Kapitel 2

Instantons

This chapter is devoted to the study of (anti)selfdual solutions of the Euclidan Yang-Mills equations. These minimize the Euclidean action in a fixed topological sector of the configuration space. We begin with the useful Hobart-Derrich-Theorem. Based on scaling arguments one may show that cenrtain euclidean field equations possess no solutions with finite action. The we turn to the Lax-Pairs of Yang-Mills systems. Then we discuss the associated linear system with the help of the powerful Factorization theorem of Birkhoff. Finally we study instantons on the 4-dimensional torus.

2.1 Hobart-Derrick-Theorem

This is rather simple but nevertheless useful theorem which is proven by scaling arguments only. We start with the euclidean action for a Yang-Mills-Higgs system with fixed background metric,

$$S=\int e\Big[\frac{1}{4g^2}F^a_{\mu\nu}F^{\mu\nu}_a+\frac{1}{2}D_\mu\phi D^\mu\phi+V(\phi)\Big]=S_{YM}+S_H.$$

It varies as follows under variations of the gauge potential and matter fields,

$$\begin{aligned}\delta S &= \int e\delta A^a_\nu\big[-\frac{1}{g^2}\{\frac{1}{e}\,\partial_\mu(eF^{a\mu\nu})+f_{abc}A^b_\mu F^{c\mu\nu}\}+J^\nu_a\big]+\\ &+ \int e\delta\phi\big[-D_\mu D^\mu\phi+\frac{\delta V}{\delta\phi}\big]+\int e\partial_\mu S^\mu.\end{aligned}$$

We have introduced the gauge current,

$$J^\nu_a=+i(T_a\phi,D^\nu\phi)-i(D^\nu\phi,T_a\phi)$$

and the vector field which determines the surface contribution to δS

$$S^\mu=\frac{1}{g^2}F^{a\mu\nu}\delta A^a_\nu+\frac{1}{2}(\delta\phi,D^\mu\phi)+\frac{1}{2}(D^\mu\phi,\delta\phi)$$

The field equations are the Yang-Mills equation

$$\frac{1}{e}\,\partial_\mu(eF_a^{\mu\nu}) + f_{abc}A_\mu^b F^{c\mu\nu} \equiv \nabla_\mu F_a^{\mu\nu} + f_{abc}A_\mu^b F^{c\mu\nu} = g^2 J_a^\nu, \tag{2.1}$$

and the Higgs equation

$$-D_\mu D^\mu \phi + V'(\phi) = 0. \tag{2.2}$$

Scale instabilities in euclidean spacetimes

The scale instability arguments were formulated by Hobart (1963) and Derrick (1964) and give necessary conditions for the existence of solutions with finite euclidean action or finite energy in flat spacetime. We assume, that

$$V(\phi) = \lambda\big[(\phi,\phi) - 1\big]^2 \geq 0.$$

Let $\Phi = (A, \phi)$ be a solution of the Yang-Mills-Higgs (YMH) field equations with finite action. Consider the simplest type of perturbation $\Phi \to \Phi_\alpha$ defined by

$$A_\alpha(x) = \alpha A(\alpha x), \quad \phi_\alpha(x) = \phi(\alpha x).$$

This perturbation is a particular diffeomorphism on the matter fields. However, the metric is not transformed, such that the action is not invariant. Indeed, the action scales as

$$S[\Phi_\alpha] = \alpha^{4-d} S_{YM} + \alpha^{2-d}\int \frac{1}{2}(D^\mu\phi)^2 + \alpha^{-d}\int V(\phi)$$

If $\Phi = \Phi_{\alpha=1}$ is a solution, then the derivative of $S[\Phi_\alpha]$ at $\alpha = 1$ must vanish:

$$0 = (d-4)S_{YM} + (d-2)\int \frac{1}{2}(D^\mu\phi)^2 + d\int V(\phi). \tag{2.3}$$

