Kapitel 2

Supersymmetric Quantum Mechanics (SQM)

In this chapter we examine simple toy models of supersymmetric field theories. These are quantum mechanical systems possessing supersymmetry [1]. Here there are no technical difficulties hiding the essential structures. Besides, such simple systems are interesting in their own right, since the dynamics of supersymmetric quantum field theories in finite volumes reduce to that of supersymmetric quantum mechanics in the infrared limit [3]. This observation maybe used to better understand non-perturbative features of field theories. A supersymmetric quantum mechanics with 16 supercharges appear in the matrix theory description of M theory [2]. In mathematical physics supersymmetric quantum mechanics has proved to be useful in proving index theorems for physically relevant differential operators [4].
Quantum mechanics can be thought of as quantum field theory in $0+1$ dimensions. The Poincaré algebra reduces to time translations generated by the Hamiltonian H. Field operators $\phi(t)$ are particular operators in the Heisenberg-picture. For example, the position x and momentum p of a point particle on $\mathbb{R}$ may be viewed as real scalar field ϕ and its conjugate momentum field π in $0+1$ dimensions. The ground state of quantum mechanics is the vacuum state of field theory.

2.1 Supersymmetric harmonic oscillator

Setting $2m=\hbar=1$ the ordinary harmonic oscillator in one dimension has Hamiltonian

$$H_{\mathrm{B}} = p^2 + \omega^2 x^2 - \omega, \tag{2.1}$$

where we have subtracted the zero-point energy $\hbar\omega$. As is wellknown one can define *lowering* and *raising* operators,

$$a = \frac{1}{\sqrt{2\omega}}\,(p - i\omega x) \quad , \quad a^\dagger = \frac{1}{\sqrt{2\omega}}\,(p + i\omega x) \tag{2.2}$$

with commutation relations

$$[a,a] = [a^\dagger, a^\dagger] = 0 \quad \text{and} \quad [a, a^\dagger] = 1. \tag{2.3}$$

These allow us to rewrite the Hamiltonian as follows,

$$H_{\mathrm{B}} = \omega\left(a^\dagger a + a a^\dagger - \omega\right) = 2\omega a^\dagger a = 2\omega N_{\mathrm{B}}\,. \tag{2.4}$$

The *ground state* is annihilated by the lowering operator a and the excited states are gotten by applying the raising operator $a^\dagger$ several times to the ground state,

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n|0\rangle, \quad a|0\rangle = 0. \tag{2.5}$$

The *number operator* counts, how many times $a^\dagger$ has been applied to the ground state,

$$N_{\rm B}|n\rangle = a^\dagger a|n\rangle = n|n\rangle, \tag{2.6}$$

and with (2.4) this determines the discrete energies $E_n = 2n\omega$ of the harmonic oscillator.
Now we 'supersymmetrize' this construction and consider the *super*-HAMILTONian

$$H = H_B + 2\omega b^\dagger b = 2\omega a^\dagger a + 2\omega b^\dagger b, \tag{2.7}$$

where the new operators b and $b^\dagger$ are *fermionic* annihilation and creation operators fulfilling anti-commutation relations,

$$\{b,b\} = \{b^\dagger,b^\dagger\} = 0 \quad \text{and} \quad \{b,b^\dagger\} = 1. \tag{2.8}$$

Since $a^\dagger a$ and $b^\dagger b$ are both non-negative, the state which is annihilated by a and b has minimal energy and hence is the ground state of H,

$$a|0\rangle = b|0\rangle = 0 \Longleftrightarrow H|0\rangle = 0. \tag{2.9}$$

The FOCK space is generated by acting with the creation operators on this state. The raising operators $(a^\dagger, b^\dagger)$ increase the bosonic and fermionic number operators

$$N_{\rm B} = a^\dagger a \quad \text{and} \quad N_{\rm F} = b^\dagger b \tag{2.10}$$

by one,

$$[N_{\rm B}, a^\dagger] = a^\dagger \quad \text{and} \quad [N_{\rm F}, b^\dagger] = b^\dagger. \tag{2.11}$$

Of course, the lowering operators decrease them by one unit. Because of the PAULI principle the only eigenvalues of $N_{\rm F}$ are 0 and 1. States with fermion number 0 are called *bosonic* and those with fermion number 1 *fermionic*. All excited states of

$$H = 2\omega(N_{\rm B} + N_{\rm F}) \tag{2.12}$$

come in pairs: the 'bosonic' eigenstate $(a^\dagger)^n|0\rangle$ and the 'fermionic' eigenstate $b^\dagger(a^\dagger)^{n-1}|0\rangle$ have the same energy $E_n = 2n\omega$, so that every bosonic state has a fermionic partner with the same energy.
In analogy with supersymmetric theories in higher dimensions we introduce the nilpotent *super-charge* $\mathcal{Q}$ and its adjoint,

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Q}^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad A = \frac{d}{dx} + \omega x. \tag{2.13}$$

