# Kapitel 3

# Quantum Fields near Black Holes

In the theory of quantum fields in curved spacetimes one treats the gravitational field classically. The structure of spacetime is described by a manifold $\mathcal{M}$ on which a metric $g_{\mu\nu}$ with Lorentz signature is defined. The matter fields propagating in classical spacetime are treated as quantum fields. For linear fields a satisfactory theory can be constructed.

The approximation in which gravity is treated classically should break down when the spacetime curvature approaches Planck scales. But it should hold for a wide variety of phenomena, including the particle creation near a black hole with Schwarzschild radius much greater than the Planck length.

The difficulties in the transition from flat to curved spacetime lies in the absence of the notion of global inertial observers or of Poincare transformations which underlie the notion of particles in Minkowski spacetime. If one accepts that quantum field theory in general curved spacetime is a *quantum theory of fields, not particles*, then one soon realizes that the the notion of global inertial observers is irrelevant for the formulation of the theory.

For a field theory the Stone-von Neumann theorem does not hold and infinitely many inequivalent irreducible representation of the canonical commutation relations exist. In flat spacetime, Poincare symmetry is used to pick out a preferred representation. This is achieved by selecting a invariant vacuum state which is equivalent to a selection of a particle notion. In a general curved spacetime there does not appear to be any preferred notion of particles. Actually, in spacetimes which are flat in the asymptotic past and the asymptotic future and for which a natural notion of particles is available in both asymptotic regions, the corresponding two representations are, in general, inequivalent.

A way out of these difficulties in picking a particular representation is to formulate the theory via the algebraic approach. No particular representation of the commutation relations need to be chosen and one needs not define a preferred notion of particles.

The framework and structure of Quantum field theory in curved spacetimes emerged from Parkers analysis of particle creation in the very early universe [1]. The theory received enormous impetus from Hawking's discovery, that black holes radiate as blackbodies due to particle creation [2]. A comprehensive summary of the work in the 1970's can be found in the book of Birrell and Davies [3] and a more up-to-date review can be found in Fulling [4].

## 3.1 The Unruh Effect

Any one-parameter group of Lorentz boost isometries in Minkowski spacetime has orbits which are timelike in a globally hyperbolic region. Such a region may be viewed as spacetime in its own right and we may construct a quantum field theory on it. When we do that, we obtain a remarkable conclusion, namely that the standard Minkowski vacuum $\Omega_M$ corresponds to a thermal state in the new construction. This means, that an accelerated observer will feel himself to be immersed in a thermal bath of particles with temperature proportional to his acceleration $a$,

$$kT = \hbar a/2\pi c.$$

The temperature tends to zero in the limit in which Planck's constant $h$ tends to zero. Such a radiation has non-zero entropy. Since the use of a accelerated frame seems to be unrelated to any statistical average, the appearance of a non-vanishing entropy is rather puzzling.

The Unruh effect shows, that at the quantum level there is deep relation between the theory of relativity and the theory of fluctuations associated with states of thermal equilibrium, two major aspects of Einstein's work: The distinction between quantum zero-point and thermal fluctuations is not an invariant one, but depends on the motion of the observer.

The Unruh effect was discovered in an attempt to gain more insight into the nature of the Hawking radiation [5]. Let us now consider a one-parameter family of Lorentz boosts in the 1-direction. Since $x^2$ and $x^3$ are not changed by such boosts, we need only consider the change of the first two coordinates $x = (T, X)^t$:

$$x = \begin{pmatrix} \cosh(au) & \sinh(au) \\ \sinh(au) & \cosh(au) \end{pmatrix} x(0) = e^{\omega u} x(0), \qquad (\omega^\mu_{\ \nu}) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Since $\dot{x}(u) = \omega x(u)$, the orbits are tangential to the *Killing field*

$$\xi = \omega x = a \begin{pmatrix} X \\ T \end{pmatrix} \quad \text{with} \quad (\xi, \xi) = -a^2 (x, x).$$

Some typical orbits are depicted in the figure (3.1). The Killing field is timelike in the regions $R, L$ and spacelike in the regions $F, P$. It is timelike future directed in the Rindler wedge $R$, defined by $X \geq |T| \geq 0$. Since

$$(\ddot{x}, \ddot{x}) = a^4 (x, x) = -a^2 (\xi, \xi),$$

where dot is the derivative with respect to the variable $u$ associated to the Killing field $\xi$, the observers following orbits of $\xi$ all undergo uniform acceleration, although this acceleration varies from orbit to orbit. Since on the orbit with $(\xi, \xi) = 1$ or $(x, x) = -1/a^2$ is $a$, it is conventional to view the orbits of $\xi$ as corresponding to a family of observers naturally associated with an observer who accelerates uniformly with acceleration $a$. The notion of 'particles' obtained from

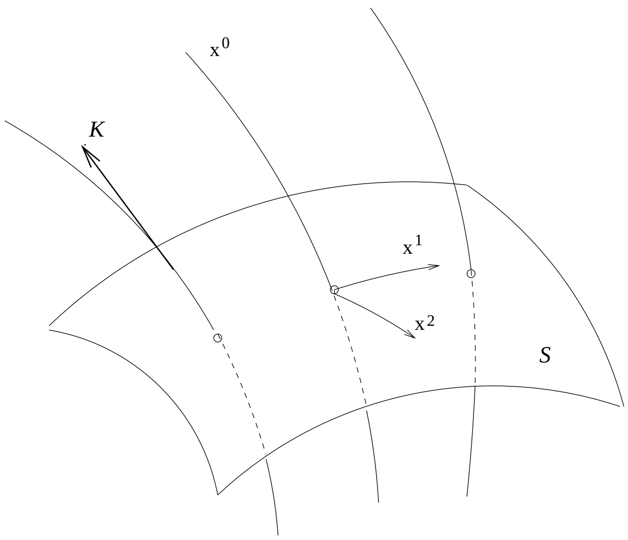


Abbildung 3.1: *A Rindler-observer sees only a quarter of Minkowski space*

this quantum field construction are referred to as the 'particles seen by an observer who undergoes uniform acceleration $a$'.

All inextendible causal curves through any point in the Rindler wedge intersect the hyperplane $\Sigma_R$ (see figure (3.2)) and this hyperplane is therefore a Cauchy surface for the globally hyperbolic Rindler wedge $R$.

The null plane $h_A$ in this figure is a *Killing horizon* for $R$. Every particle which has left the Rindler wedge (through $h_A$) cannot return to it.

We coordinatize the wedge by the affine parameter $u$ on the orbits and by the invariant 'distance' $(x,x)$ of the orbits from the origin. The latter is negative on the Rindler wedge and thus we set $(x,x) = -e^{av}/a$ so that $v \in R$ and the acceleration on the orbit with $v = 0$ is $a$. The transformation from $x$ to $(u,v)$ reads

$$T = \frac{1}{a}e^{av}\sinh au \quad , \quad X = \frac{1}{a}e^{av}\cosh au,$$

where we took $u = 0$ for $T = 0$. The inverse transformation is

$$u = \frac{1}{a}\text{artanh}\frac{T}{X} \quad \text{and} \quad v = \frac{1}{2a}\log a^2\big[X^2 - T^2\big].$$

The Rindler wedge is covered by $(u,v) \in R^2$ and the future event horizon has $u = \infty$. To make the problem simple, we begin with a free zero-mass scalar field $\phi$ in 2-dimensional Minkowski space. The Lagrangian $\mathcal{L}$ and Hamiltonian $\mathcal{H}$ are given by

$$\mathcal{L} = \frac{1}{2}\int_{-\infty}^{\infty}\Big[(\frac{\partial\phi}{\partial T})^2 - (\frac{\partial\phi}{\partial X})^2\Big]dX, \quad \mathcal{H} = \frac{1}{2}\int_{-\infty}^{\infty}\Big[\pi^2 + (\frac{\partial\phi}{\partial X})^2\Big]dX,$$

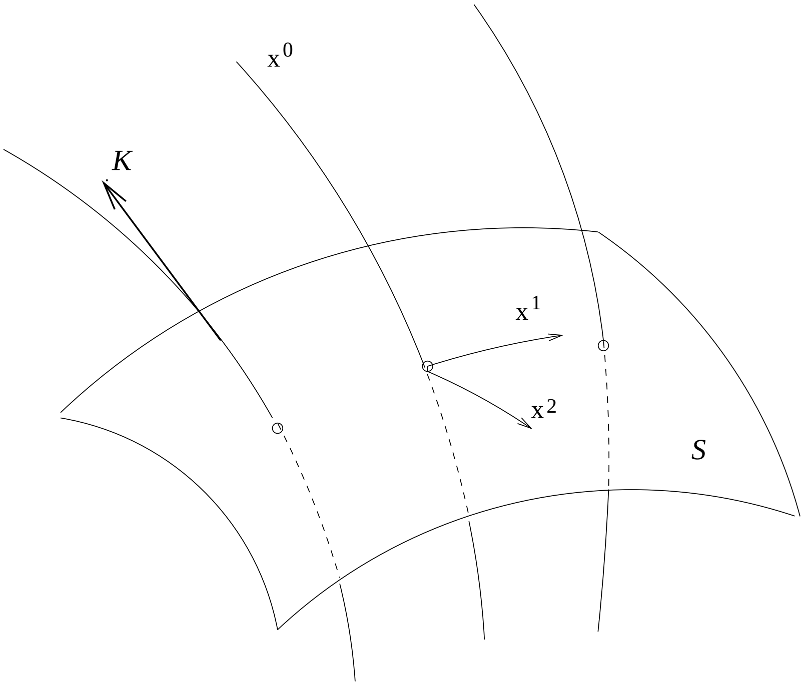


Abbildung 3.2: *A Cauchy surface $\Sigma_R$ and the horizons.*

where $\pi = \partial\phi/\partial T$ is the momentum-field conjugate to $\phi$. At equal Minkowski time $T$, we have the usual commutation relation

$$[\phi(T,X), \pi(T,X')] = i\delta(X - X'). \tag{3.1}$$

The transition from Minkowski- to Rindler space is a conformal transformation,

$$ds^2 = dx^\mu dx_\mu = e^{2\sigma}\big(du^2 - dv^2\big), \qquad \text{where} \quad \sigma = av,$$

and as a consequence the Lagrangian and Hamiltonian in Rindler space have the same form as in Minkowski spacetime

$$\begin{aligned} \tilde{\mathcal{L}} &= \frac{1}{2}\int_{-\infty}^{\infty} \Big[\big(\frac{\partial\phi}{\partial u}\big)^2 - \big(\frac{\partial\phi}{\partial v}\big)^2\Big] dX \\ \tilde{\mathcal{H}} &= \frac{1}{2}\int_{-\infty}^{\infty} \Big[\tilde{\pi}^2 + \big(\frac{\partial\phi}{\partial v}\big)^2\Big] dX, \end{aligned}$$

where now $\tilde{\pi} = \partial\phi/\partial u$. The equal-time commutation relation in Rindler space is

$$[\phi(u,v), \tilde{\pi}(u,v')] = i\delta(v-v'). \tag{3.2}$$

The field equations in both $M$ and $R$ take on the identical forms

$$\frac{\partial^2\phi}{\partial T^2} - \frac{\partial^2\phi}{\partial X^2} = 0 \quad \text{and} \quad \frac{\partial^2\phi}{\partial u^2} - \frac{\partial^2\phi}{\partial v^2} = 0.$$

In the Heisenberg picture, the expansions in terms of annihilation and creation operators are

$$\phi(T,X) = \int \frac{dk}{\sqrt{2\omega}}\Big(a_k f_k(t,x) + h.c.\Big), \quad \text{where} \quad f_k = \frac{1}{\sqrt{2\pi}} e^{-i\omega t + ikx}, \quad \omega = |k|$$

and

$$\phi(u,v) = \int \frac{dp}{\sqrt{2\epsilon}}\Big(\tilde{a}_p \tilde{f}_p(u,v) + h.c.\Big), \quad \text{where} \quad \tilde{f}_p = \frac{1}{\sqrt{2\pi}} e^{-i\epsilon u + ipv}, \quad \epsilon = |p|.$$

From the equal time commutators (3.1) and (3.2) one derives the following commutation relation for the annihilation and creation operators

$$[a_k, a^\dagger_{k'}] = \delta(k-k'), \qquad [a_p, a^\dagger_{p'}] = \delta(p-p'), \qquad [a_k, a_{k'}] = [a_p, a_{p'}] = 0.$$

The *vacuum state* in Minkowski spacetime is characterized by

$$a_k \Omega_M = 0 \quad \text{for all} \quad k$$

Assuming that this is the state of the system, the expectation value of the occupation number as defined by the Rindler observer, $n_p \equiv a^\dagger_p a_p$, is found to be

$$\boxed{\big(\Omega_M, n_p \Omega_M\big) = \text{volume} \times \frac{1}{e^{2\pi\epsilon/a} - 1}.} \tag{3.3}$$

Thus for an accelerated observer the quantum field seems to be in an equilibrium state with temperature proportional to $a$. This puzzling result is the *Unruh effect*. We now give a proof of this important result.