From this interesting relation we draw the following consequences: if (A, ϕ) solves the Yang-Mills-Higgs equation, then

- A pure YM-theory in $d < 4$ dimensions has only the trivial solution $F = 0$. Hence, there are no instantons in less then 4 dimensions.
- An euclidean solution in 4 dimensions is gauge-equivalent to a pure YM-solution. Indeed, in 4 dimensions

  $$\int (D^\mu\phi)^2 + 4\int V(\phi) = 0 \Longrightarrow D\phi = 0,\ |\phi| = 1.$$

  Such a Higgs field can be gauged to a constant field.
- In more as 4 dimensions there are no nontrivial euclidean solutions of the YMH equations with finite action.

- In 2 dimensions and for $\lambda = 0$ the only solution is $F = 0$ and $|\phi| = 1$. This result is relevant for superconductivity. There are no vortex solutions with finite energy and $\lambda = 0$. Indeed, for $\lambda = 0$ it follows that $F = 0$. Hence A can be gauged to zero and $\triangle\phi$ must vanish. But the only harmonic ϕ with $d\phi \in L_2$ is ϕ =constant.

- If $|\phi| = 1$ for $d < 4$, then $D\phi = 0, F = 0$. To see that, we note that V' vanishes at these values of ϕ and hence $D^2\phi = 0$. Therefore

$$0 = \triangle|\phi|^2 = 2(D^2\phi, \phi) + 2(D_\mu\phi, D^\mu\phi) = 2(D_\mu\phi, D^\mu\phi).$$

  From (2.3) we conclude that $F = 0$ and that there are no interesting static solutions with constant ϕ.

2.2 Instantons - Introduction

Instantons are solution with finite action of the pure YM-equations

$$d^*F - iA\wedge^* F + i^*F\wedge A = 0 \Longleftrightarrow \frac{1}{e}\,\partial_\mu(eF_a^{\mu\nu}) + f_{abc}A_\mu^b F^{c\mu\nu} = 0$$

in spacetimes with euclidean signature. One furthermore demands, that the field strength is dual or antiselfdual

$$F =^* F \quad \text{or} \quad F = -^*F \Longleftrightarrow F_{\mu\nu} = \pm\frac{1}{2}\eta_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{2.4}$$

This condition on the field strength is diffeomorphism and Weyl invariant. Because of the Bianchi-identity

$$dF - iA\wedge F + iF\wedge A = 0$$

a (anti)selfdual configuration automatically solves the Yang-Mills equation.
Because of

$$\alpha\wedge^*\beta = \frac{1}{p!}\alpha_{\mu_1\dots\mu_p}\beta^{\mu\dots\mu_p}\eta.$$

one has the following scalar product for p-forms with values in the Lie algebra of a (compact) group[1]:

$$(\alpha,\beta) = \int \mathrm{tr}\,\alpha\wedge^*\beta$$

In particular

$$0 \le (F\pm^* F, F\pm^* F) = 2(F,F) \pm 2(F,^* F)$$

[1] here one needs the euclidean signature

from which we conclude, that

$$\boxed{S_{YM} = \frac{1}{2g^2}\int F\wedge {}^* F \geq \frac{1}{2g^2}|\int F\wedge F| = \frac{1}{2g^2}|\oint \omega_3|} \tag{2.5}$$

where we have used our earlier result, namely that $\int F\wedge F$ is a surface integral. Clearly, the inequality becomes an equality if and only if F is (anti)selfdual. If F is (anti)selfdual, then

$$S_{YM} = \pm\frac{1}{2g^2}\oint \omega_3. \tag{2.6}$$

In a conformally flat spacetime the components of F and ${}^*F$ are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad , \quad {}^*F^{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}.$$

Thus, the field strength is selfdual, if $\vec{E} = \vec{B}$ and it is antiselfdual if $\vec{E} = -\vec{B}$.