The non-hermitean A is proportional to a and we obtain

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_{\rm B} & 0 \\ 0 & H_{\rm F} \end{pmatrix} \equiv H \tag{2.14}$$

where $H_{\rm B}$ and $H_{\rm F}$ are the restriction of the super-HAMILTONian H to the bosonic and fermionic subspaces, respectively,

$$H_{\rm B} = H|_{N_{\rm F}=0} \quad \text{and} \quad H_{\rm F} = H|_{N_{\rm F}=1}.$$

$\mathcal{Q}$ is nilpotent and commutes with the super-HAMILTONian

$$\{\mathcal{Q}, \mathcal{Q}\} = 0 \quad \text{and} \quad [\mathcal{Q}, H] = 0. \tag{2.15}$$

It generates the supersymmetry of the quantum mechanical system.

2.2 Pairing and ground states for SQM

The supersymmetric harmonic oscillator is the simplest example of a susy quantum mechanics. In less simple systems the raising and lowering operators are replaced by first order differential operators and one introduces the nilpotent supercharges (2.13) with

$$A = \frac{d}{dx} + W(x) \quad \text{and} \quad A^\dagger = -\frac{d}{dx} + W(x). \tag{2.16}$$

For a linear function $W(x)$ the operators A and $A^\dagger$ are proportional to the bosonic annihilation and creation operators of the supersymmetric harmonic oscillator. The arbitrary real function $W(x)$ is called *super potential*. The super-HAMILTONian is defined as in (2.14) with

$$\begin{aligned} H_\mathrm{B} &= A^\dagger A = -\frac{d^2}{dx^2} + W^2(x) - W'(x) \\ H_\mathrm{F} &= AA^\dagger = -\frac{d^2}{dx^2} + W^2(x) + W'(x). \end{aligned} \tag{2.17}$$

Since both superpartner HAMILTONian are non-negative, all energies are zero or positive. A bosonic zero-energy state is annihilated by A and a fermionic zero-energy state is annihilated by $A^\dagger$,

$$H_\mathrm{B}|0\rangle = 0 \Longleftrightarrow A|0\rangle = 0 \quad , \quad H_\mathrm{F}|0\rangle = 0 \Longleftrightarrow A^\dagger|0\rangle = 0 \tag{2.18}$$

The excited modes come always in pairs, similarly as for the supersymmetric harmonic oscillator. To prove this statement assume that $|\psi_\mathrm{B}\rangle$ is a bosonic eigenfunction with energy E,

$$H_\mathrm{B}|\psi_\mathrm{B}\rangle = A^\dagger A|\psi_\mathrm{B}\rangle = E|\psi_\mathrm{B}\rangle.$$

It follows, that $A|\psi_\mathrm{B}\rangle$ is a fermionic eigenfunction with the same energy,

$$H_\mathrm{F}\big(A|\psi_\mathrm{B}\rangle\big) = (AA^\dagger)A|\psi_\mathrm{B}\rangle = A(A^\dagger A)|\psi_\mathrm{B}\rangle = AH_\mathrm{B}|\psi_\mathrm{B}\rangle = E\big(A|\psi_\mathrm{B}\rangle\big).$$

The bosonic state $|\psi_\mathrm{B}\rangle$ and its partner state

$$|\psi_\mathrm{F}\rangle = \frac{1}{\sqrt{E}}\,A|\psi_\mathrm{B}\rangle \tag{2.19}$$

have identical norm,

$$\langle\psi_\mathrm{F}|\psi_\mathrm{F}\rangle = \frac{1}{E}\big\langle\psi_\mathrm{B}|A^\dagger A|\psi_\mathrm{B}\big\rangle = \langle\psi_\mathrm{B}|\psi_\mathrm{B}\rangle \tag{2.20}$$

and this proves, that the partner state of any excited state is never the null-vector. Likewise, the nontrivial partner state of any fermionic eigenstate $|\psi_\mathrm{F}\rangle$ with positive Energy E is

$$|\psi_\mathrm{B}\rangle = \frac{1}{\sqrt{E}}\,A^\dagger|\psi_\mathrm{F}\rangle. \tag{2.21}$$

This then proves that H_B and H_F have identical spectra, up to possible zero-modes.
We may calculate the ground state(s) of the super-HAMILTONian in position space explicitly. With (2.18) we must study the first order differential equations

$$\begin{aligned} A\psi_\mathrm{B}(x) &= \left(\frac{d}{dx} + W(x)\right)\psi_\mathrm{B}(x) = 0 \\ A^\dagger\psi_\mathrm{F}(x) &= \left(-\frac{d}{dx} + W(x)\right)\psi_\mathrm{F}(x) = 0 \end{aligned} \tag{2.22}$$