First we express the annihilation and creation operators in Rindler space in terms of the field operator and its $u$-derivative as

$$\begin{aligned}
\frac{1}{\sqrt{2\epsilon}}\Big(\tilde{a}_p e^{-i\epsilon u} + \tilde{a}^\dagger_{-p} e^{i\epsilon u}\Big) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dv\, \phi(u,v) e^{-ipv} \\
i\sqrt{\frac{\epsilon}{2}}\Big(-\tilde{a}_p e^{-i\epsilon u} + \tilde{a}^\dagger_{-p} e^{i\epsilon u}\Big) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dv\, \frac{\partial\phi(u,v)}{\partial u} e^{-ipv}.
\end{aligned}$$

We insert the expansion of the field operator in terms of the creation and annihilation operators in Minkowski spacetime. Using, that at $u = 0$

$$T = 0, \qquad X = \frac{1}{a} e^{av}, \qquad \frac{\partial T}{\partial u} = e^{av} \quad \text{and} \quad \frac{\partial X}{\partial u} = 0$$

one arrives at the following *Bogolubov transformation* relating $(a_k, a_k^\dagger)$ with $(\tilde{a}_p, \tilde{a}_p^\dagger)$:

$$\begin{aligned}\tilde{a}_p + \tilde{a}_{-p}^\dagger &= \int_{-\infty}^{\infty} dv dk \sqrt{\frac{\epsilon}{\omega}} \Big( a_k f(k,p,v) + a_k^\dagger f(-k,p,v) \Big) \\ \tilde{a}_p - \tilde{a}_{-p}^\dagger &= \int_{-\infty}^{\infty} dv dk \sqrt{\frac{\omega}{\epsilon}} \Big( a_k f(k,p,v) - a_k^\dagger f(-k,p,v) \Big) e^{av}.\end{aligned}$$

We have introduced the function

$$f(k,p,v) = \frac{1}{2\pi} \exp\big( i[\frac{k}{a} e^{av} - pv] \big).$$

The Bogolubov transformation can be solved for the annihilation operators in Rindler space:

$$\tilde{a}_p = \int dk dv \Big( \big[ \sqrt{\frac{\epsilon}{\omega}} + \sqrt{\frac{\omega}{\epsilon}} e^{av} \big] a_k f(k,p,v) + \big[ \sqrt{\frac{\epsilon}{\omega}} - \sqrt{\frac{\omega}{\epsilon}} e^{av} \big] a_k^\dagger f(-k,p,v) \Big).$$

Setting $y = \exp(av)$ and using the formula

$$\int_0^{\infty} dx\, x^{\nu-1} e^{-(\alpha+i\beta)x} = \Gamma(\nu)(\alpha^2+\beta^2)^{-\nu/2} e^{-i\nu \arctan(\beta/\alpha)}$$

valid for $\alpha \geq 0$ and $0 < \nu < 1$ we find for $\nu \to 0$

$$\begin{aligned}Y(k,p) = \int dv f(k,p,v) &= \frac{1}{2\pi a} \int dy e^{iky/a} y^{-1-ip/a} \\ &= \begin{cases} \frac{1}{2\pi a} \left(\frac{\omega}{a}\right)^{ip/a} \Gamma(-\frac{ip}{a}) e^{\pi p/2a} & \text{if } k > 0 \\ \frac{1}{2\pi a} \left(\frac{\omega}{a}\right)^{ip/a} \Gamma(-\frac{ip}{a}) e^{-\pi p/2a} & \text{if } k < 0. \end{cases}\end{aligned}$$

Analogously, one finds

$$\int dv\, e^{av} f(k,p,v) = \frac{p}{k} Y(k,p).$$

A short calculation shows that

$$\begin{aligned}\tilde{a}_p &= 2 \int_0^{\infty} \sqrt{\frac{\epsilon}{\omega}} \Big( Y(k,p) a_k + Y(-k,p) a_k^\dagger \Big), \quad p > 0 \\ \tilde{a}_p &= 2 \int_0^{\infty} \sqrt{\frac{\epsilon}{\omega}} \Big( Y(-k,p) a_{-k} + Y(k,p) a_{-k}^\dagger \Big), \quad p < 0.\end{aligned}$$

Using the commutator-relation for the $a_k$ and $a_{k'}^\dagger$ and that the $a_k$ annihilate the Minkowski vacuum allows us to calculate the expectation value $(\Omega_M, n_p \Omega_M)$. Using finally that

$$|\Gamma(iy)|^2 = \Gamma(iy)\Gamma(-iy) = \frac{\pi}{y \sinh(\pi y)}$$

one finds the following expression for this expectation value

$$\boxed{(\Omega_M, n_p \Omega_M) \sim \frac{1}{e^{2\pi\epsilon/a} - 1}.} \tag{3.4}$$

We will come back to the Unruh effect and its physical interpretation later on.

### 3.1.1 Bogolubov Transformations

Using the Klein-Gordon field equation it is easily seen, that the inner product

$$(u_1, u_2) \equiv i \int\limits_{\Sigma} \Big( \bar{u}_1 n^\mu \nabla_\mu u_2 - (n^\mu \nabla_\mu \bar{u}_1) u_2 \Big) \sqrt{h}\, d^3x$$

is conserved for two (complex) solutions. Here $\Sigma$ is some spacelike Cauchy hypersurface in space-time, $n^\mu$ is the future directed unit-vector field normal to $\Sigma$ and $h_{\mu\nu}$ the induced metric on the hypersurface. This inner product will not be positive definite for boson fields. Let us introduce a complete set of conjugate pairs of solutions $(u_k, u_k^\dagger)$ of the Klein-Gordon equation[1] satisfying the following orthonormality conditions

$$(u_k, u_{k'}) = \delta(k, k') \Rightarrow (\bar{u}_k, \bar{u}_{k'}) = -\delta(k, k') \quad (u_k, \bar{u}_{k'}) = 0.$$

There will be an infinity of such sets. Now we expand the field operator in terms of these modes:

$$\phi = \int d\mu(k) \Big( a_k u_k + a_k^\dagger \bar{u}_k \Big),$$

so that

$$(u_k, \phi) = a_k \quad \text{and} \quad (\bar{u}_k, \phi) = -a_k^\dagger.$$

By using the canonical commutation relations it is then easy to show that the operator coefficients $(a_k, a_k^\dagger)$ satisfy the usual commutation relations.

If $(v_p, \bar{v}_p)$ is a second set of basis functions we may as well expand the field operator in terms of this set

$$\phi = \int d\mu(p) \Big( b_p v_p + b_p^\dagger \bar{v}_p \Big).$$

The second set will be linearly related to the first one by

$$\begin{aligned} v_p &= \int d\mu(k) \Big( (u_k, v_p) u_k - (\bar{u}_k, v_p) \bar{u}_k \Big) = \int d\mu(k) \Big( \alpha(p,k) u_k + \beta(p,k) \bar{u}_k \Big) \\ \bar{v}_p &= \int d\mu(k) \Big( (u_k, \bar{v}_p) u_k - (\bar{u}_k, \bar{v}_p) \bar{u}_k \Big) = \int d\mu(k) \Big( \bar{\beta}(p,k) u_k + \bar{\alpha}(p,k) \bar{u}_k \Big). \end{aligned}$$

[1] the $k$ are any labels, not necessarily the momentum

The inverse transformation reads

$$\begin{aligned} u_k &= \int d\mu(p)\Big(v_p\bar{\alpha}(p,k) - \bar{v}_p\beta(p,k)\Big) \\ \bar{u}_k &= \int d\mu(p)\Big(-v_p\bar{\beta}(p,k) + \bar{v}_p\alpha(p,k)\Big). \end{aligned}$$

If the $\beta(k,p)$ vanish the 'vacuum' is left unchanged, but if they do not vanish we have a nontrivial *Bogolubov transformations*

$$(\, a \quad a^\dagger\,) = (\, b \quad b^\dagger\,)\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{3.5}$$

which mixes the annihilation and creations operators. If one defines a Fock space and a 'vacuum' corresponding to the first mode expansion,

$$a_k\Omega_u = 0,$$

then the expectation of the number operator $b_p^\dagger b_p$ defined with respect to the second mode expansion is

$$\big(\Omega_u, b_p^\dagger b_p\Omega_u\big) = \int d\mu(k)|\beta(p,k)|^2.$$

That is, the old vacuum contains new particles. It may even contain an infinite number of new particles, in which case the two Fock spaces cannot be related by a unitary transformation.

### 3.1.2 Green functions

The Green functions of the Klein-Gordon operator, generically denoted by $G(x,x')$, are solutions of

$$(\Box + m^2)G(x,x') = \delta^4(x-x') \tag{3.6}$$

and allow for a solution of the Klein-Gordon equation with sources:

$$(\Box + m^2)\phi(x) = j(x) \Longrightarrow \phi(x) = \phi^{(0)}(x) + \int d^4x' G(x,x')j(x'),$$

where $\phi^{(0)}$ obey the homogeneous equation and is chosen in such a way that $\phi$ satisfies the boundary conditions.

Making use of translation invariance, (3.6) is solved through a Fourier transformation. Setting

$$G(\xi) = \frac{1}{(4\pi)^2}\int d^4p\, e^{-ip\xi}\tilde{G}(p)$$

we get

$$(-p^2 + m^2)\tilde{G}(p) = 1.$$

In classical field theory the *retarded and advanced Green functions*[2]

$$G_{\substack{ret\\adv}}(x) = -\frac{1}{(2\pi)^4}\int d^4p\,\frac{e^{-ipx}}{(p_0 \pm i\epsilon)^2 - \vec{p}^2 - m^2}$$

play an important role. These distributions are Lorentz invariant. $G_{ret}$ vanishes outside the forward light cone and $G_{adv}$ outside of the backward light cone. Both Green functions are real, with $G_{adv}(x) = G_{ret}(-x)$. For massless particles

$$G_{\substack{ret\\adv}}(x) = \frac{1}{2\pi}\theta(\pm x^0)\delta(x^2).$$

The mass term is responsible for the fact that the support is not concentrated on the light cone, but also involves signals propagating at a speed smaller than one.

These Green functions can be gotten from the *Pauli-Jordan (Schwinger) function*

$$iG(x,x') = \big(\Omega_M, [\phi(x), \phi(x')]\Omega_M\big)$$

as follows

$$G_{ret}(x,x') = -\theta(t-t')G(x,x') \quad \text{and} \quad G_{adv}(x,x') = \theta(t'-t)G(x,x').$$

$G$ is the difference of its positive and negative frequency parts,

$$iG = G^+ - G^-,$$

and these parts are just the *Wightman functions*

$$G^+(x,x') = (\Omega_M, \phi(x)\phi(x')\Omega_M) \quad \text{and} \quad G^-(x,x') = (\Omega_M, \phi(x')\phi(x)\Omega_M).$$

In the quantum theory one encounters another solution to the same equation, first introduced by *Stueckelberg and Feynman*:

$$iG_F(x,x') = \Big(\Omega_M, T\big((\phi(x)\phi(x')\big)\Omega_M\Big) = \frac{i}{(2\pi)^4}\int d^4p\,\frac{e^{-ipx}}{p^2 - m^2 + i\epsilon}.$$

Contrary to the retarded and advanced Green functions $G_F$ is complex and has an exponential tail for negative $x^2$. In terms of the Wightman functions it is

$$iG_F = \theta(t-t')G^+(x,x') + \theta(t'-t)G^-(x,x').$$

The Feynman propagator $G_F$ obeys the differential equation

$$(\Box + m^2)G_F(x,x') = -\delta(x-x').$$

For massless particles the Feynman propagator and Hadamard's elementary function $G^{(1)} = G^+ + G^-$ become

[2] better: distributions

$$G_F(x,x') = \frac{i}{4\pi^2}\frac{1}{\xi^2 - i\epsilon} \quad \text{and} \quad G^{(1)}(x,x') = -\frac{1}{2\pi^2\xi^2}, \tag{3.7}$$

where $\xi = x - x'$. For massive field the Feynman propagator is given in term of Hankel functions. In curved spacetime some, but not all, of these Green functions are intrinsically determined by the manifold. The Pauli-Jordan commutator function

$$iG(x,x') = \big(\Omega_M, [\phi(x), \phi(x')]\Omega_M\big)$$

is a $c$-number calculable from field equation and the canonical commutation relations. The retarded and advanced Green functions are uniquely defined by the purely geometrical restrictions on their supports. However, the positive frequency part of $G$, i.e. the Wightman function, requires for its definition either a distinguished vacuum vector of a notion of positive frequency. These elements are either absent (for time-dependent models) or ambiguous (as in Rindler spacetime). Similar remarks apply to the Feynman propagator.