Now we shall see, that for configurations with finite action the surface integral in (2.6) is a multiple of an integer, the so-called *instanton number*. The following arguments apply to flat spacetime[2]. Since the euclidean action should be finite, we expect, that

$$F \longrightarrow 0 \quad \text{or} \quad A \longrightarrow igdg^{-1} \quad \text{for} \quad |x| \longrightarrow \infty$$

It follows, that asymptotically

$$\omega_3 = \frac{1}{3}\text{tr}\left(gdg^{-1}\wedge gdg^{-1}\wedge gdg^{-1}\right).$$

Let us assume, that $G = SU(2) \sim S^3$. In this case the map

$$\hat{x} \longrightarrow \lim_{r\to\infty} g(x = r\hat{x})$$

is a map from S^3 to S^3. Continuous maps from S^3 to S^3 may have winding numbers. To understand what I mean by winding numbers, let us first consider the simpler example of an Abelian gauge potential on R^2:

$$R^2 \ni x \longrightarrow A(x).$$

We demand, that for $r\to\infty$ the potential is pure gauge, $A \to igdg^{-1}$ with $g\in U(1)\sim S^1$. The following maps are of this type:

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = r\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \longrightarrow A = \frac{r^2}{a^2+r^2} igdg^{-1}, \quad \text{where} \quad g = e^{in\varphi}. \tag{2.7}$$

The Cartesian components of the gauge potential are

$$(A_\mu) = \frac{n}{a^2+r^2}\begin{pmatrix} -x^1 \\ x^0 \end{pmatrix}.$$

[2]In other compact or non-compact spacetimes modifications maybe in order

For large r we have

$$A \sim igdg^{-1} = nd\varphi.$$

Clearly, if $\hat{x}$ rotates once by a full circle, the $g(\varphi)$ rotates n-times. The winding number of g is

$$n = \frac{i}{2\pi} \oint d\varphi \, gdg^{-1} = \frac{1}{2\pi} \int_0^{2\pi} nd\varphi = \oint A = \int F.$$

Note that the function multiplying $igdg^{-1}$ in (2.7) has a zero at $r = 0$. This must be the case, since the gauge potential must unwind at the origin.

Let us now discuss the maps $S^3 \to SU(2)$. We parametrize $SU(2)$ according to

$$g = V^0 + iV^i\sigma_i, \quad \sum V^\mu V_\mu = 1 \Longrightarrow g^{-1} = V^0 - iV^i\sigma_i,$$

where the σ_i denote the Pauli-matrices. One easily finds

$$gdg^{-1} = iW^j\sigma_j, \quad \text{where} \quad W^j = dV^0V^j - V^0dV^j + \epsilon^j{}_{pq}V^pdV^q,$$

so that

$$\mathrm{tr}\,(gdg^{-1})^3 = -2\epsilon_{ijk}W^i \wedge W^j \wedge W^k = 2\epsilon_{\alpha\mu\nu\sigma}V^\alpha dV^\mu \wedge dV^\nu \wedge dV^\sigma.$$

Now we assume, that the boundary (e.g. the sphere at infinity) over which ω_3 is integrated is parametrized by three parameters ξ^i, i.e. $x^\mu = x^\mu(\xi)$. Then

$$\oint \omega_3 = \frac{2}{3}\epsilon_{\alpha\mu\nu\sigma}\epsilon^{ijk} \oint V^\alpha V^\mu{}_{,i} V^\nu{}_{,j} V^\sigma{}_{,k} \; d^3\xi. \tag{2.8}$$

One can show, that the integrand on the right hand side satisfies the identity

$$\Big[\epsilon_{\alpha\mu\nu\sigma}\epsilon^{ijk}V^\alpha V^\mu{}_{,i} V^\nu{}_{,j} V^\sigma{}_{,k}\Big]^2 = 36 \det\Big[\frac{\partial V^\mu}{\partial \xi^i}\frac{\partial V^\mu}{\partial \xi^j}\Big] = 36\det(g_{ij}),$$

i.e. is proportional to the determinant of the induced metric on the boundary. Hence we end up with

$$\oint \omega_3 = 4 \lim_{R\to\infty} \int_{S^3_R} d^3\xi \sqrt{\det(g_{ij})} = 8\pi^2 q,$$

where q is the instanton number. While the point ξ covers the spheres S^3_R once, the vector V^α can cover the sphere any q number of times, each contributing a 4-dimensional solid angle $\int d^3\xi = 2\pi^2$.