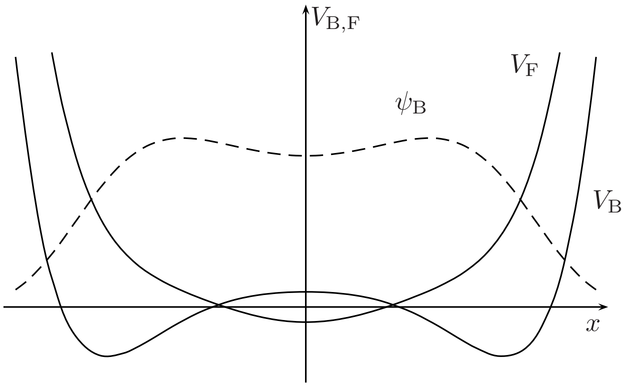
The solutions of these first order differential equations are

$$\psi_{\rm B}(x)\propto\exp\left(-\int^x W(x')dx'\right)\quad\text{and}\quad\psi_{\rm F}(x)\propto\exp\left(\int^x W(x')dx'\right). \tag{2.23}$$

If one of these two functions is normalisable, then susy is *unbroken*. Since $\psi_{\rm B}(x)\cdot\psi_{\rm F}(x)$ is constant, there exists at most one *normalisable* state with zero energy. As superpotential with unbroken supersymmetry we have chosen $W=\lambda x(x^2-a^2)$. The partner-potentials

$$\begin{aligned} V_{\rm B} &= \lambda^2x^2(x^2-a^2)^2-\lambda(3x^2-a^2)\\ V_{\rm F} &= \lambda^2x^2(x^2-a^2)^2+\lambda(3x^2-a^2) \end{aligned} \tag{2.24}$$

are plotted for $\lambda=a=1$ in the following figure.



The corresponding partner Hamilton operators have the same positive eigenvalues. There is one normalisable bosonic state with zero energy,

$$\psi_{\rm B}=\text{const.}\cdot\exp\left(-\lambda x^2\left(\tfrac14\lambda x^2-\tfrac13 a^2\right)\right). \tag{2.25}$$

This wave function is shown in the above figure.
A long time ago, Schrödinger asked the following question [5]: Given a general Hamiltonian in position space

$$\tilde H_B=-\frac{d^2}{dx^2}+V(x). \tag{2.26}$$

Is there always a first order differential operator A, such that $\tilde H_B=A^\dagger A$? This is the co-called *factorisation-problem.* Since the ground state energy E_0 of $\tilde H_B$ is in general not zero, in contrast to the ground state energy of $A^\dagger A$, we need to subtract E_0 from $\tilde H_B$ for the factorisation to work. Hence we set

$$H_B=\tilde H_B-E_0=A^\dagger A \tag{2.27}$$

Comparing with (2.17) this problem leads to the nonlinear differential equation of Ricatti,

$$V(x)-E_0=W^2(x)-W'(x). \tag{2.28}$$

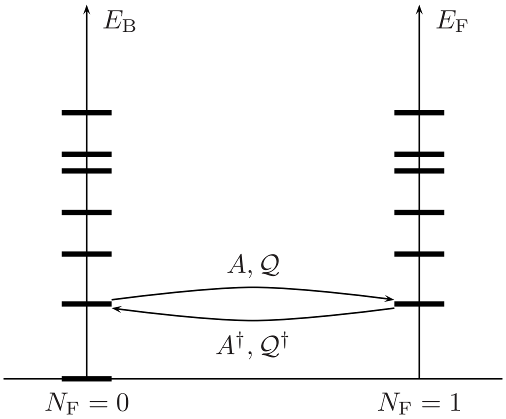
This equation is solved by the following well-known trick: setting

$$W(x)=-\frac{\psi_0'(x)}{\psi_0(x)}=-\frac{d}{dx}\log\psi_0(x), \tag{2.29}$$

the RICATTI equation transforms into the linear SCHRÖDINGER equation for ψ_0,

$$-\psi_0'' + V\psi_0 = E_0\psi_0. \tag{2.30}$$

Since the ground state ψ_0 has no node, the superpotential W is real and regular, as required. Of course, the transformation (2.29) is just the relation (2.23) between the superpotential and the ground state wave function in the bosonic sector.



The pairing of the non-zero energies and eigenfunctions in supersymmetric quantum mechanics is depicted in the figure on the left. The supercharge $\mathcal{Q}$ maps bosonic eigenfunctions into fermion ones and $\mathcal{Q}^\dagger$ maps fermionic eigenfunctions into bosonic ones. For potentials with scattering states there is a corresponding relation between the transmission and reflection coefficients of H_{B} and H_{F}. From the scattering data of H_{B} one can calculate those of its partner-HAMILTONian H_{F}. The explicit formulae are given below.

2.3 SUSY breaking in SQM

Supersymmetry requires the existence of a fermion for every boson, and vice-versa. If supersymmetry were unbroken, we would expect the super-partners to have equal mass. However, in a system with broken susy, this might not be the case, even if the partner particles did still exist.

The fact that susy has not been observed in nature so far does not imply that there are no practical uses for supersymmetric theories. It could be that every occurring supersymmetry is a broken one. We still would have a supercharge and super-HAMILTONian obeying the super algebra. But the symmetry could be spontaneously broken, in which case there is no invariant vacuum state.