## 3.2 The KMS condition

Consider an arbitrary quantum mechanical system with time-independent Hamiltonian $H$. The time evolution of an observable, represented by $A$, in the Heisenberg picture is

$$A(z) = e^{izH}Ae^{-izH},$$

where $z = t + i\tau$ is a complex time. If $\tau = 0$ then this is the time-evolution in a static spacetime with Lorentzian signature, if $t = 0$ then it is the time-evolution in the corresponding static spacetime with euclidean signature. If $\exp(-\beta H), \beta > 0$ is trace class, one can define the equilibrium state of temperature $T = 1/\beta$:

$$\langle A\rangle_\beta = \frac{1}{Z}\operatorname{tr} e^{-\beta H}A, \qquad Z = \operatorname{tr} e^{-\beta H}. \tag{3.8}$$

For two observables $A$ and $B$ we define the thermal expectation values

$$\begin{aligned} G_+^\beta(z,A,B) &= \langle A(z_2)B(z_1)\rangle_\beta = \frac{1}{Z}\operatorname{tr}\Big(e^{-\beta H}e^{iz_2H}Ae^{-i(z_2-z_1)H}Be^{-iz_1H}\Big) \\ &= \frac{1}{Z}\operatorname{tr}\Big(e^{i(z+i\beta)H}Ae^{-izH}B\Big) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} G_-^\beta(z,A,B) &= \langle B(z_1)A(z_2)\rangle_\beta = \frac{1}{Z}\operatorname{tr}\Big(e^{-\beta H}e^{iz_1H}Be^{-i(z_1-z_2)H}Ae^{-iz_2H}\Big) \\ &= \frac{1}{Z}\operatorname{tr}\Big(Be^{izH}Ae^{-i(z-i\beta)H}\Big) \end{aligned} \tag{3.10}$$

where we have used the cyclicity of the trace and introduced $z = z_2 - z_1$. Both exponents in (3.9) have negative real parts if $-\beta < \tau < 0$; for (3.10) the condition is $0 < \tau < \beta$. Therefore, these two formulas define holomorphic functions in those respective strips. $G^{\beta}_{\pm}(t, A, B)$ are their boundary values. From (3.9,3.10) it follows immediately, that

$$G^{\beta}_{-}(z, A, B) = G^{\beta}_{+}(z - i\beta, A, B) \tag{3.11}$$

For $z = t$ this reads

$$\langle BA(t)\rangle_{\beta} = \langle A(t - i\beta)B\rangle_{\beta}. \tag{3.12}$$

Condition (3.11) is called the KMS condition after Kubo, Martin and Schwinger [9]. It can be given a precise sense in terms of $C^*$ algebras and their states for systems for which $\exp(-\beta H)$ is not trace-class. The KMS-condition is now accepted as a definition of 'thermal equilibrium at temperature $1/\beta$'.

So far the analytic functions $G_{\pm}$ have been defined in disjoint, adjacent trips in the complex time plane. The KMS-condition states that one of these is the translate of the other and this allows us to define a periodic function throughout the complex plane, with the possible exception of the lines $\tau = \Im(z) = n\beta$. Suppose, that

$$[A(t), B] = 0 \quad \text{for} \quad t \in I \subset R.$$

Then the boundary values of $G^{\beta}_{+}$ and $G^{\beta}_{-}$ coincide on $I$ and we conclude (by the edge-of-the-wedge theorem) that $G^{\beta}_{\pm}$ are restrictions of a single holomorphic, periodic function, $\mathcal{G}^{\beta}(z, A, B)$, defined in a connected region in the complex time plane except parts of the lines $\tau = n\beta$.

## 3.3 Static spacetimes

In a static spacetime we may choose coordinates, such that the metric has the form

$$(g_{\mu\nu}) = \begin{pmatrix} g_{00} & 0 \\ 0 & g_{ij} \end{pmatrix}$$

with time-independent entries. Such a metric is conformally equivalent to a ultra-static metric $\hat{g}_{\mu\nu}$,

$$g_{\mu\nu} = g_{00}\hat{g}_{\mu\nu} = g_{00}\begin{pmatrix} 1 & 0 \\ 0 & -h_{ij} \end{pmatrix}.$$

Since

$$\Box_g + m^2 = g_{00}^{-(d+2)/4}\Big(\Box_{\hat{g}} + g_{00}m^2 + \text{curvature terms}\Big)g_{00}^{(d-2)/4}$$

the Klein-Gordon equation

$$(\Box_g + m^2)\phi = 0$$

is equivalent to

$$\frac{\partial^2}{\partial t^2}u = Ku, \quad \text{where} \quad K = -\triangle_{\hat{g}} + V(\vec{x}) \quad \text{and} \quad \phi = (g_{00})^{(2-d)/4}u. \tag{3.13}$$

The solutions of this equation have the form

$$u_\nu(t,\vec{x}) = e^{-i\omega_\nu t}\psi_\nu(\vec{x})$$

where the $\psi_\nu$ are normalized eigenfunctions of the hermitian operator $K$:

$$K\psi_\nu = \omega_\nu^2\psi_\nu.$$

The conserved Klein-Gordon inner product

$$(\phi_{\nu_1},\phi_{\nu_2}) \equiv i\int\limits_{\Sigma}\left(\phi_{\nu_1}^\dagger n^\mu\nabla_\mu\phi_{\nu_2} - \text{c.c.}\right)\sqrt{-\det(g_{ij})}\,d^3x$$

is then proportional to the $L_2$-scalar product on the hypersurfaces $t$ =constant:

$$(\phi_{\nu_1},\phi_{\nu_2}) = 2\omega_\nu\int d^3x\sqrt{h}\,\psi_{\nu_1}^\dagger\psi_{\nu_2} = 2\omega_\nu\langle\psi_{\nu_1},\psi_{\nu_2}\rangle.$$

The Green functions of the field operator

$$\phi(x) = \sum\frac{1}{\sqrt{\omega_\nu}}\Big(\psi_\nu e^{-i\omega_\nu t}a_\nu + \psi_\nu^\dagger e^{i\omega_\nu t}a_\nu^d agger \tag{3.14}$$

namely

$$\begin{aligned} G_+^\infty(t,x,y) &= \langle 0|\phi(x)\phi(y)|0\rangle = \sum\psi_\nu(\vec{x})\psi_\nu^\dagger(\vec{y})e^{-i\omega_\nu t} \\ G_-^\infty(t,x,y) &= \langle 0|\phi(y)\phi(x)|0\rangle = \sum\psi_\nu(\vec{y})\psi_\nu^\dagger(\vec{x})e^{i\omega_\nu t}, \end{aligned}$$

where $t = x^0 - y^0$, are each analytic function on a half plane. $G_+$ is analytic in $z = t + i\tau$ for $\tau < 0$ and $G_-$ for $\tau > 0$. The distributions $G_\pm^\infty$ are boundary values of these analytic functions as the real axis is approached from their respective directions. If $\vec{x} \neq \vec{y}$ then the $x$ and $y$ will be spacelike for sufficiently small $t = x^0 - y^0$. Since $\phi(x)$ and $\phi(y)$ must commute for spacelike separated $x, y$, we have

$$G_+^\infty(z,\vec{x},\vec{y}) = G_-^\infty(z,x,y)$$

for $z$ on a certain interval $(-d,d)$ of the real axis. Therefore, each of these functions is an analytic continuation of the other. That is, for fixed $\vec{x} \neq \vec{y}$ there is a single holomorphic $\mathcal{G}^\infty(z,\vec{x},\vec{y})$, defined on a connected region of the complex time plane, such that

$$\mathcal{G}^\infty(z,\vec{x},\vec{y}) = \begin{cases} G_+^\infty(z,\vec{x},\vec{y}) & \text{if } \Im(z) < 0 \\ G_-^\infty(z,\vec{x},\vec{y}) & \text{if } \Im(z) > 0, \end{cases}$$

and both equalities hold on the interval on the real axis. In general there will be branch cuts along the real axis from $z = \pm d$ to $z = \pm\infty$. On the imaginary axis

$$G^\infty(\tau,\vec{x},\vec{y}) = \mathcal{G}^\infty(i\tau,\vec{x},\vec{y}) = \sum\psi_\nu(\vec{x})\psi_\nu^\dagger(\vec{y})\;e^{-\omega_\nu|\tau|}$$

and this euclidean Green function (often called two-point Schwinger function) is the unique solution of

$$\Big(\frac{\partial^2}{\partial\tau^2} + K)G^\infty(x,y) = \delta(\tau)\delta(\vec{x}-\vec{y})\frac{1}{\sqrt{\gamma}}, \quad \tau = x^0 - y^0,$$

which decays for $|\tau| \to \infty$.

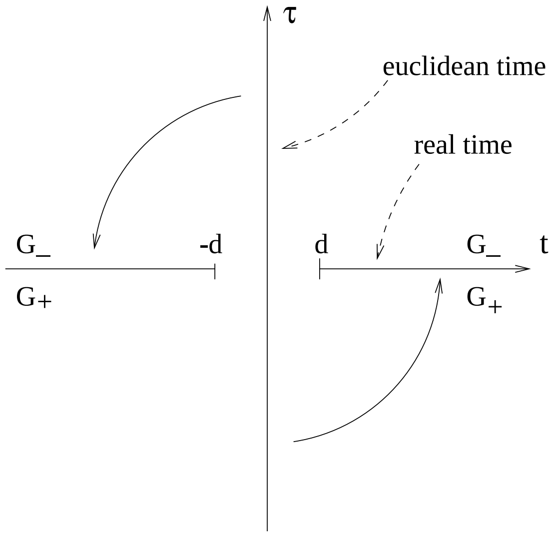


Abbildung 3.3: *The various two-point functions are boundary values of the analytic $\mathcal{G}^\infty$.*

### 3.3.1 Flat spacetime

For simplicity we consider a massless field. Then

$$G^\infty(\tau,\vec{\xi}) = \frac{1}{4\pi^2}\frac{1}{\tau^2+\vec{\xi}^2}, \qquad \vec{\xi} = \vec{x}-\vec{y}$$

from which follows, that

$$\mathcal{G}(z,\vec{\xi}) = -\frac{1}{4\pi^2}\frac{1}{z^2-\vec{\xi}^2}.$$

From this we read of, that

$$-4\pi^2 G_+^\infty(x,y) = \begin{cases} \big[(x-y)^2 - i\epsilon\big]^{-1} & \text{if } x^0 > y^0 \\ \big[(x-y)^2 + i\epsilon\big]^{-1} & \text{if } x^0 < y^0, \end{cases}$$

and

$$-4\pi^2 G_-^{\infty}(\tau,\vec{\xi}) = \begin{cases} \left[(x-y)^2 + i\epsilon\right]^{-1} & \text{if } x^0 > y^0 \\ \left[(x-y)^2 - i\epsilon\right]^{-1} & \text{if } x^0 < y^0. \end{cases}$$

All other Green functions are obtained similarly. For example, the Feynman-Greenfunction

$$i\Delta_F(x,y) = \langle 0|T\big(\phi(x)\phi(y)\big)|0\rangle$$

is

$$\Delta_F(x,y) = \frac{i}{4\pi^2}\frac{1}{(x-y)^2 - i\epsilon}.$$

**Application to the Rindler wedge** Let $p$ and $p'$ be two events on the world line of an accelerated observer with fixed $v$. The event $p$ happens at Rindler time $u$ and the earlier event $p'$ at Rindler time $u'$. Expressed in Rindler coordinates, the invariant distance between $p$ and $p'$ is

$$(\Delta T)^2 - (\Delta X^1)^2 = \frac{4}{a^2} e^{2av} \sinh^2 \frac{a}{2}(u-u')$$

and the free massless Feynman propagator, used by an inertial observer, is

$$G_F(p,p') = \frac{i}{16\pi^2} a^2 e^{-2av} \frac{1}{\sinh^2 \frac{a}{2}(u-u') - i\epsilon}. \tag{3.15}$$

Now let us rewrite the right hand side. First, specializing the results for the massless Feynman propagator to points on the world line of an accelerated observer, we immediately obtain

$$\int d^4p\, e^{-iE(u-u')} \frac{i}{p^2 + i\epsilon} = -\frac{4\pi^2}{(u-u')^2 - i\epsilon}.$$

Now we integrate in

$$\int d^4p\, e^{-iE(u-u')} \frac{2\pi\delta(p^2)}{e^{\beta|E|} - 1}$$

over $p^0 = E$, introduce polar coordinates, and expand the resulting denominator in powers of $e^{-\beta p}$. This way the integral becomes

$$4\pi^2 \int_0^{\infty} dp\, \frac{p}{e^{\beta p} - 1}\left(e^{ip(u-u')} + e^{-ip(u-u')}\right) = -4\pi^2 \sum_{n\neq 0} \frac{1}{(u-u'+in\beta)^2}.$$

With

$$\frac{a^2}{4}\frac{1}{\sinh^2 \frac{a}{2}(u-u') - i\epsilon} = \frac{1}{(u-u')^2 - i\epsilon} + \sum_{n\neq 1}^{\infty} \frac{1}{(u-u'+in\beta)^2},$$

where $\beta = 2\pi/a$, we finally end up with the following spectral representation of the Feynman-propagator as seen by the Rindler observer

$$G_F(p,p') = \frac{e^{-2av}}{(2\pi)^4}\int d^4p\, e^{-iE(u-u')}\Big(\frac{1}{p^2+i\epsilon} - 2\pi i\frac{\delta(p^2)}{e^{\beta|E|}-1}\Big). \tag{3.16}$$

This is the finite temperature propagator. It follows, that, in equilibrium, atoms dragged along the world line find their excited levels populated as predicted by temperature $\beta^{-1} = a/2\pi$.