We investigate the map

$$x = \begin{pmatrix} \cos\varphi \\ \sin\alpha\cos\beta\sin\varphi \\ \sin\alpha\sin\beta\sin\varphi \\ \cos\alpha\sin\varphi \end{pmatrix} \longrightarrow V^{(n)} = \begin{pmatrix} \cos n\varphi \\ \sin\alpha\cos\beta\sin n\varphi \\ \sin\alpha\sin\beta\sin n\varphi \\ \cos\alpha\sin n\varphi \end{pmatrix},$$

where $\beta \in [0, 2\pi]$ and $\alpha, \varphi \in [0, \pi]$. Setting $(\xi^1, \xi^2, \xi^3) = (\varphi, \alpha, \beta)$ the induced metric reads

$$g_{ij} = \begin{pmatrix} n^2 & 0 & 0 \\ 0 & \sin^2 n\varphi & 0 \\ 0 & 0 & \sin^2\alpha \sin^2 n\varphi \end{pmatrix} \Longrightarrow \sqrt{(g_{ij})} = n \sin^2 n\varphi \sin\alpha$$

A short calculation yields then

$$\int \sqrt{\det(g_{ij})}\, d^3\xi = 2n\pi^2.$$

Note, that the group element belonging to $V^{(n)}$ is just the n'th power of that belonging to $V^{(1)}$.

Degree of a mapping. The instanton number is the degree of the mapping $S^3 \to SU(2)$. The concept of a degree of a mapping is of course more general and it is helpful to know this generalization when one deals with other dimensions (vortices, monopoles, textures). In instanton context the following theorem due to Bott is helpful:

Theorem: Any continuous mapping of S^3 into a simple Lie group G can be continuously deformed into a mapping into an $SU(2)$ subgroup of G.

Therefore, for a Yang-Mills theory with simple gauge group it is only necessary to consider $S^3 \to SU(2)$. As a preparation, we need the following

Theorem: Let M be a orientable compact and connected d-dimensional manifold and ω a d-form, such that

$$\int_M \omega = 0.$$

Then ω is exact. For a proof see S. Sternberg, Lectures on Differential Geometry, p. 120.

It follows immediately, that the dimension of $H^d(M)$ (closed modulo exact d forms) is one: Let η be the closed (but not exact) volume form and ω another d-form. Then there is a real number λ such that

$$0 = \int_M \eta - \lambda \int_M \omega = 0 \Longrightarrow \eta = \lambda\omega + d\theta,$$

and hence $\eta \sim \omega$.

Let M and $\tilde{M}$ be two orientable compact and connected d-dimensional manifolds, $\tilde{\eta}$ a volume form on $\tilde{M}$ ($\int \tilde{\eta} \neq 0$) and

$$f : M \longrightarrow \tilde{M}$$

a differentiable map. The degree of this map is

$$\deg f = \frac{\int_M f^*\tilde{\eta}}{\int_{\tilde{M}} \tilde{\eta}}. \tag{2.9}$$

The degree is independent of the choice of the volume form. Let $\tilde{\omega}$ be another volume form of $\tilde{M}$

which we normalize, such that it leads to the same volume as $\tilde{\eta}$. Then $\tilde{\omega} = \tilde{\eta} + d\tilde{\theta}$. Recalling, that the pullback commutes with the exterior differential, we have

$$f^*\tilde{\omega} = f^*\tilde{\eta} + d(f^*\tilde{\theta}),$$

which prove the statement. If f and g are homotopic to each other, then deg(f) =deg(g). The following theorem shows, that the degree is an integer:

Theorem: If q is a regular value of f, then $f^{-1}(q)$ is a finite set, and

$$\deg(f) = \sum_{p \in f^{-1}(q)} [\text{signature of } \det(f_*)_p].$$

Here $\det(f_*)_p$ is the determinant of the matrix of the differential $(f_*)_p$ of f with respect to a positive local coordinate system at each p. If $f^{-1}(q)$ is empty, then the degree vanishes.