In order for supersymmetry to exist and be unbroken, we require a ground state such that $H_{\mathrm{B}}|0\rangle = H_{\mathrm{F}}|0\rangle = 0|0\rangle$. This requires that the ground state is annihilated by the generators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ of supersymmetry. Thus we have

$$\text{susy unbroken} \Longleftrightarrow \text{exist normalizable } |0\rangle \text{ with } Q|0\rangle = Q^\dagger|0\rangle = 0.$$

The Witten Index: WITTEN defined an index to determine whether susy was broken in supersymmetric field theories. This index is

$$\Delta = \mathrm{Tr}\,(-1)^{N_{\mathrm{F}}}, \tag{2.31}$$

where N_{F} is the *fermion number*. For simplicity we assume that the spectrum of H is discrete and use the energy eigenfunctions to calculate Δ. Let us first assume that supersymmetry is broken, which means, that there is *no* normalizable zero-energy state. Then all eigenstates of H have positive energies and come in pairs: one bosonic state with $N_{\mathrm{F}} = 0$ and one fermionic state with $N_{\mathrm{F}} = 1$ having the same energy. Their contribution to Δ cancel. Since all states with positive energy are paired we obtain $\Delta = 0$.

Now we assume that there are ground states with zero energy, n_B bosonic ones and n_F fermionic ones. Their contribution to the WITTEN index is $n_B - n_F$. Since the contribution of the excited states cancel pairwise we obtain

$$\Delta = n_B - n_F. \tag{2.32}$$

This yields a efficient method to determine whether susy is broken,

$$\Delta \neq 0 \Longrightarrow \text{supersymmetry is unbroken.} \tag{2.33}$$

The converse need not be true. It could be that susy is unbroken but the number of zero-energy states in the bosonic and fermionic sectors are equal so that Δ vanishes. This does not happen in one-dimensional SQM, so that

$$\Delta \neq 0 \Longleftrightarrow \text{supersymmetry is unbroken in SQM.} \tag{2.34}$$

Already in SQM the operator $(-)^{N_\mathrm{F}}$ is not trace class and its trace must be regulated for the WITTEN index to be well defined. A natural definition is

$$\Delta = \lim_{\alpha \downarrow 0} \Delta(\alpha), \qquad \Delta(\alpha) = \mathrm{Tr}\left((-1)^{N_\mathrm{F}} e^{-\alpha H}\right). \tag{2.35}$$

In SQM with discrete spectrum $\Delta(\alpha)$ does not depend on α, since the contribution of all super partners cancel in (2.35). The contribution of the zero-energy states is still $n_B - n_F$. In field theories the exited states should still cancel in $\Delta(\alpha)$ in which case it is α-independent. Since $\Delta(\alpha)$ is constant, is may be evaluated at small α. But for $\alpha \to 0$ one may use the asymptotic expansion for the *heat kernel* of $\exp(-\alpha H)$ to actually calculate the WITTEN index.

2.4 Scattering states

Let us now see, how supersymmetry relates the *transmission* and *refection coefficients* of H_B and H_F for potentials supporting scattering states [6]. Thus we assume that the superpotential tends to constant values for large $|x|$,

$$\lim_{x\to\pm\infty} W(x) = W_\pm, \quad \text{such that} \quad \lim_{x\to\pm\infty} V_\mathrm{B}(x) = \lim_{x\to\pm\infty} V_\mathrm{F}(x) = W_\pm^2. \tag{2.36}$$

We consider an *incoming* plane wave from the left. The asymptotic wave function for scattering from the one-dimensional potential V_B is given by

$$\psi_\mathrm{B}(k,x) \longrightarrow \begin{cases} e^{ikx} + R_\mathrm{B} e^{-ikx} & x \to -\infty \\ T_\mathrm{B} e^{-ikx} & x \to +\infty \end{cases}, \tag{2.37}$$

where R_B and T_B are the reflection and transmission coefficient in the bosonic sector. The properly normalized fermionic partner state has the asymptotic forms

$$\psi_\mathrm{F}(k,x) = \frac{1}{ik + W_-} A\psi_\mathrm{B}(x) \longrightarrow \begin{cases} e^{ikx} + R_\mathrm{F} e^{-ikx} & x \to -\infty \\ T_\mathrm{F} e^{-ikx} & x \to +\infty \end{cases}$$

with the following reflection and transmission coefficients,

$$R_\mathrm{F} = \frac{W_- - ik}{W_- + ik} R_\mathrm{B} \quad \text{and} \quad T_\mathrm{F} = \frac{W_+ + ik}{W_- + ik} T_\mathrm{B}. \tag{2.38}$$

The scattering data for the supersymmetric partners are not the same but they are related in this simple way. As example we consider the superpotential

$$W(x) = a\tanh(bx) \quad \text{with} \quad W_\pm = \pm a,$$

which gives rise to the two partner potentials

$$V_\mathrm{B}(a,b;x) = a\left(a - \frac{a+b}{\cosh^2 bx}\right) \quad \text{and} \quad V_\mathrm{F}(a,b;x) = a\left(a - \frac{a-b}{\cosh^2 bx}\right) \tag{2.39}$$