The propagator is a sum of amplitudes for the path connecting $p$ with $p'$. We shall continue to Euclidean spacetime, in which

$$G_E(p,p') = \langle X|\frac{1}{-\triangle}|X'\rangle = \int\limits_0^\infty ds\,\langle X|e^{s\triangle}|X'\rangle = \int\limits_0^\infty ds\int \mathcal{D}X^\mu \exp\big(-\frac{1}{4}\int\limits_0^s \dot{X}^2\big),$$

where the paths start at $p'$ and end at $p$. The path integral splits into pieces, each piece corresponding to an integration over paths whose projection on the $X^0, X^1$ plane winds $n$ times around the origin. If $p, p'$ lie in the $X^0, X^1$ plane, then

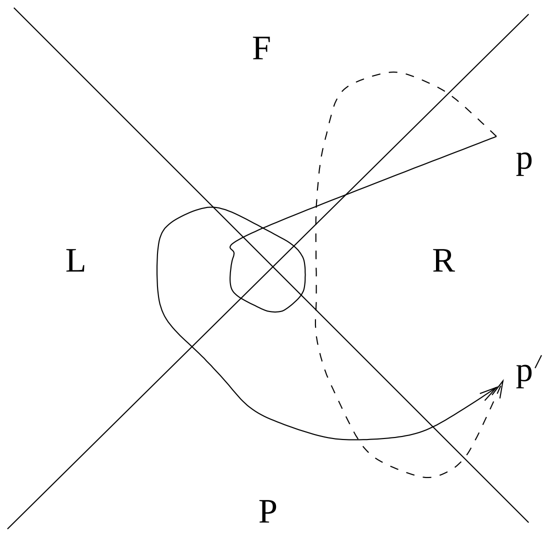


Abbildung 3.4: *Path corresponding to winding number +2 and 0*

$$\int \mathcal{D}X^\mu e^{-\frac{1}{4}\int \dot{X}_\mu \dot{X}^\mu} = \frac{1}{4\pi s}\sum_n \int \mathcal{D}X\,\delta(n(X)-n-\phi)\exp\big(-\frac{1}{4}\int\limits_0^s \dot{X}^2\big),$$

where the last path integral is only over path in the $(X^0, X^1)$ plane,

$$n(X) = \int_0^s \frac{X_0\dot{X}_1 - X_1\dot{X}_0}{X_0^2+X_1^2}$$

is the winding number of the path $X(s)$ and $\phi$ the angle between its endpoints. The path integral for path with fixed windings can be calculated and then continued back to Minkowski spacetime.

The result is

$$G_n[(u,v,0,0),(u',v',0,0)] = -\frac{1}{4\pi^2}\frac{1}{e^{2av}-e^{2av'}}\frac{1}{(u-u'-in\beta)^2}.$$

Specializing to the two events on the world line of an observer with $v=0$, one has the result

$$G_n[(u,0,0,0),(u',0,0,0)] = -\frac{1}{4\pi^2}\frac{1}{(u-u'-in\beta)^2}.$$

which is what we wanted to show. Let us interpret the result for winding number 2 in figure (3.4). Limiting ourselves to the Rindler wedge, we see a line being absorbed at $p'$ coming from $u=-\infty$, one being emitted at $p$ going to $u=\infty$ and an extra spectator going from $u=-\infty$ to $u=\infty$. This can be extended to general values of $n$.

Earlier on we have already argued, that the Feynman propagator requires for its definition either a distinguished vacuum or a notion of positive frequency. Hence, the difficulty with the Unruh-effect cannot be resolved merely by shifting attention from annihilation-creation operators to Green functions. Indeed, we have just seen that the two methods yield the same result: an accelerated observer will 'see' thermal radiation, even though the field $\phi$ is in the vacuum state $\Omega_M$ and an inertial observer detects no particles. Since both the accelerated and unaccelerated observer agree that the stress-energy-momentum of $\phi$ vanishes this has led to the description 'quasi' or 'fictious' particles for the quanta that excite the accelerated detector. Later we shall reconsider the Unruh effect and will have to say more about interpretational issues.

## 3.4 Quantum Fields in Curved Spacetime

Since no analog of either a plane wave basis or a choice of a 'positive frequency subspace' is available in a general curved spacetime, we reformulate quantum field theory without using a plane wave expansion. A particle interpretation can be given in a stationary, curved spacetime. But in a general, non-stationary spacetime, the states of the quantum field will not possess a physically meaningful particle interpretation. It is necessary, that the causal structure of spacetime is well behaved so that the space of classical solutions have the same basic structure as in Minkowski spacetime. The conditions of *global hyperbolicity* ensures that this is the case.

In an arbitrary curved spacetime, the properties of the classical solutions can be very different form those in Minkowski spacetime. Let us have a look at two examples:

1. Let $\mathcal{M}$ be a flat 4-torus, with spatial periodicity $L$ and time periodicity $T$. Then $\exp(-i\omega t + i\vec{k}\vec{x})$ is a periodic solution of the Klein-Gordon equation with $m=0$ only if

   $$\omega = \frac{2\pi m_0}{T}, \quad \vec{k} = \frac{2\pi\vec{m}}{L} \quad \text{and} \quad \frac{m_0^2}{T^2} = \frac{\vec{m}^2}{L^2}, \quad \text{where} \quad m_\mu \in Z.$$

   Thus, for irrational $T^2/L^2$ only the solution $\phi$ =constant is admitted.

2. Consider any spacetime with a 'timelike singularity'. Since anything can emerge from such a singularity, uniqueness for solutions of the field equation with given initial conditions on a spacelike hypersurface cannot hold.

Fortunately, there is a simple condition on $(\mathcal{M}, g_{\mu\nu})$ which guarantees that the field equations have a well posed initial value formulation. First, we assume that spacetime is *time orientable*, such that a continuous choice can be made throughout spacetime of which half of each light cone constitutes the 'future' direction and which half the 'past'. Let $\Sigma \subset \mathcal{M}$ be achronal hypersurface[3]. We define the *domain of dependence* of $\Sigma$ by

$$\mathrm{D}(\Sigma) = \{p \in \mathcal{M} | \text{every inextendible causal curve through p intersects } \Sigma\}.$$

Recall, that a curve is causal if its tangent is everywhere either timelike or null. If $\mathrm{D}(\Sigma) = \mathcal{M}$, then $\Sigma$ is called a *Cauchy surface* for the spacetime (which is automatically $C^0$.) If $\mathcal{M}$ admits a Cauchy surface, then it is said to be *globally hyperbolic.* Then the following theorem, due to Geroch (1970), holds:

**Theorem** If $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic with Cauchy surface $\Sigma$, then $\mathcal{M}$ has topology $R \times \Sigma$. Furthermore, $\mathcal{M}$ can be *foliated* by a one-parameter family of smooth Cauchy surfaces $\Sigma_t$, i.e. a smooth 'time coordinate' $t$ can be chosen on $\mathcal{M}$ such that each surface of constant $t$ is a Cauchy surface.

In a globally hyperbolic spacetime with smooth, spacelike Cauchy surface $\Sigma$ there is a *well posed initial value problem* for the Klein-Gordon equation (Hawking and Ellis 1973): Given smooth initial data $\phi_0, \dot{\phi}_0 \in C^\infty(\Sigma)$, then there exists a unique solution $\phi$ of (3.17), defined on all of $\mathcal{M}$, such that on $\Sigma$ we have

$$\phi = \phi_0 \quad \text{and} \quad n^\mu \nabla_\mu \phi = \dot{\phi}_0,$$

where $n^\mu$ is the unit future-directed normal to $\Sigma$. In addition, $\phi$ is smooth and varies continuously with the initial data.

The classical action of a minimally coupled scalar field without self-interaction is is

$$S_\phi = \frac{1}{2} \int \eta \Big( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \Big), \qquad \eta = \sqrt{-g} d^4 x$$

and the curved spacetime version of the Klein-Gordon equation reads

$$\nabla^\mu \nabla_\mu \phi + m^2 \phi = 0. \tag{3.17}$$

For the phase-space formulation of the Klein-Gordon field we introduce a 'slicing' of $\mathcal{M}$ by spacelike Cauchy surfaces $\Sigma_t$. Let $n^\mu$ be the unit normal vector field to the hypersurfaces $\Sigma_t$. The spacetime metric $g_{\mu\nu}$ induces a spatial (three-dimensional Riemannian) metric $(-h_{\mu\nu})$ on each $\Sigma_t$ by the formula

$$h_{\mu\nu} = n_\mu n_\nu - g_{\mu\nu}.$$

Let $t^\mu$ be a 'time evolution' vector field on $\mathcal{M}$ satisfying $t^\mu \nabla_\mu t = 1$. We decompose it into its parts normal and tangential to $\Sigma_t$,

$$t^\mu = g^{\mu\nu} t_\nu = n^\mu (n, t) - h^{\mu\nu} t_\nu \equiv N n^\mu - N^\mu,$$

[3]No pair of points $p, q \in \Sigma$ can be joined by a timelike curve.

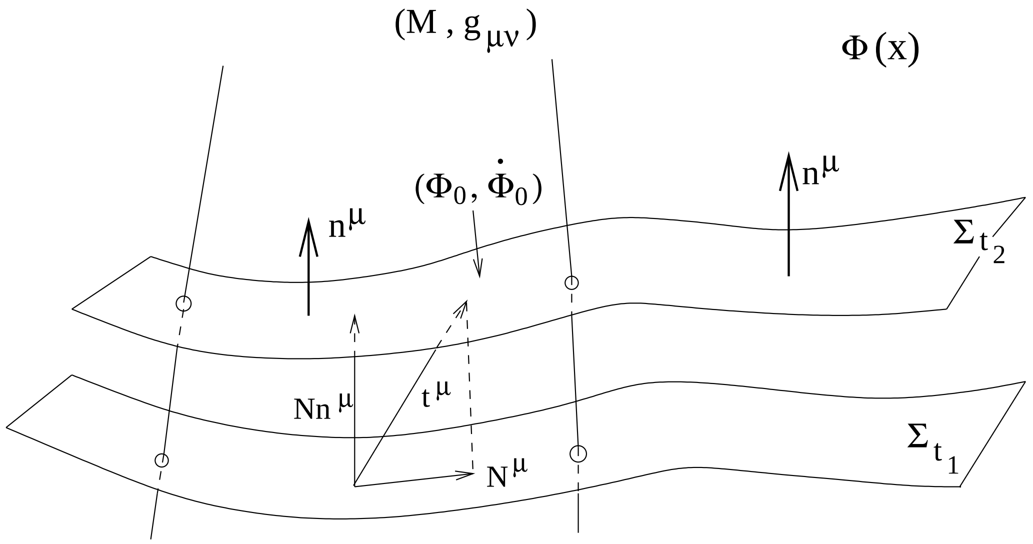


Abbildung 3.5: *A globally hyperbolic spacetime with Cauchy hypersurface has a well-posed initial value problem.* 3+1 *split.*

where we have defined the *lapse function* $N = (n, t)$ and the *shift vector* $N^\mu = h^{\mu\nu} t_\nu$ tangential to the $\Sigma_t$. Now we introduce local coordinates $x^\mu = (t, x^i), i = 1, 2, 3$ with $t^\mu \nabla_\mu x^i = 0$, so that $t^\mu \nabla_\mu = \partial_t$ and $N^\mu \partial_\mu = N^i \partial_i$. The metric coefficients in this coordinate system are

$$\begin{aligned} g_{00} &= g(\partial_t, \partial_t) = t^\mu t^\nu g(\partial_\mu, \partial_\nu) = N^2 + N^i N_i \\ g_{0i} &= g(\partial_t, \partial_i) = N(n^\mu \partial_\mu, \partial_i) - N^\mu(\partial_\mu, \partial_i) = N^j h_{ji} \equiv N_i, \end{aligned}$$

so that

$$(g_{\mu\nu}) = \begin{pmatrix} N^2 + N_i N^i & N_i \\ N_i & -h_{ij} \end{pmatrix} \quad \text{and} \quad (g^{\mu\nu}) = \frac{1}{N^2} \begin{pmatrix} 1 & -N^i \\ -N^i & N^i N^j - N^2 h^{ij} \end{pmatrix}.$$

The determinant of the metric is $\det(g_{\mu\nu}) = N^2 \det(-h_{ij})$. Inserting these decompositions into the Klein-Gordon action one obtains

$$S = \int \mathcal{L} dt$$

with

$$\mathcal{L} = \frac{1}{2} \int_{\Sigma_t} \Big\{ \frac{1}{N^2} \Big( \dot\phi - N^i \partial_i \phi \Big)^2 + h^{ij} \partial_i \phi \partial_j \phi - m^2 \phi^2 \Big) \Big\} N \sqrt{h} d^3 x.$$

We find that the momentum density, $\pi$, conjugate to the configuration variable $\phi$ on $\Sigma_t$ is given by

$$\pi = \frac{\delta S}{\delta \dot\phi} = \frac{\sqrt{h}}{N} \big( \dot\phi - N^i \partial_i \phi \big) = \sqrt{h} \big( n^\mu \partial_\mu \phi \big).$$

A point in classical phase space $\mathcal{P}$ of the Klein-Gordon theory consists of the specification of functions $(\phi(x), \pi(x))$ on a Cauchy surface $\Sigma_0$. If we specify $\mathcal{P}$ precisely by requiring that $(\phi, \pi) \in C_0^\infty$ (smooth and of compact support) then, by the result of Hawking and Ellis above, they give rise to a unique solution to (3.17). The space of solutions $\mathcal{S}$ is independent on the choice of the Cauchy surface.