The signature in this formula is 1 if the map $p \to q$ is orientation preserving and it is -1 if the orientation is reversed.

Let us now return to the instantons. Together with (2.5) we conclude, that the euclidean action is bounded below by the instanton number as follows:

$$S_{YM} \geq \frac{4\pi^2}{g^2}|q|.$$

The instanton number q is the integer-valued winding number.

We have got the following picture: The gauge fields with finite euclidean action must be pure gauge at infinity. Hence we can assign an integer instanton number to any such configuration and the configuration space decomposes into homotopy classes characterized by winding numbers q, the number of times the S^3 covers the group manifold $SU(2)$. The absolute minimum in each sector is achieved by an (anti)selfdual configuration.

To continue, we need some results about *double-null coordinates,* since in these coordinates the (anti)-selfduality condition takes a particularly simple form. Since we want to deal with the Euclidean, Minkowskian and ultrahyperbolic cases at the same time, we introduce the complexified Minkowski spacetime with metric

$$ds^2 = 2\big(dzd\tilde{z} - dwd\tilde{w}\big) \tag{2.10}$$

and the volume element

$$\nu = dz \wedge d\tilde{z} \wedge dw \wedge d\tilde{w}. \tag{2.11}$$

The coordinate vectors $\partial_z, \partial_w, \partial_{\tilde{z}}, \partial_{\tilde{w}}$ form a null-tetrad at each point. A general null tetrad is a basis of $4-$vectors $\{Z, W, \tilde{Z}, \tilde{W}\}$ such that

$$\eta(Z, \tilde{Z}) = -\eta(W, \tilde{W}) = 1 \quad \text{and} \quad \nu(Z, \tilde{Z}, W, \tilde{W}) = 1,$$

where η is the metric tensor, and such that all the other inner products vanish. We recover the various real slices by imposing reality conditions on the complex coordinates as follows:

- Euclidean real slice E:

$$B = \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 - ix^1 & x^2 + ix^3 \\ -(x^2 - ix^3) & x^0 + ix^2 \end{pmatrix},$$

where x^μ are real Cartesian coordinates. E is picked out by the reality conditions $\tilde{w} = -\bar{w}, \tilde{z} = \bar{z}$. The normalized matrices B span $SU(2)$,

$$\sigma_2 B \sigma_2 = \bar{B}, \quad \det(B) = \frac{1}{2} r^2, \qquad r^2 = (x^0)^2 + (x^3)^2 + (x^1)^2 + (x^2)^2.$$

- Minkowski real slice M:

$$B = \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 - x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 + x^1 \end{pmatrix},$$

with real x^μ. The reality conditions are that $z, \tilde{z}$ are real and $\bar{w} = \tilde{w}$. We have

$$B = B^\dagger \quad \text{and} \quad \det B = \frac{1}{2} r^2, \qquad r^2 = (x^0)^2 - (x^3)^2 - (x^1)^2 - (x^2)^2.$$

- Ultrahyperbolic real slice U:

$$B = \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 - ix^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 + ix^1 \end{pmatrix},$$

with real x^μ. The reality conditions are $\tilde{z} = \bar{z}, \tilde{w} = \bar{w}$. The (normalized) matrices B span $SU(1,1)$:

$$\sigma_1 B \sigma_1 = \bar{B}, \quad \det B = \frac{1}{2} r^2 \sigma_3, \qquad r^2 = (x^0)^2 + (x^3)^2 - (x^1)^2 - (x^2)^2.$$