Supersymmetry, together with the socalled *shape-invariance* [7]

$$V_\mathrm{F}(a,b;x) = V_\mathrm{B}(a-b,b;x) + 2ab - b^2 \tag{2.40}$$

allows one to find the scattering data for an infinite tower of Pöschl-Teller potentials. Let us assume, that we know the scattering data $R_\mathrm{B}(a,b)$ and $T_\mathrm{B}(a,b)$ for the parameters a and b. It follows that

$$\begin{aligned} R_\mathrm{B}(a-b,b;k) = R_\mathrm{F}(a,b;k) &= +\frac{a+\mathrm{i}k}{a-\mathrm{i}k} R_\mathrm{B}(a,b) \\ T_\mathrm{B}(a-b,b;k) = T_\mathrm{F}(a,b;k) &= -\frac{a+\mathrm{i}k}{a-\mathrm{i}k} T_\mathrm{B}(a,b). \end{aligned} \tag{2.41}$$

The iteration of this relations yields

$$\begin{aligned} R_\mathrm{B}(a,b;k) &= \prod_{n=0}^{N-1} \frac{a-nb-ik}{a-nb+ik}\, R_\mathrm{B}(a-Nb,b;k) \\ T_\mathrm{B}(a,b;k) &= (-)^N \prod_{n=0}^{N-1} \frac{a-nb-ik}{a-nb+ik}\, T_\mathrm{B}(a-Nb,b;k) \end{aligned} \tag{2.42}$$

Now we set $a = Nb$ in these relations. Then the coefficients R_B and T_B on the right hand sides are 0 and 1 and we read off the following scattering coefficients (after setting $N - n \equiv m$)

$$T_\mathrm{B}(Nb,b;k) = (-)^N \prod_{m=1}^{N} \frac{mb-ik}{mb+ik} \quad \text{and} \quad R_\mathrm{B}(Nb,b;k) = 0. \tag{2.43}$$

for the Pöschl-Teller potentials,

$$V_\mathrm{B} = b^2\left(N^2 - \frac{N(N+1)}{\cosh^2 bx}\right), \qquad W = Nb\tanh(bx). \tag{2.44}$$

The poles $k_m = -mb$ of the transmission coefficient yield the energies of the bound states,

$$E_m = k_m^2 + b^2N^2 = b^2\left(N^2 - m^2\right) \qquad m = 1,\dots,N. \tag{2.45}$$

Supersymmetry is unbroken, since the ground state has a energy zero.

2.5 Isospectral deformations

Let us assume that V_B supports n bound states. By using supersymmetry one can easily construct an n-parameter family of potentials $V(\lambda_1,\dots,\lambda_n;x)$ for which the Hamiltonian has the same energies and scattering coefficients as $H = -\triangle + V_{\rm B}$. The existence of such families of isospectral potentials has been known for a long time from the inverse scattering approach [11], but the GELFAND-LEVITAN approach for constructing them is rechnically more involved than the supersymmetry approach described here. Here we show how a *one-parameter* isospectral family of potentials is obtained by first deleting and the re-inserting the ground state of $V_{\rm B}$ using the DARBOUX-procedure [12]. The generalization to an n-parameter family is described in [13].

Suppose that $|\psi_{\rm B}\rangle$ is a normalizable zero-energy groundstate of the bosonic HAMILTONian with potential $V_{\rm B} = W^2 - W'$. Its explicit form in position space is

$$\psi_{\rm B}(x) = \exp\left(-\int^x W(x')dx'\right). \tag{2.46}$$

Suppose that the partner potential $V_{\rm F} = W^2 + W'$ is kept fixed. A natural question is whether there are other superpotentials leading to the *same* potential $V_{\rm F}$. So let us assume that there exists a second solution $\hat{W} = W + \phi$ giving rise to the same potential. This requirement leads to

$$0 = \big(\hat{W}^2 + \hat{W}'\big) - \big(W^2 + W'\big) = \phi^2 + 2W\phi + \phi' \tag{2.47}$$

The transformation $\phi = (\log F)'$ leads to the linear differential equation for F'

$$F'' + 2WF' = 0 \tag{2.48}$$

with solution

$$F'(x) = \exp\left(-2\int^x W(x')dx'\right) = \psi_{\rm B}^2(x). \tag{2.49}$$

The integration constant is just the lower bound of the integral in the exponent or equivalently the norm of the bosonic ground state. A further integration yields F and hence ϕ and introduces another constant λ which is identified with the deformation parameter,

$$\phi(x) = \frac{d}{dx}\log\big(I(x) + \lambda\big), \qquad I(x) = \int\limits_{-\infty}^{x} \psi_{\rm B}^2(x')dx' \tag{2.50}$$

In this formula for ϕ we could change the lower integration bound or multiply I with any non-vanishing constant. This is equivalent to a redefinition of the constant λ.