### 3.4.1 Stationary Spacetimes

At least technically one may generalize the well-known flat spacetime construction of a Fock space if spacetime is stationary, i.e. possesses a global timelike Killing vector field $K$ which generates a flow of isometries. Then we may choose basic functions $u_k$ that satisfy

$$iL_K u_k = \omega(k) u_k \quad \text{and} \quad iL_K u_k^\dagger = -\omega(k) u{\dagger}_k,$$

where the $\omega(k) > 0$ are constant. If $K^\mu$ is globally timelike, then one may introduce a coordinate $t$ upon which the metric does not depend and with respect to which $K^\mu$ takes the form $K = \partial_t$. The covariant components of $K$ are $K_\mu = g_{\mu 0}$. Since $ds^2 = g_{00} dt^2 + \ldots = (K, K) dt^2 + \ldots$, the coordinate $t$ is in general not the proper time of observers moving with the flow of $K$. However, since

$$\nabla_K (K, K) = 2 K^\mu K^\alpha \nabla_\alpha K_\mu = K^\mu K^\alpha L_K g_{\mu\nu} = 0,$$

the norm of $K$ is constant along the orbits of $K$. Therefore we may scale $K$ such that $t$ gives directly the proper time measured by at least one co-moving clock. The $\omega(k)$ are the frequencies relative to that clock. and the $u_k$ and $u_k^\dagger$ are the positive and negative frequency solutions, or positive and negative energy solutions, respectively. Now the construction of the vacuum, one-particle space and Fock space is done in the usual way:

Gibbons[4] has given the following covariant constructions: The quantity

$$T_K = \int_\Sigma T^{\mu\nu} K_\nu d\Sigma_\mu$$

is conserved on account of the Killing equation and covariant conservation of the energy-momentum tensor. Although it is an ill-defined operator, it possesses well defined commutation relations with the components of the field

$$[T_K, \phi] = iL_K \phi.$$

One can make $T_K$ well defined, by normal ordering it with respect to the above chosen $a$ and $a^\dagger$,

$$E = \int_\Sigma : T^{\mu\nu} : K_\nu d\Sigma_\mu.$$

The vacuum will then be the zero reference point for energy

$$E\Omega = 0.$$

[4] G.W. Gibbons, 1974

The $a$ and $a^\dagger$ will be energy-raising and lowering operators

$$[E, a_k] = -\omega(k) a_k.$$

If there is another Killing vector $L$ that commutes with $K$,

$$[K, L] = 0 \Longleftrightarrow [L_K, L_L] = 0,$$

then the basis functions may be chosen so as to satisfy also

$$iL_L u_k = \lambda(k) u_k,$$

where the $\lambda(k)$ are constants. The $a_k$ and $a_k^\dagger$ then become raising and lowering operators for the associated quantity

$$T_L = \int\limits_{\Sigma} : T^{\mu\nu} : L_\nu d\Sigma_\mu,$$

i.e.

$$[T_L, a_k] = \lambda(k) a_k.$$

More generally, if there is a set of independent Killing vectors generating a Lie algebra, the $u_k$ may be selected to yield a irreducible representation of that algebra.

**Problems with this procedure:**

1. There may be no Killing vector at all. One probably has to give up the particle picture in this generic situation.

2. There may be a global Killing vector, but it may not be everywhere timelike. Then one may exclude the non-timelike region from space time. This corresponds to the imposition of boundary conditions. One may also try to retain the non-timelike region but attempt to define a meaningful vacuum by invoking physical argument.

3. Spacetime may be stationary only in limited regions. If each region possesses a complete Cauchy hypersurface, then a local timelike Killing field may be set up in each and a vacuum defined for each. With respect to the basis function of which region should the stress tensor be normal ordered? It is not possible to define the stress tensor so that (a) it is normal ordered in both regions (b) its matrix elements are smooth functions, and (c) it satisfies the divergence equation

   $$T^{\mu\nu}_{\ \ ;\nu} = 0$$

   everywhere.

### 3.4.2 The energy inner product

The following construction is do to Ashtekar and Magnon and to Kay[5]. Let

$$S = \int d\mu \left(\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) = \int d\mu\, \mathcal{L}, \qquad d\mu = \sqrt{|g|}d^d x$$

be the action of a scalar field. Using

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{|g|}\,\delta g^{\mu\nu}g_{\mu\nu},$$

one easily finds for the variation of the action

$$\delta S = \int d\mu \Big(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\mathcal{L}\Big)\delta g^{\mu\nu},$$

and the metric stress-energy tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}}$$

has the simple form

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}.$$

Being the variation of the action with respect to the symmetric metric it is symmetric. Under a infinitesimal one-parameter group of diffeomorphism, the orbits of which are tangential to the vector field $X(x)$, the metric and field transform as $\delta g^{\mu\nu} = L_X g^{\mu\nu}$ and $\delta\phi = L_X\phi$. A diffeomorphism-invariant action does not change under such variations, so that

$$\begin{aligned} 0 &= \delta_X S = \int d^d x\, \frac{\delta S}{\delta g^{\mu\nu}(x)} L_X g^{\mu\nu}(x) + \int d^d x\, \frac{\delta S}{\delta\phi(x)} L_X\phi(x) \\ &= \frac{1}{2}\int d\mu\, T_{\mu\nu}(\nabla^\mu X^\nu + \nabla^\nu X^\mu) - \int d\mu\,(\Box\phi + V'(\phi))L_X\phi \\ &= -\int d\mu T_{\mu\nu}^{\ \ ;\nu}X^\mu - \int d\mu\big(\Box\phi + V'(\phi)\big)L_X\phi. \end{aligned}$$

If the scalar field fulfills the field equation (is *on shell*) then the metric energy momentum tensor is automatically conserved on account of the diffeomorphism invariance of the action. The conservation of $T^{\mu\nu}$ can also directly be proved by using the Klein-Gordon equation.

Let $\mathcal{S}$ be the space of solutions of the free Klein-Gordon equation. We complexify $\mathcal{S}$ to $\mathcal{S}^C$ and define an 'energy inner product' as above

$$(\phi_1, \phi_2) = \int_\Sigma T_{\mu\nu}(\phi_1, \phi_2)K^\nu n^\mu\sqrt{h}d^3x$$

[5] Ashtekar, A. (1975), Proc. Roy. Soc. London A346, 375; Kay, B. (1978), Commun. Math. Phys. 62, 55.

where the bilinear-form defined by the 'stress tensor' is extended to $\mathcal{S}^C$ as

$$T_{\mu\nu}(\phi_1,\phi_2) = \frac{1}{2}\Big(\phi_1^\dagger{}_{,\mu}\,\phi_2{}_{,\nu} + \phi_1^\dagger{}_{,\nu}\,\phi_2{}_{,\mu} - g_{\mu\nu}\big(\nabla\phi_1^\dagger\nabla\phi_2 - m^2\phi_1^\dagger\phi_2\big)\Big).$$

We assume that $m^2 > 0$ so that $(.,.)$ is positive for compact $\Sigma$. Since the $\phi_i$ are solutions of the free Klein-Gordon equation, the stress tensor is conserved and since $K$ is a Killing field $\nabla_\mu(T^{\mu\nu}K_\nu) = 0$. Hence, using Gauss's law, we see that $(.,.)$ is independent of the choice of Cauchy surface $\Sigma$. In particular, let

$$\alpha_t : \mathcal{M} \longrightarrow \mathcal{M}$$

be the one-parameter group of isometries generated by the timelike Killing field $K$. Then $(.,.)$ is invariant under the time translation map $\alpha_t^* : \mathcal{S}^C \to \mathcal{S}^C$ defined by

$$\alpha_t^*(\phi) = \phi \circ \alpha_t \quad \text{or} \quad \big(\alpha_t^*(\phi)\big)(x) = \phi\big(\alpha_t(x)\big),$$

since applying $\alpha_t^*$ to solutions is equivalent to applying $\alpha_t$ to $\Sigma$. Next we complete $\mathcal{S}^C$ in the 'energy-norm' to get a complex Hilbert space $\tilde{\mathcal{H}}$ (this is not yet the Hilbert space we seek). The time translation map $\alpha_t^*$ extends to $\tilde{\mathcal{H}}$ and defines a strongly continuous, one-parameter, unitary group, also denoted by $\alpha_t^*$. By Stone's theorem

$$\alpha_t^* = e^{i\tilde{h}t}, \qquad \tilde{h} \quad \text{selfadjoint}.$$

Note, that from the definition of the Lie derivative,

$$\frac{d}{dt}(\alpha_t^*\phi)|_{t=0} = -L_X\phi$$

we have for all $\phi \in \mathcal{S}^C$

$$\tilde{h}\phi = iL_K\phi.$$

Now we recall, that

$$\Omega\big([\phi_1,\pi_1],[\phi_2,\pi_2]\big) = \int\limits_{\Sigma_0} \big(\pi_1\phi_2 - \pi_2\phi_1\big)\, d^3x, \quad \pi = \sqrt{h}n^\mu\nabla_\mu\phi,$$

is conserved on solutions and hence may be viewed as bilinear map on $\mathcal{S}^C$, if we extend $\Omega$ by (complex) linearity in each variable. Now one can prove, that

$$|\Omega(\phi_1^\dagger,\phi_2)| \le C\|\phi_1\|\|\phi_2\|$$

from which follows, that $\Omega(\phi_1^\dagger,\phi_2)$ extends continuously to a quadratic form on $\tilde{\mathcal{H}}$. It can also be shown, that

$$\Omega(\phi_1^\dagger,\tilde{h}\phi_2) = 2i(\phi_1,\phi_2),$$

and that $\tilde{h}$ is bounded away from zero. Now, let $\tilde{\mathcal{H}}^+$ be the positive spectral subspace of $\tilde{\mathcal{H}}$ and let $K$ be the projection map $K : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^+$. For all real solutions we may now define the *scalar product*

$$\mu(\phi_1,\phi_2) = \Im\,\Omega(K\phi_1,K\phi_2) = 2\Re(K\phi_1,\tilde{h}^{-1}K\phi_2).$$

$\Omega$ is conserved for solutions of the Klein-Gordon equation, so we may view it as a bilinear map on $\mathcal{S}$, i.e. $\Omega : \mathcal{S} \times \mathcal{S} \to R$.

The one-particle Hilbert space $\mathcal{H}$ is just the completion of the space $\tilde{\mathcal{H}}^+$, of 'positive frequency solutions' in the Klein-Gordon inner product. This construction avoids any direct attempt to take the 'time Fourier transform' of solutions along the orbits of the Killing field. Such Fourier transform may exist only in a distributional sense.

## 3.5 The Stress-Energy Tensor

Beside the (smeared) fields there are many additional operators in which one is interested in quantum theory. Primary among these is the stress-energy tensor. It is of interest since it describes the local energy, momentum and stress properties of the field. It is relevant for describing the back-reaction of the quantum field on the spacetime geometry. Semiclassically one would expect that back-reaction is described by the 'semiclassical Einstein equation'

$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle_\omega .$$

Thus, it is of considerable interest to determine the expectation value of the stress-energy tensor in physically relevant states $\omega$. Some restrictions should be expected on the class of states on which $\langle T_{\mu\nu} \rangle$ can be defined. We shall see that the *Hadamard condition* provides a restriction of exactly this sort of states.