One should keep in mind that the volume form ν (with our convention) is real on E and U but imaginary on M:

$$\nu = \begin{cases} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 & \text{on E} \\ -dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 & \text{on U} \\ i dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 & \text{on M.} \end{cases}$$

We shall denote a general coordinate system on CM by x^a. Then

$$ds^2 = \eta_{ab} dx^a dx^b \quad , \quad \nu = \Delta \cdot dx^0 \wedge dx^1 \wedge dx^1 \wedge dx^3, \quad \Delta = \sqrt{\det(\eta_{ab})}.$$

The exterior product, derivative and Lie derivative are defined as in the real case we discussed earlier. The Hodge dual of a p-form is

$$^*\alpha = \frac{1}{(d-p)!} {}^*\alpha_{ef\ldots} dx^e \wedge dx^f \ldots, \quad {}^*\alpha_{ef\ldots} = \frac{1}{p!} \Delta \epsilon_{cd\ldots ef\ldots} \alpha^{cd\ldots}.$$

For 2-forms in 4 dimensions

$$^*\alpha_{ab} = \frac{1}{2} \Delta \epsilon_{abcd} \eta^{ce} \eta^{df} \alpha_{ef},$$

and * is idempotent. Given a double-null coordinate system, for which

$$g_{z\tilde{z}} = g^{z\tilde{z}} = -g_{w\tilde{w}} = -g^{w\tilde{w}} = 1 \quad \text{and} \quad \Delta\epsilon_{z\tilde{z}w\tilde{w}} = 1$$

we can decompose a general 1-form and 2-form as

$$\begin{aligned}
\alpha &= (\alpha_z dz + \alpha_w dw) + (\alpha_{\tilde{z}} d\tilde{z} + \alpha_{\tilde{w}} d\tilde{w}) \\
\beta &= (\beta_{zw} dz \wedge dw) + (\beta_{\tilde{z}\tilde{w}} d\tilde{z} \wedge d\tilde{w}) \\
&\quad + (\beta_{z\tilde{z}} dz \wedge d\tilde{z} + \beta_{w\tilde{w}} dw \wedge d\tilde{w} + \beta_{z\tilde{w}} dz \wedge d\tilde{w} + \beta_{w\tilde{z}} dw \wedge d\tilde{z}).
\end{aligned}$$

The one-form α decomposes into $\alpha_{(1,0)} + \alpha_{0,1)}$ and the two-form β into $\beta_{2,0} + \beta_{0,2} + \beta_{1,1}$. The decomposition depends on the choice of coordinates, but is invariant under transformations which preserve the foliation by constant z, w and the foliation by surfaces of constant $\tilde{z}, \tilde{w}$. With the above formula for the dual one finds

$$^*\alpha = \left(\alpha_{(1,0)} - \alpha_{(0,1)}\right) \wedge \omega, \quad \omega = dz \wedge d\tilde{z} - dw \wedge d\tilde{w}$$

and

$$\begin{aligned}
^*\beta &= (\beta_{zw} dz \wedge dw) + (\beta_{\tilde{z}\tilde{w}} d\tilde{z} \wedge d\tilde{w}) \\
&\quad - (\beta_{z\tilde{z}} dw \wedge d\tilde{w} + \beta_{w\tilde{w}} dz \wedge d\tilde{z} + \beta_{z\tilde{w}} dz \wedge d\tilde{w} + \beta_{w\tilde{z}} dw \wedge d\tilde{z})
\end{aligned}$$

from which follows, that

$$\alpha = dz \wedge dw, \quad \tilde{\alpha} = d\tilde{z} \wedge d\tilde{w} \quad \text{and} \quad \omega = dz \wedge d\tilde{z} - dw \wedge d\tilde{w}$$

span the space of selfdual 2-forms and

$$dz \wedge d\tilde{w}, \quad dw \wedge d\tilde{z} \quad \text{and} \quad dz \wedge d\tilde{z} + dw \wedge d\tilde{w}$$

span the space of anti-selfdual forms.