By construction W and $\hat{W} = W + \phi$ lead to the same $V_{\rm F}$. But the corresponding partner potentials are different,

$$\hat{W}^2 - \hat{W}' = W^2 - W' + \phi^2 + 2W\phi - \phi' \overset{(2.47)}{=} V_{\rm B} - 2\phi' = V_{\rm B} + 2\phi^2 + 4W\phi. \tag{2.51}$$

Thus the bosonic HAMILTON operators with superpotentials W and $W + \phi$ are unequal. But since they have the same partner HAMILTONian $H_{\rm F}$ they must have the same spectrum, up to possible zero modes. This then proves that the one-parameter family of HAMILTON operators

$$H_{\rm B}(\lambda) = -\frac{d^2}{dx^2} + V_{\rm B}(\lambda; x), \qquad V_{\rm B}(\lambda; x) = V_{\rm B}(x) - 2\frac{d^2}{dx^2}\log\big(I(x) + \lambda\big) \tag{2.52}$$

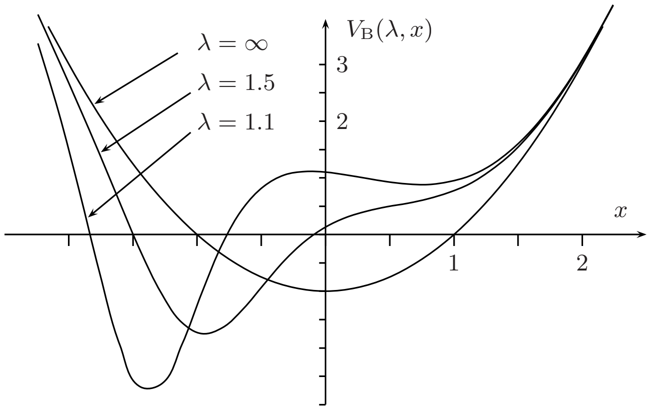
all have the same spectrum, up to possible zero modes. The deformation depends via $I(x)$ in (2.50) on the ground state wave function of the undeformed operator $H_{\rm B}$.

Deformation of the harmonic oscillator: Let us see how the deformation looks like for the simple harmonic oszillator with ground state wave function $\psi_{\rm B}(x)\propto\exp(-\omega x^2/2)$. We obtain

$$\phi(\lambda,x)=2\sqrt{\frac{\omega}{\pi}}\,\frac{e^{-\omega x^2}}{\mathrm{erf}\big(\sqrt{\omega}\,x\big)+\lambda},\quad \text{where}\quad \mathrm{erf}(y)=\frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}dt \tag{2.53}$$

is the error function, and this leads to the deformation

$$V_{\rm B}(\lambda;x)=\omega^2x^2-\omega+4\omega x\,\phi(\lambda,x)+2\phi^2(\lambda,x)=V_{\rm B}(-\lambda;-x). \tag{2.54}$$



In the figure on the left we have plotted the potential of the harmonic oscillator and the two deformed potentials for $\lambda=1.5$ and $\lambda=1.1$. We have set $\omega=1$. For the deformed potential to be regular we must assume $|\lambda|>1$. For $\lambda\to\pm\infty$ the potential tends to the potential of the harmonic oscillator. For $|\lambda|\downarrow 1$ the deviation from the oscillator potential become significant near the origin.

Deformation of reflectionless Pöschl-Teller potentials: We deform the Potential

$$V_{\rm B}=b^2\left(1-2\cosh^{-2}bx\right), \tag{2.55}$$

that is the PÖSCHL-TELLER potential (2.44) with *one bound state*. The ground state

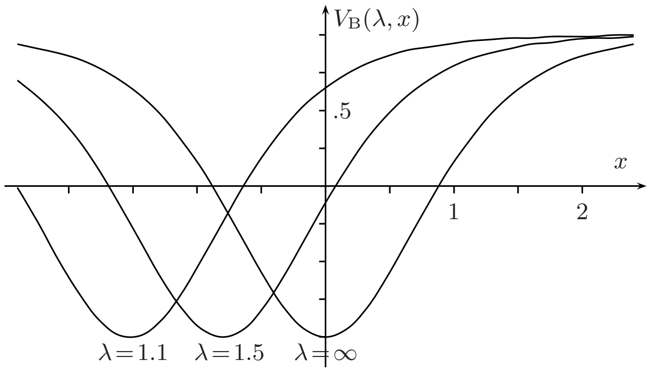
$$\psi_{\rm B}(x)=\frac{b}{\cosh bx} \tag{2.56}$$

has zero energy. Since $\int\psi_{\rm B}^2=b\tan bx$ we obtain

$$\phi(x)=\frac{b^2}{\cosh^2 bx}\,\frac{1}{b\tanh bx+\lambda} \tag{2.57}$$

This leads to the deformation

$$V_{\rm B}(\lambda,x)=b^2\left(1-2\cosh^{-2}bx\right)+4b\tanh(bx)\phi(x)+2\phi^2(x)=V_{\rm B}(-\lambda,-x). \tag{2.58}$$



In the figure on the left we have plotted the refectionless PÖSCHL-TELLER potential with one bound state and two isospectral potentials for $\lambda=1.5$ and 1.1. We have set $b=1$. For the deformed potential to be regular we must assume $|\lambda|>1$. For $\lambda\to\pm\infty$ the potential tends to the PÖSCHL-TELLER potential. For $\lambda\downarrow 1$ the minimum of the potential tends to $-\infty$ and for $\lambda\uparrow -1$ to ∞.