### 3.5.1 Hadamard states

In the following we shall assume, that $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic. Then the Cauchy problem for the Klein-Gordon equation for any Cauchy surface $\Sigma$ has a unique solution. It follows, that there are unique retarded and advanced Green functions

$$\Delta_{ret}(x,y) \quad , \quad \Delta_{adv}(x,y) \quad \text{with} \qquad \mathrm{supp}(\Delta_{ret}) = \{(x,y); x \in J_+(y)\}.$$

Hadamard states are states, for which the two-point function has the following singularity structure

$$\omega\big(\phi(x)\phi(y)\big) \equiv \omega_2(x,y) = \frac{u}{\sigma} + v \log \sigma + w, \quad \text{where} \tag{3.18}$$

$\sigma(x,y)$ is the square of the geodesic distance of $x$ and $y$ and $u, v, w$ are smooth functions on $\mathcal{M}$. It has been shown that if $\omega_2$ has the Hadamard singularity structure in a neighborhood of a Cauchy-surface, then it has his form everywhere [6]. To show that, one observes, that $\omega_2$ satisfies the wave equation. This result can then be used to show, that on a globally hyperbolic spacetime there is a class of states, forming a dense subspace of a Hilbert space, whose two-point functions have the Hadamard singularity structure.

The two-point function must be positive,

$$\omega\big(\phi(f)^\dagger \phi(f)\big) = \int d\mu(x) d\mu(y) \omega\big(\phi(x)\omega(y)\big) \bar{f}(x) f(y) \geq 0,$$

and must obey the Klein-Gordon equation. These requirements determine $u$ and $v$ uniquely and puts stringent conditions on the form of $w_2$. The *Feynman Greenfunction* is related to $\omega_2$ and the retarded Greenfunction as

$$i\Delta_F(x,y) = \omega_2(x,y) + \Delta_{ret}(y,x).$$

Since $\Delta_{ret}$ is unique, the ambiguities of $\Delta_F$ are the same as those of $\omega_2$. The *propagator function*

$$i\Delta(x,y) = [\phi(x),\phi(y)] = i\Delta_{ret}(x,y) - i\Delta_{adv}(x,y)$$

determines the antisymmetric part of $\omega_2$,

$$\omega_2(x,y) - \omega_2(y,x) = i\Delta(x,y),$$

so that this part is without ambiguities. For a scalar field without self-interaction we expect, that

$$\begin{aligned}\omega\big(\phi(x_1)\dots\phi(x_n)\big) &= 0 \qquad \text{for odd} \quad n\\ \omega\big(\phi(x_1)\dots\phi(x_{2n})\big) &= \sum_{\substack{i_1<i_2\dots<i_n\\ j_1<j_2\dots j_n}} \prod_{k=1}^{n} \omega\big(\phi(x_{i_k})\phi(x_{j_k})\big).\end{aligned}$$

A state fulfilling these conditions is called *quasifree*. Now one can show, that any choice of $\omega_2(x,y)$ fulfilling the properties listed above give rise to a well-defined Hilbert space, i.e. a Fock space over a quasifree vacuum state. The Hilbert space is

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \tag{3.19}$$

where the scalar-products on the 'n-particle subspace' $\mathcal{H}_n$ in

$$\mathcal{H}_n = \{\psi \in \mathcal{D}(\mathcal{M}^n)_{symm}/\mathcal{N}\}^{completeon} \tag{3.20}$$

is just

$$(\psi_1,\psi_2) = \int d\mu(x_1,..,x_n,y_1,..,y_n) \prod_{i=1}^{n} \omega_2(x_i,y_i)\bar{\psi}_1(x_1,..,x_n)\psi_2(y_1,..,y_n),$$

where we introduced the abbreviation $d\mu(x_1,x_2,..) = d\mu(x_1)d\mu(x_2)\dots$. Since

$$(\Box + m^2)\omega_2(x,y) = 0,$$

the functions in the image of $\Box + m^2$ have zero norm. The set $\mathcal{N}$ of zero-norm states has been divided out in (3.20) in order to end up with a positive definite Hilbert space.

The smeared field operator is now defined in the usual way:

$$\phi(f) = a(f)^\dagger + a(\bar{f}),$$

where

$$\begin{aligned}\big(a(\bar{f})\psi\big)_n(x_1,..,x_n) &= \sqrt{n+1}\int d\mu(x,y)\omega_2(x,y)f(x)\psi_{n+1}(y,x_1,..,x_n)\\ \big(a(f)^\dagger\psi\big)_n(x_1,..,x_n) &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} f(x_k)\psi_{n-1}(x_1,..,\check{x}_k,..,x_n), \quad n>0\end{aligned}$$

and $(a(f)^\dagger\psi)_0 = 0$. It is now easy to see, that $\omega_2$ is just the Wightman function of $\phi$ in the vacuum state $\psi_0$:

$$\omega_2(x,y) = \big(\psi_0, \phi(x)\phi(y)\psi_0\big).$$

### 3.5.2 The Wald axioms for the stress-energy tensor

The difficulties with defining

$$\langle T_{\mu\nu}\rangle$$

are present already in Minkowski spacetime. The divergences are due to the zero-point energies of the infinite collection of harmonic oscillators which comprise the quantum field. A simple cure for this difficulty is the *normal ordering* prescription:

$$\omega\big(:T_{\mu\nu}:\big) = \omega\big(T_{\nu\nu}\big) - (\Omega_M, T_{\mu\nu}\Omega_M).$$

The so defined vacuum expectation value of the stress-energy tensor vanishes. On curved spacetime there is no satisfactory generalization of this prescription since there is

1. No preferred vacuum state
2. Due to vacuum polarization effects we do not expect that the stress-energy of the vacuum (assuming there is a natural one) vanishes.

To make progress let us look at an alternative formulation of the normal ordering prescription without doing the Fourier transformation. We first consider the ill-defined object $\phi^2(x)$, which is part of the stress-energy tensor. We may split the points and consider first the object $\omega(\phi(x)\phi(y))$ which solves the Klein-Gordon equation. This bi-distribution makes perfectly good sense. For physically reasonable states $\psi$ in the Fock space (e.g. states with a finite number of particles) the singular behavior of this bi-distribution is the same as that belonging to the vacuum state, $\omega_0\big(\phi(x)\phi(y)\big)$. For such states the difference

$$F(x,y) = \omega\big(\phi(x)\phi(y)\big) - \omega_0\big(\phi(x)\phi(y)\big)$$

is a smooth function of its arguments. Hence, after performing this 'vacuum subtraction' the coincidence limit may be taken. We then define

$$\omega\big(\phi^2(x)\big) = \lim_{x\to y} F(x,y).$$

The same prescription can be used for the stress-energy tensor instead of $\phi^2$. We define

$$\omega\big(T_{\mu\nu}(x)\big) = \lim_{x \to x'} \Big(\partial_\mu \partial_{\nu'} - \frac{1}{2} g_{\mu\nu} \big[\partial_\mu \partial^{\nu'} - m^2\big]\Big) F(x, x').$$

After what has been said above, we do not believe that this is a physical definition of expectation values of the stress-energy tensor. However, note that the point-splitting prescription sensibly defines the *differences* of the expected stress energy between two states,

$$\omega_1\big(T_{\mu\nu}\big) - \omega_2\big(T_{\mu\nu}\big).$$

In the absence of an obvious prescription to define the expectation values, it is useful to take an axiomatic approach. Wald showed that a renormalized stress tensor satisfying certain reasonable physical requirements is essentially unique [7]. Its ambiguity can be absorbed into redefinitions of the coupling constants in the gravitational field equation. Wald argues that one expects this operator to have the following properties:

1. **Consistency:** Whenever $\omega_1(\phi(x)\phi(y)) - \omega_2(\phi(x)\phi(y))$ is a smooth function, then $\omega_1(T_{\mu\nu}) - \omega_2(T_{\mu\nu})$ is well-defined and should be given by the above 'point-splitting' prescription.

2. **Conservation:** In the classical theory the stress-energy tensor is conserved. If the regularization needed to define a stress-energy tensor respects the diffeomorphism invariance, then

$$\nabla_\nu T^{\mu\nu} = 0$$

   must also hold in the quantized theory. This property is needed for consistency of Einstein's gravitational field equation.

3. In Minkowski spacetime, we have $(\Omega_M, T_{\mu\nu}\Omega_M) = 0$.

4. **Causality:** We assume, that spacetime is asymptotically static. For a fixed in-state, $\omega_{in}\big(T_{\mu\nu}(x)\big)$ is independent of variations of $g_{\mu\nu}$ outside the past light cone of $x$. For a fixed out-state, $\omega_{out}\big(T_{\mu\nu}\big)$ is independent of metric variations outside the future light cone of $x$.

The first and last properties are the key ones, since they uniquely determine the expected stress-energy tensor up to the addition of local curvature terms. This fact is contained in the

**Uniqueness theorem (Wald):** Let $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ be operators on globally hyperbolic spacetime satisfying the axioms of Wald. Then the difference

$$U_{\mu\nu} = T_{\mu\nu} - \tilde{T}_{\mu\nu}$$

has the following properties.

1. $U_{\mu\nu}$ is a multiple of the identity operator.

2. It is conserved, $\nabla_\nu U^{\mu\nu} = 0$.

3. It is a local tensor of the metric. That is, it depends only on the metric and its derivatives, via the curvature tensor, at the same point $x$.

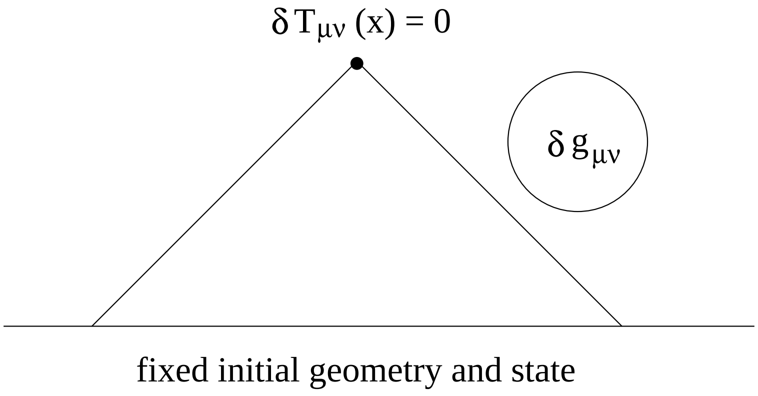


Abbildung 3.6: *Changes outside the past light cone do not affect* $\langle T_{\mu\nu}(x)\rangle$.

As a consequence of the properties,

$$\omega(T_{\mu\nu}) - \omega(\tilde{T}_{\mu\nu})$$

is independent on the state $\omega$ and depends only locally on curvature invariants. The Causality axiom can be replaced by a locality property, which does not assume an asymptotically static spacetime. The proofs of these properties are rather simple and can be found in the standard textbooks.

### 3.5.3 Calculating the stress-energy tensor

A 'point-splitting' prescription where one subtracts from $\omega(\phi(x)\phi(y))$ the expectation value $\omega_0(\phi(x)\phi(y))$ in some fixed state $\omega_0$ will fulfills the consistency requirement, but cannot fulfill the first and third axiom at the same time. However, if one subtracts a locally constructed bi-distribution $H(x,y)$ which satisfies the wave equation, has a suitable singularity structure and is equals to $(\Omega_M, \phi(x)\phi(y)\Omega_M)$ in Minkowski spacetime, then all four properties will be satisfied.

To find a suitable bi-distribution one recalls the singularity structure (3.18) of $\omega_2(x,y)$. In Minkowski spacetime and for massless fields $w = 0$ and this suggests that we take the bi-distribution

$$H(x,y) = \frac{U(x,y)}{\sigma} + V(x,y)\log\sigma$$

The resulting stress-energy obeys almost all properties, besides that for massive fields on Minkowski spacetime we still find a non-vanishing vacuum expectation value, and that

$$\nabla_\nu\,\omega\big(T^{\mu\nu}\big) = \nabla_\nu Q,$$

where $Q$ is a scalar density, locally dependent on the metric. Hence we may modify our prescription by simply subtracting $(Q+c)g_{\mu\nu}$ from $T_{\mu\nu}$. The constant $c$ is chosen, such that on Minkowski spacetime the vacuum expectation value vanishes.

### 3.5.4 Effective action

We have seen, that the classical metric energy momentum tensor is automatically conserved if it is gotten by variation of a diffeomorphism-invariant classical action. If we could construct a diffeomorphism-invariant *quantum action* $\Gamma(g_{\mu\nu},\phi)$, whose variation with respect to the metric yields an expectation value of the energy momentum tensor,

$$\langle T_{\mu\nu}(x)\rangle = \frac{2}{\sqrt{|g|}}\frac{\delta\Gamma}{\delta g^{\mu\nu}},$$

then $\langle T_{\mu\nu}\rangle$ would be conserved by construction. Let us look at a particular example.

