \sigma_{,\mu}\sigma_{,\nu} + \eta_{\mu\nu}(\nabla\sigma)^2\right) - \eta_{\mu\nu}\triangle\sigma \\ \mathcal{R} &= -(d-1)e^{-2\sigma}\left(2\triangle\sigma + (d-2)(\nabla\sigma)^2\right). \end{aligned} \tag{3.52}$$

In particular for the AdS-metric the RICCI tensor is proportional to the metric and the RICCI scalar is constant,

$$\mathcal{R}_{\mu\nu} = \frac{d-1}{R^2} g_{\mu\nu} \quad \text{and} \quad \mathcal{R} = \frac{d(d-1)}{R^2}. \tag{3.53}$$

as one might expect for a hyperboloid. For such RICCI *flat* space-times the EINSTEIN tensor is proportional to the metric as well,

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\mathcal{R} = -\frac{(d-1)(d-2)}{2R^2}\, g_{\mu\nu},$$

such that AdS is as solution of EINSTEIN' vacuum-equation with negative *cosmological constant*

$$G_{\mu\nu} = \Lambda g_{\mu\nu}, \qquad \Lambda = -\frac{(d-1)(d-2)}{2R^2}. \tag{3.54}$$

By construction AdS_d has isometry group $SO(2, d-1)$ and this group acts linearly in the embedding space with coordinates ξ^m. They transform under infinitesimal transformation as follows,

$$\delta\xi^m = \omega^m_{\ n}\xi^n. \tag{3.55}$$

Since (ω_{mn}) is antisymmetric this can be written as

$$\begin{aligned} \delta\xi^\alpha &= \omega^\alpha_{\ \beta}\xi^\beta + a^\alpha\xi^{d-1} + b^\beta\xi^d \\ \delta\xi^{d-1} &= a_\alpha\xi^\alpha + \omega\,\xi^d \\ \delta\xi^d &= -b_\alpha\xi^\alpha + \omega\,\xi^{d-1} \end{aligned} \tag{3.56}$$

For the local coordinates in (3.48) this reads

$$\begin{aligned} \delta y^\alpha &= \omega^\alpha_{\ \beta}y^\beta + \frac{1}{2R}\delta^\alpha\left(y^\beta y_\beta - z^2\right) - \frac{1}{R}y^\alpha\delta_\beta y^\beta - \omega y^\alpha + R\sigma^\alpha, \\ \delta z &= -\frac{1}{R}\delta_\beta y^\beta\, z - \omega z, \quad \text{with} \quad \delta^\alpha = (a-b)^\alpha, \;\; \sigma^\alpha = \tfrac{1}{2}(a+b)^\alpha \end{aligned} \tag{3.57}$$

This transformations are infinitesimal symmetry transformations generated KILLING vector fields. To each parameter ω_{mn} there is one vector field. Their explicit form reads

$$\begin{aligned} X(\omega) &= -\left(z\partial_z + y^\alpha\partial_\alpha\right) \\ X(\delta^\alpha) &= \tfrac{1}{R}y_\alpha X(\omega) + \tfrac{1}{2R}\left(y^\beta y_\beta - z^2\right)\partial_\alpha \\ X(\sigma^\alpha) &= R\partial_\alpha \\ X(\omega^\alpha{}_\beta) &= \eta^{\beta\gamma}\left(y_\gamma\partial_\alpha - y_\alpha\partial_\gamma\right) \end{aligned} \tag{3.58}$$

They form a nonlinear realisation of the $so(2,d-1)$ symmetry algebra on the Anti-DESITTER spacetime, as expected. In particular, the last set of vector fields generate an $SO(1,d-2)$ subgroup. We leave the proof as an exercise.

Suppose that you only know the metric (3.49). To find the symmetries of space-time, you should solve the KILLING equation-equation

$$(L_X g)_{\mu\nu} = X^\rho g_{\mu\nu,\rho} + g_{\rho\nu}X^\rho_{,\mu} + g_{\mu\rho}X^\rho_{,\nu} = 0. \tag{3.59}$$

One can prove, that the above vector fields are the only solutions of these equations.

Anti-DESITTER spaces are not globally hyperbolic, and hence there are no CAUCHY-hypersufaces. The corresponding problems for (quantum)field theories on AdS-spaces have been discussed by AVIS et.al [15]. In passing we note, that AdS_d spaces possess a boundary which maybe identified with the $d-1$-dimensional MINKOWSKI spacetime. The isometry group $SO(d,2)$ maps points on the boundary into points on the boundary. Its restriction to the boundary of AdS_d is isomorphic to the conformal group of $d-1$-dimensional MINKOWSKI spacetime.