The potential $V_{\rm B}$ maybe viewed as soliton with center at the minimum. For $\lambda=1$ the soliton is at $x=-\infty$ and moves to the origin for $\lambda\to\infty$. For $\lambda=-1$ the soliton is centered at ∞ and moves with decreasing λ to the left. For $\lambda=-\infty$ it reaches the origin. Actually one show that after a change of variables, $\lambda=\lambda(t)$, the function $V_{\rm B}(t,x)$ solves the KORTEWEG-DEVRIES equation.

2.6 SQM in higher dimensions

Supersymmetric quantum mechanical systems also exist in higher dimensions. The construction is based on the following rewriting of the supercharge (2.13):

$$\mathcal{Q}=\psi A \quad\text{and}\quad \mathcal{Q}^\dagger=\psi^\dagger A^\dagger \tag{2.59}$$

containing the *fermionic* operators

$$\psi=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \psi^\dagger=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad\text{with}\quad \{\psi,\psi\}=\{\psi^\dagger,\psi^\dagger\}=0,\quad \{\psi,\psi^\dagger\}=\mathbb{1} \tag{2.60}$$

and the operators $A,A^\dagger$ in (2.16). The super-HAMILTONian (2.14) takes the form

$$H=p^2+W^2+W'[\psi,\psi^\dagger]. \tag{2.61}$$

In [8] this construction has been generalised to higher dimensions. In d dimensions one has d fermionic annihilation operators ψ_i and d creation operators $\psi_i^\dagger$,

$$\{\psi_i,\psi_j\}=\{\psi_i^\dagger,\psi_j^\dagger\}=0\quad\text{and}\quad\{\psi_i,\psi_j^\dagger\}=\delta_{ij},\quad i=1,\dots,d. \tag{2.62}$$

For the supercharge we make the ansatz $Q=\sum_i\psi_i(\partial_i+W_i)$. It is nilpotent if $\partial_iW_j-\partial_jW_i=0$. Locally this is equivalent to the existence of a potential χ with $W_i=\partial_i\chi$. Thus we are lead to the following *nilpotent* supercharge and its adjoint,

$$Q=\psi_i\left(\partial_i+\partial_i\chi\right)\quad\text{and}\quad Q^\dagger=\psi_i^\dagger\left(-\partial_i+\partial_i\chi\right). \tag{2.63}$$

The super HAMILTONian takes the simple form

$$H=\{Q,Q^\dagger\}=-\triangle+(\nabla\chi,\nabla\chi)+[\psi_i,\psi_j^\dagger]\,\partial_i\partial_j\chi \tag{2.64}$$

and generalizes the operator (2.61) to higher dimensions. It commutes with the *number operator*

$$N_{\rm F}=\sum_i\psi_i^\dagger\psi_i\Longrightarrow[N_{\rm F},\psi_i]=-\psi_i,\quad[N_{\rm F},\psi_i^\dagger]=\psi_i^\dagger. \tag{2.65}$$

The most direct way to find a representation for the fermionic operators makes use of the FOCK construction over a 'vacuum'-state $|0\rangle$ which is annihilated by all ψ_i,

$$\psi_i|0\rangle=0,\quad i=1,\dots,d. \tag{2.66}$$

Acting with the $\psi_i^\dagger$ on $|0\rangle$ yields states with $N_{\rm F}=1,2,\dots$. When counting these states we should take into account, that the $\psi_i^\dagger$ anticommute such that

$$|ij\dots\rangle=\psi_i^\dagger\psi_j^\dagger\cdots|0\rangle \tag{2.67}$$

is antisymmetric in $i, j, \ldots$. The states, the eigenvalues of N_F together with their degeneracies are listed in the following table:

states:	$\vert 0\rangle$	$\vert i\rangle$	$\vert i,j\rangle$	$\cdots$	$\vert 1,2,\ldots,d\rangle$
N_F	0	1	2	$\ldots$	d
# of states	$\binom{d}{0}=1$	$\binom{d}{1}=d$	$\binom{d}{2}$	$\cdots$	$\binom{d}{d}=1$