**Conformally flat spacetimes**

A conformally coupled scalar field propagating on a spacetime $\mathcal{M}$ has classical action

$$S[\phi] = \int_{\mathcal{M}} \sqrt{g}\Big(-\frac{1}{2}\phi\triangle_c\phi\Big), \quad \text{where} \quad \triangle_c = \triangle - \frac{d-2}{4(d-1)}R$$

is the Weyl-covariant wave operator. Note, that for a vacuum solution of the Einstein equation the Ricci scalar vanishes and there is no distinction between conformal and minimal coupling. Formally, the expectation value (which one?) of the stress-energy tensor is

$$\langle T_{\mu\nu}(x)\rangle = -\frac{1}{Z[g]}\int \mathcal{D}\phi\, \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}e^{-S[\phi]} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\Gamma[\phi],$$

where we have introduced the effective action

$$\Gamma[g] = -\log Z[g] = -\log\int \mathcal{D}\phi\; e^{-S[\phi]} = \frac{1}{2}\log\det(-\triangle_c).$$

For arbitrary spacetimes the spectrum of $\triangle_c$ is not known. However, the variation of $\Gamma$ with respect to $\sigma$ in

$$g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu},$$

which is proportional to the expectation value of the trace of the stress-energy tensor,

$$\frac{\delta\Gamma}{\delta\sigma(x)} = -2g^{\mu\nu}\frac{\delta\Gamma}{\delta g^{\mu\nu}} = -\sqrt{g}\langle T^\mu_{\ \mu}\rangle$$

can be calculated. The non-vanishing of this trace in the quantized theory is the so-called trace-anomaly and this anomaly is known. It follows, that the difference $\Gamma[g]-\Gamma[\hat{g}]$ can be calculated by integrating the trace anomaly To do that explicitly, we recall (see our discussion of the Weyl-transformation) that

$$\triangle_c = e^{-\frac{1}{2}(d+2)\sigma}\hat{\triangle}_c e^{\frac{1}{2}(d-2)\sigma}.$$

Now we interpolate between the reference metric $\hat{g}$ and $g$ by the one-parametric family of metrics

$$g^{(\tau)}_{\mu\nu} = e^{2\tau\sigma}\hat{g}_{\mu\nu}.$$

Using the manifestly covariant $\zeta$-function techniques, the $\tau$-variation of

$$\Gamma[g^{(\tau)}] = \frac{1}{2}\log\det(-\triangle_c^{(\tau)}), \qquad \triangle_c^{(\tau)} = e^{\frac{1}{2}(d+2)\sigma\tau}\hat{\triangle}_c e^{\frac{1}{2}(d-2)\sigma\tau}$$

can easily be calculated[6]. Integrating from $\tau = 0$ to $\tau = 1$ yields

$$\Gamma[g] - \Gamma[\hat{g}] = \frac{2}{(4\pi)^{d/2}} \int_0^1 d\tau \int_{\mathcal{M}} \sqrt{g^{(\tau)}} a_{d/2}^{(\tau)}(x)\sigma(x), \tag{3.21}$$

where $a_{d/2}^{(\tau)}$ is the coefficient of dimension $L^{-d}$ in the asymptotic small $t$-expansion of the heat kernel,

$$\langle x|\exp\left(t\triangle_{g^{(\tau)}}\right)|x\rangle \sim \frac{1}{(4\pi t)^{\frac{d}{2}}}\sum_{n=0} t^n\, a_n^{(\tau)}(x).$$

The Seeley-deWitt coefficients $a_n$ are local scalar functions of the metric, have length-dimension $-2n$ and have been calculated up to $a_5$.

**2 dimensions**

Every 2-dimensional spacetime is conformally flat and we may assume that

$$g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu} \quad \text{or} \quad \hat{g}_{\mu\nu} = \delta_{\mu\nu}.$$

It follows, that, up to the metric-independent effective action $\Gamma[\delta]$ the effective action can be calculated. The result is the *Polyakov effective action*

$$\Gamma[g] - \Gamma[\delta] = \frac{c}{96\pi}\int \sqrt{g}R\frac{1}{\triangle}R,$$

where $c$ is the *central charge* which is 1 for uncharged scalars. The expectation value of $T_{\mu\nu}$ is gotten by differentiation with respect to the metric. The result is

$$\begin{aligned} \langle T_{\mu\nu}\rangle &= \frac{c}{24\pi}\Big(g_{\mu\nu}R - \nabla_\mu\nabla_\nu\frac{1}{\triangle}R\Big) \\ &+ \frac{c}{48\pi}\Big(\nabla_\mu\frac{1}{\triangle}R\cdot\nabla_\nu\frac{1}{\triangle}R - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\frac{1}{\triangle}R\cdot\nabla_\alpha\frac{1}{\triangle}R\Big). \end{aligned} \tag{3.22}$$

This is indeed conserved and has trace $cR/24\pi$. In isothermal coordinates

$$R = -2\triangle\sigma = -2e^{-2\sigma}\triangle_0\sigma, \qquad \Gamma^\alpha_{\mu\nu} = \delta^\alpha_\mu\sigma_{,\nu} + \delta^\alpha_\nu\sigma_{,\nu} - \delta_{\mu\nu}\sigma^{,\alpha}$$

and hence

$$T_{\mu\nu} = \frac{c}{24\pi}\Big(\delta_{\mu\nu}\big[(\nabla\sigma)^2 - 2\triangle_0\sigma\big] + 2\big[\partial_\mu\partial_\nu\sigma - \partial_\mu\sigma\partial_\nu\sigma\big]\Big).$$

[6] We assume, that the Euclidean wave operator possesses no zero-modes

Or, if we introduce complex coordinates $z = \frac{1}{2}(x^0 + ix^1)$ we obtain

$$\langle T_{zz}\rangle = -\frac{c}{12\pi}e^{\sigma}\partial_z^2 e^{-\sigma} \quad \text{and} \quad \langle T_{z\bar{z}}\rangle = -\frac{c}{12\pi}\triangle_0\sigma.$$

Note, that $\sigma$ is not determined by $R$. We may always add a harmonic piece to $\sigma$, without affecting $R$. In a Lorentzian spacetime, the corresponding result is

$$\boxed{\langle T_{uu,vv}\rangle = -\frac{c}{12\pi}e^{\sigma}\partial_{u,v}^2 e^{-\sigma} + t_{u,v}, \qquad \langle T_{uv}\rangle = -\frac{c}{12\pi}\Box_0\sigma,} \tag{3.23}$$

where we introduced the light cone variables $u = \frac{1}{2}(x^0 - x^1)$ and $v = \frac{1}{2}(x^0 + x^1)$. The prescription to invert the wave operator in $\Box$ (3.22) shows up in the free function $t_{u,v}$. A choice of these functions is equivalent to the choice of the quantum state whose stress-energy is calculated. Let us now apply these results to a toy black hole in 2 dimensional spacetime with metric

$$ds^2 = \big(1 - 2M/r\big)dt^2 - \frac{1}{1 - 2M/r}dr^2.$$

This is just the $(t, r)$-part of the Schwarzschild metric in 4 spacetime dimensions. To find isothermal coordinates in which light rays travel on $45^0$ lines, we note, that null geodesics satisfy

$$ds^2 = 0 \Longrightarrow \big(\frac{dt}{dr}\big)^2 = \frac{1}{\alpha^2}.$$

Thus, on null geodesics

$$t = \pm r_* + constant,$$

where the 'Regge-Wheeler tortoise coordinate' $r_*$ is defined by

$$r_* = r + 2M\log\big(\frac{r}{M} - 2\big). \tag{3.24}$$

Note, that the event horizon at $r = 2M$ has tortoise coordinate $r_* = -\infty$. In the coordinate system $(t, r_*)$ the metric becomes conformally flat by construction,

$$ds^2 = \big(1 - 2M/r\big)\big(dt^2 - dr_*^2\big) \equiv \alpha\big(dt^2 - dr_*^2\big). \tag{3.25}$$

As above we introduce null-coordinates

$$u = \frac{1}{2}(t - r_*) \quad \text{and} \quad v = \frac{1}{2}(t + r_*).$$

Using that $\partial_v = \partial_t + \partial_{r_*}$ and that $pa_{r_*} = \alpha\partial_r$ we obtain

$$\Box_0\sigma = \frac{2M}{r^3}\alpha, \quad e^{\sigma}\partial_{u,v}^2 e^{-\sigma} = \frac{2M}{r^3}\alpha + \frac{M^2}{r^4}$$

and the light-cone components (3.23) of the energy momentum tensor read

$$\langle T_{uu,vv}\rangle = -\frac{c}{12\pi}\Big(\frac{2M\alpha}{r^3} + \frac{M^2}{r^4}\Big) + t_{u,v}, \quad \langle T_{uv}\rangle = -\frac{c}{12\pi}\frac{2M\alpha}{r^3}.$$

With

$$T_{uu(vv)} = T_{00} + T_{11} - (+)2T_{01} \quad \text{and} \quad T_{uv} = T_{00} - T_{11}$$

we find for $\langle T_{\mu\nu}\rangle$ in the $x^\mu = (t, r_*)$ coordinate system[7]

$$\langle T_\mu{}^\nu\rangle = -\frac{cM}{24\pi r^4}\begin{pmatrix} 4r + M/\alpha & 0 \\ 0 & -M/\alpha \end{pmatrix} + \frac{1}{4\alpha}\begin{pmatrix} t_v + t_u & t_v - t_u \\ -t_v + t_u & -t_v - t_u \end{pmatrix} \tag{3.26}$$

The *Boulware state* is the state appropriate to a vacuum around a static star and contains no radiation at spatial infinity $\mathcal{J}^\pm$. Hence the terms $t_v(v)$ and $t_u(u)$ must vanish and the tensor simplifies to

$$\langle O_s|T_\mu{}^\nu|O_s\rangle = -\frac{cM}{24\pi r^4}\begin{pmatrix} 4r + M/\alpha & 0 \\ 0 & -M/\alpha \end{pmatrix}. \tag{3.27}$$

However, this state is singular at the horizon. To see that more explicitly, let us recall, how to introduce Kruskal coordinates which cover the whole spacetime and are regular at the *event horizon* $r = 2M$. The metric (3.25)

$$ds^2 = \frac{8M}{r}e^{-r/2M}e^{(v-u)/2M}dudv \quad \text{where we used} \quad \alpha = \frac{2M}{r}e^{(r_*-r)/2M}$$

suggests, that we introduce

$$U = -e^{-u/2M} \quad \text{and} \quad V = e^{v/2M}$$

so that the metric is regular on the horizons:

$$ds^2 = \frac{32M^3}{r}e^{-r/2M}dUdV.$$

With respect to these regular coordinates the energy momentum takes the form

$$\begin{aligned}
\langle T_{UU}\rangle &= (\frac{M}{U})^2\Big(4t_u - \frac{c}{3\pi}(\frac{2M\alpha}{r^3} + \frac{M^2}{r^4})\Big) \\
\langle T_{VV}\rangle &= (\frac{M}{V})^2\Big(4t_v - \frac{c}{3\pi}(\frac{2M\alpha}{r^3} + \frac{M^2}{r^4})\Big) \\
\langle T_{UV}\rangle &= \frac{M^2}{UV}\frac{c}{3\pi}\frac{2M\alpha}{r^3}.
\end{aligned}$$

The component $\langle T_{UU}\rangle$ is regular at the horizon $U = 0$ if $M^2 t_u = c/192\pi$ and $\langle T_{VV}\rangle$ is regular at the horizon $V = 0$, if $M^2 t_v = c/192\pi$ holds. The corresponding state is called the *Israel-Hartle-Hawking state.* In this state the asymptotic form of the energy-momentum tensor is

$$\langle 0_{HH}|T^\mu_\nu|0_{HH}\rangle = \frac{c}{384\pi M^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{c\pi}{6}(kT)^2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.28}$$

with $T = 1/8\pi kM = \kappa/2\pi k$. This is the stress-tensor of a *bath* of thermal radiation at temperature

[7]comparison with Birrell and Davies, p. 283: $M^2 t_v = 1/192 + 4K - 2Q$ and $M^2 t_u = 1/192 - 2Q$

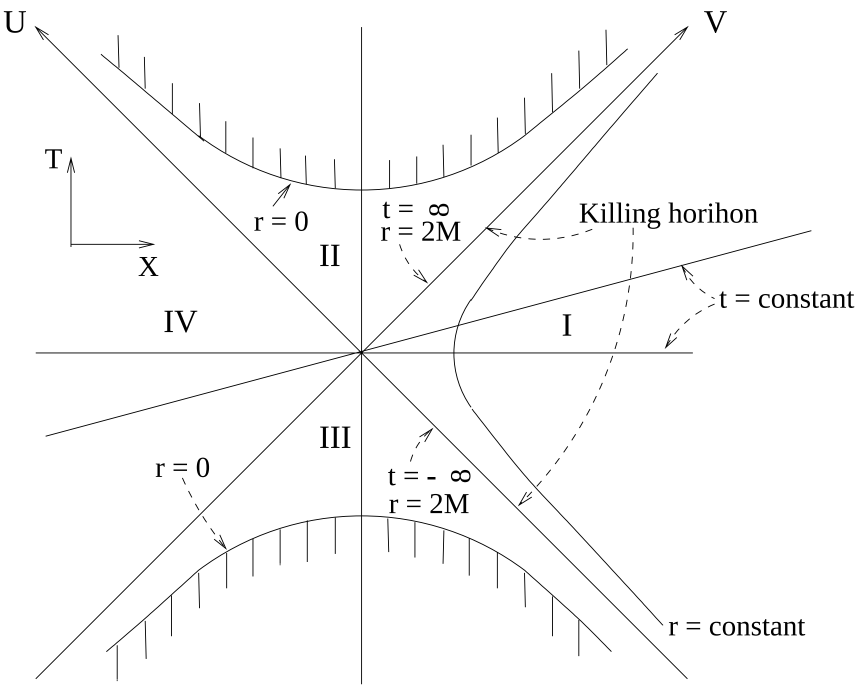


Abbildung 3.7: *The Kruskal extension of Schwarzschild spacetime*

$T$. Finally, demanding that energy-momentum is regular at the future horizon and that there is no incoming radiation, i.e.

$$t_u = \frac{c}{192\pi M^2} \quad \text{and} \quad t_v = 0$$

results in

$$\langle 0_U | T^\mu_\nu | 0_U \rangle = \frac{c}{768\pi M^2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \frac{c\pi}{12}(kT)^2 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \tag{3.29}$$

The *Unruh state* is regular on the future horizon and singular at the past horizon. It describes the Hawking evaporation process with only outward flux of thermal radiation.