Since there are

$$\sum_{p=0}^{d}\binom{d}{p}=2^d$$

independent states we obtain a 2^d-dimensional representation of the fermionic algebra (2.62). For example, for one-dimensional systems and the orthonormal basis

$$e_1=|0\rangle \quad \text{and} \quad e_2=\psi^\dagger|0\rangle$$

the annihilation operator reads

$$\psi=\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{2.68}$$

In 2 dimensions ψ_1 and ψ_2 are 4-dimensional matrices. For the orthonormal basis

$$\{|0\rangle,|1\rangle,|2\rangle,|12\rangle\}$$

they take the form

$$\psi_1=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \quad \text{and} \quad \psi_2=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{2.69}$$

With the help of (2.62) and (2.66) one may calculate the matrix elements of ψ_i between any two FOCK states (2.67) in arbitrary dimensions.
Taking into account the x-dependency of the states, the HILBERT space of SQM in d dimensions is

$$\mathcal{H}=L_2\big(\mathbb{R}^d\big)\otimes\mathbb{C}^{2^d} \tag{2.70}$$

and decomposes into sectors with different fermion numbers,

$$\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1\oplus\ldots\oplus\mathcal{H}_d \quad \text{with} \quad N_\mathrm{F}\big|_{\mathcal{H}_p}=p\mathbb{1}. \tag{2.71}$$

An arbitrary element in $\mathcal{H}$ has the expansion

$$|\psi\rangle=f(x)|0\rangle+f_i(x)|i\rangle+\frac{1}{2}f_{ij}|ij\rangle+\frac{1}{3!}f_{ijk}|ijk\rangle+\ldots. \tag{2.72}$$

The supercharge and its adjoint in (2.63) decrease and increase the fermion number by one,

$$[N_\mathrm{F},Q]=-Q \quad \text{and} \quad [N_\mathrm{F},Q^\dagger]=Q^\dagger. \tag{2.73}$$

Since the super-HAMILTONian commutes with $N_{\rm F}$ it has block-diagonal form in the basis adapted to the decomposition (2.71) of the HILBERT space,

$$N_{\rm F} = \begin{pmatrix} 0_{\mathcal{H}_0} & & & & \\ & 1_{\mathcal{H}_1} & & & \\ & & 2_{\mathcal{H}_2} & & \\ & & & \ddots & \\ & & & & d_{\mathcal{H}_d} \end{pmatrix} \quad , \quad H = \begin{pmatrix} H_0 & & & & \\ & H_1 & & & \\ & & H_2 & & \\ & & & \ddots & \\ & & & & H_d \end{pmatrix} \tag{2.74}$$

SQM in higher dimensions with a nilpotent supercharge defines a *complex* of the following structure:

$$\mathcal{H}_0 \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_1 \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_2 \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \bullet\;\bullet\;\bullet\;\bullet \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_{d-1} \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_d \quad (Q^2 = 0)$$

It is very similar to the DE RHAM complex for differential forms. The nilpotent charge $Q^\dagger$ is the analog of the exterior differential d and Q of the co-differential δ. The super HAMILTONian $H = \{Q, Q^\dagger\}$ corresponds to the LAPLACE-BELTRAMI operator $-\triangle = d\delta + \delta d$.

2.7 The supersymmetric hydrogen atom

For the simple function $\chi = -\lambda r$ we obtain the following super-HAMILTONian

$$\begin{aligned} H &= -\triangle + \lambda^2 - (d-1)\frac{\lambda}{r} + \frac{2\lambda}{r} N_{\rm F} - 2\lambda \frac{(x,\psi^\dagger)(x,\psi)}{r^3} \\ &= -\triangle + \lambda^2 + (d-1)\frac{\lambda}{r} - \frac{2\lambda}{r} \tilde{N}_{\rm F} + 2\lambda \frac{(x,\psi)(x,\psi^\dagger)}{r^3} \end{aligned} \tag{2.75}$$

where we have introduced the operators

$$(x,\psi^\dagger) = \sum_i x_i \psi_i^\dagger, \quad (x,\psi) = \sum_i x_i \psi_i \quad \text{and} \quad \tilde{N}_{\rm F} = \sum_i \psi_i \psi_i^\dagger. \tag{2.76}$$

Since all states in $\mathcal{H}_0$ are annihilated by the ψ_i and all states in $\mathcal{H}_d$ by the $\psi_i^\dagger$ we find the following HAMILTON operators in these subspaces,

$$\begin{aligned} H_0 &= -\triangle + \lambda^2 - (d-1)\frac{\lambda}{r} \\ H_d &= -\triangle + \lambda^2 + (d-1)\frac{\lambda}{r}. \end{aligned} \tag{2.77}$$

Hence, the SCHRÖDINGER operators for both the electron-proton and positron-proton systems are part of the super-HAMILTONian H. In [9] a detailed analysis of the supersymmetric hydrogen atom has been given. In this paper is shown that the conserved RUNGE-LENZ vector of the COULOMB-problem can be supersymmetrized. Together with the angular momentum vector it generates an $SO(4)$ symmetry. This large *dynamical symmetry* group allows for a purely algebraic solution of the supersymmetric hydrogen atom, very similar to the algebraic solution of the ordinary hydrogen atom by W. PAULI [10].