## 3.6 Hawking radiation

and hence The most dramatic result arising from investigation of particle creation near black holes was Hawking's discovery that particle creation also occurs near a Schwarzschild black hole, resulting in 'emission' of a thermal spectrum of particles [2]. We give the main steps of the derivation and the discussion of this result. Before doing that, we recall some facts about spherically symmetric spacetimes and the Schwarzschild metric.

### 3.6.1 Spherically symmetric spacetimes

A spacetime is spherically symmetric if its isometry group contains a subgroup isomorphic of the rotation group $SO(3)$, and if the orbits of this subgroup are two-dimensional spheres. The induced metric on each orbit must be a multiple of the metric on a unit 2-sphere, and thus can be characterized by the total area $A$, of the sphere. It is convenient to introduce the function $r$, referred to as 'radial coordinate', defined by $r = (A/4\pi)^{1/2}$. Thus in spherical coordinates $(\theta, \varphi)$ the metric on each orbit takes the form

$$ds^2 = r^2 d\Omega^2 = r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big).$$

The three linearly independent spacelike Killing vector fields with closed orbits and which satisfy the $so(3)$ commutation relations are

$$-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}, \quad \cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}, \quad \frac{\partial}{\partial\varphi}.$$

In adapted coordinates the line element of a spherically symmetric spacetime has the canonical form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\Omega^2, \qquad \nu = \nu(t,r), \quad \lambda = \lambda(t,r). \tag{3.30}$$

The most general vacuum solution of Einsteins field equation is given by the well-known Schwarzschild line element

$$ds^2 = \alpha dt^2 - \frac{1}{\alpha}dr^2 - r^2 d\Omega^2, \qquad \alpha = 1 - 2M/r. \tag{3.31}$$

This solution is stationary with timelike Killing field $\partial_t$ which is orthogonal to the Hypersurfaces $t = const$. Hence every spherically symmetric vacuum solution is automatically static. The advance of the perihelion of mercury, the bending of light by the sun, the time delay of light and the gravitational redshift have been used to test this line element. These and some more recent observations are in good agreement with the theoretical predictions of Einsteins theory of relativity.

In the following spacelike, timelike and null-hypersurfaces will be important. A hyperplane $\Sigma$ with normal vector field $n$ at a point $p$ is spacelike, null or timelike at this point if $(n,n)$ is negative, zero or positive at $p$. For example, the vector field normal to the hypersurfaces $x^0$ = constant obeys

$$(n, \partial_i) = n^\mu g_{\mu i} = n_i = 0, \quad i = 1,2,3.$$

If we set $n_0 = 1$, then

$$(n,n) = g^{00}.$$

A coordinate hypersurface $x^\mu$=const. is spacelike, null or timelike if $g^{\mu\mu}$ is negative, zero or positive. We see that the hypersurfaces with constant $\varphi$ or constant $\theta$ are space like at all points. Since

$$g^{rr} = -\alpha(r),$$

the surface of constant $r$ is spacelike outside the event horizon at $r = 2M$, timelike inside and null on the horizon.

### 3.6.2 The Kruskal Extension

The singularity of the Schwarzschild metric at the Schwarzschild radius $r = 2M$ is a coordinate singularity, whereas the singularity at $r = 0$ is a true singularity. The coordinate singularity is very similar to the coordinate singularity if one uses Rindler coordinates on the Rindler wedge. Introducing the tortoise coordinate as above, the line element reads

$$ds^2 = \alpha(r)\big(dt^2 - dr_*^2\big) - r^2 d\Omega^2,$$

where $r = r(r^*)$ is given in (3.24). As above, we introduce the null-coordinates $u, v$ so that

$$ds^2 = \frac{8M}{r} e^{-r/2M} e^{(n-u)/2M} du dv - r^2 d\Omega^2.$$

If one introduces the Kruskal coordinates $U, V$ the metric becomes

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} dU dV - r^2 d\Omega^2$$

and is regular at the horizon. If we finally set

$$U = T - X \quad \text{and} \quad V = T + X$$

then the Schwarzschild metric takes the final form given by Kruskal (1960)

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \big(dT^2 - dX^2\big) - r^2 d\Omega^2. \tag{3.32}$$

The transformation from the Kruskal coordinates $(T, X)$ to the Schwarzschild coordinates $(t, r)$ is explicitly given by

$$\begin{aligned} X^2 - T^2 &= -UV = e^{(v-u)/2M} = e^{r_*/2M} = e^{r/2M}\big(\frac{r}{2M} - 1\big) \\ \log\frac{T+X}{X-T} &= \log\big(-\frac{V}{U}\big) = \log e^{(u+v)/2M} = \frac{t}{2M}. \end{aligned}$$

It follows, that

$$\frac{XdT - TdX}{X^2 - T^2} = \frac{dt}{4M} \quad \text{and} \quad XdX - TdT = \frac{r}{8M^2} e^{r/2M} dr.$$

The allowed range of the Kruskal coordinates is given by the condition $r > 0$, which yields $T^2 - X^2 < 1$. The spacetime diagram for the Kruskal extension is shown in figure (3.7). By construction all radial null geodesics are $45^0$ lines. There are spacelike physical singularities in the extended region at $T = \pm\sqrt{1 + X^2}$. The wedge $I$ in with positive $X$ and $|T| \leq X$ corresponds to the exterior Schwarzschild solution.

### 3.6.3 Wave equation in Schwarzschild spacetime

According to the general discussion we need to study the classical wave propagation of a Klein-Gordon scalar field in region $I$ of the extended Schwarzschild spacetime (3.8). One might expect,

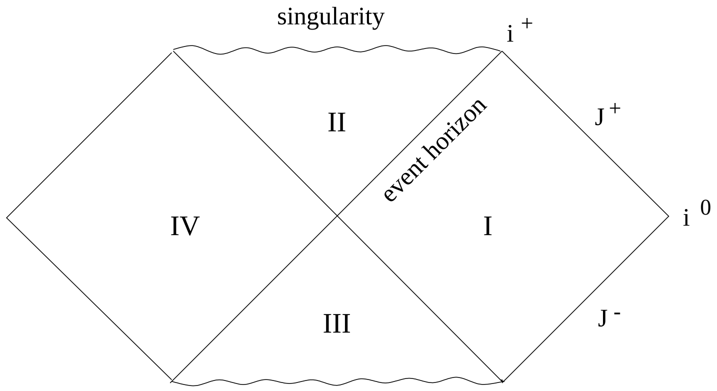


Abbildung 3.8: *Conformal diagram of the extended Schwarzschild spacetime.*

that any solution in this region must have started from infinity or must have entered region $I$ from the white hole region $III$. At late times, one expects that every solution will propagate into the black hole region $II$ and/or propagate back to infinity. For the investigation we use, that in in Schwarzschild coordinates the Laplacian reads

$$\Box = \frac{1}{\alpha}\frac{\partial^2}{\partial t^2} - \Big(\alpha\,\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \alpha' \frac{\partial}{\partial r}\Big) + \frac{\vec{L}^2}{r^2}. \tag{3.33}$$

Since spacetime is spherically symmetric we can expand the field in spherical harmonics and write the wave equation $(\Box + m^2)\phi$ for each mode of the form Setting

$$\phi = \frac{f(t,r)}{r} Y_{lm} e^{-i\omega t}.$$

We obtain

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r_*^2} - \Big(1 - \frac{2M}{r}\Big)\Big(\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + m^2\Big) f = 0, \tag{3.34}$$

where the tortoise coordinate $r_*$ has been defined above, $M$ is the mass of the black hole and $m$ is the mass of the Klein-Gordon field. This equation can be identified with the wave equation for a massless scalar field in 2-dimensional flat spacetime with scalar potential

$$V(r_*) = \Big(1 - \frac{2M}{r}\Big)\Big(\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + m^2\Big).$$

As $r_* \to -\infty$ (i.e. $r \to 2M$) the potential falls off exponentially, $V \sim \exp(r_*/2M)$, and as $r_* \to \infty$ the potential behaves as $\sim m^2 - 2Mm^2/r_*$ in the massive case and $\sim l(l+1)/r^2$ in the massless case. In the asymptotic region $r \to \infty$ this equation possesses outgoing solution $\sim e^{i\omega r_*}$ and ingoing solutions $\sim e^{-i\omega r_*}$. In terms of the null-coordinates the asymptotic solutions look like

$$\text{outgoing: } \frac{1}{r}e^{-i\omega u}, \qquad \text{ingoing: } \frac{1}{r}e^{-i\omega v}. \tag{3.35}$$

Because of the potential term in (3.34) the incoming waves will partially scatter off the gravitational field to become a superposition of incoming and outgoing waves.

We decompose $\phi$ into a complete set of positive frequency modeled denoted by $u_{\omega lm}$:

$$\phi = \sum_{l,m} \int d\omega \Big( a_{\omega lm} u_{\omega lm} + a^\dagger_{\omega lm} u^\dagger_{\omega lm} \Big),$$

which are normalized according to

$$\big( u_{\omega_1 l_1 m_1}, u_{\omega_2 l_2 m_2} \big) = \delta(\omega_1 - \omega_2)\delta_{l_1 l_2}\delta_{m_1 m_2},$$

where we used the conserved 'norm' introduced earlier,

$$(u_1, u_2) = i \int_\Sigma \Big( u_1^\dagger n^\mu \nabla_\mu u_2 - (n^\mu \nabla_\mu u_1^\dagger) u_2 \Big) \sqrt{h} d^3x,$$

and are chosen to reduce to the incoming spherical modes (3.35) in the remote past. The state should correspond to the absence of incoming radiation,

$$a_{\omega lm} \psi_0 = 0. \tag{3.36}$$

# Literaturverzeichnis


[1] Parker, L. (1969), 'Quantised Fields and Particle Creation in Expanding Universes', Phys. Rev. D 183, 1057

[2] Hawking, S. (1975), 'Particle Creation. by Black Holes', Commun. Math. Phys. 43, 199.

[3] Birrell, N.D. and Davies, P.C.W. (1982), 'Quantum Fields in Curved Space', CUP.

[4] Fulling, S.A. (1989), 'Aspects of Quantum Field Theory in Curves Spacetime', CUP.

[5] Unruh, W.G. (1976), 'Notes on Black Hole Evaporation', Phys. Rev. D14, 870.

[6] Fulling, S.A., Sweeny, M., and Wald, R.M (1978), 'Singularity structure of the two-point function in quantum field theory in curved spacetime', Commun. Math. Phys.', 63, 257.

[7] Wald, R.M. (1984), 'On the trace anomaly of a conformally invariant quantum field in curved spacetime', Phys. Rev. D 17, 1477.

[8] Wald, R.M. (1994), 'Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics', Chicago Lectures in Physics.

[9] Kubo, R. (1957), 'Statistical-Mechanical Theory of Irreversible Processes I, H. Phys. Soc. Japan 12, 570; Martin, P.C. and Schwinger, J. (1959), 'Theory of Many-Particle Systems', Phys. Rev. 115, 1342.