The *exterior derivative* decomposes into a a 'holomorphic' and 'antiholomorphic' piece[3], $d = \partial + \bar{\partial}$ of two operators,

$$\partial = dz\partial_z + dw\partial_w, \quad \tilde{\partial} = d\tilde{z}\partial_{\tilde{z}} + d\tilde{w}\partial_{\tilde{w}}. \tag{2.12}$$

All instantons in euclidean spacetime or equivalently on S^4 are known. The general solutions have been found by Atiyah, Drinfeld, Hitchin and Manin (ADHM) and depend on $8q - 3$ parameters. The first solution has been found by Belavin, Polyakov, Tyupkin and Schwartz. The BPS instanton has the form

$$A = if(r)\, gdg^{-1}, \quad \text{where} \quad g = \frac{\sqrt{2}}{r} B$$

so that

$$F = if' dr \wedge gdg^{-1} + i(f - f^2) d(gdg^{-1})$$

[3] Only on E and U are z, w holomorphic coordinates

with

$$gdg^{-1} = \frac{1}{r^2}\begin{pmatrix} \tilde z dz - z d\tilde z + \tilde w dw - w d\tilde w & -2\tilde z dw + 2w d\tilde z \\ 2\tilde w dz - 2z d\tilde w & -\tilde z dz + z d\tilde z - \tilde w dw + w d\tilde w \end{pmatrix}$$

and

$$d(gdg^{-1}) = -\frac{2dr}{r} gdg^{-1} + \frac{2}{r^2}\begin{pmatrix} d\tilde z dz + d\tilde w dw & -2d\tilde z dw \\ 2d\tilde w dz & -d\tilde z dz - d\tilde w dw \end{pmatrix}$$

we obtain

$$\begin{aligned} F &= i\big[f' + \frac{2f(f-1)}{r}\big] dr \wedge gdg^{-1} \\ &- \frac{2if(f-1)}{r^2}\begin{pmatrix} d\tilde z dz + d\tilde w dw & -2d\tilde z dw \\ 2d\tilde w dz & -d\tilde z dz - d\tilde w dw \end{pmatrix} \end{aligned}$$

The last term is anti-selfdual. Hence, if

$$rf' + 2f(f-1) = 0 \Longleftrightarrow (\frac{r^2}{f})' = 2r \Longleftrightarrow f = \frac{r^2}{r^2+\mu^2},$$

then F is anti-selfdual.

The ansatz

$$A = if(r)\, gdg^{-1}, \qquad g = \frac{\sqrt{2}}{r} B^{-1}$$

leads to a selfdual instanton. Indeed,

$$\begin{aligned} F &= i\big[f' + \frac{2f(f-1)}{r}\big] dr \wedge gdg^{-1} \\ &- \frac{2if(f-1)}{r^2}\begin{pmatrix} -d\tilde z dz + d\tilde w dw & 2dz dw \\ 2d\tilde z d\tilde w & d\tilde z dz + d\tilde w dw \end{pmatrix} \end{aligned}$$

is selfdual, if f fulfills the same equation as above. To calculate the topological charge, we calculate

$$F \wedge F = \pm 24 \frac{\mu^4}{(r^2+\mu^2)^4} \sigma_3^2 dz d\tilde z dw d\tilde w,$$

where the plus (minus) sign holds for the selfdual (anti-selfdual) instanton. Thus we end up with the following topological charge density in euclidean spacetime

$$\mathrm{tr}\, F \wedge F = \pm 48 \mu^4 \frac{r^3}{(\mu^2+r^2)^4} dr d\Omega,$$

with the plus (minus) sign for the selfdual (anti-selfdual) configuration. Using that

$$\int \frac{r^3 dr}{(r^2+\mu^2)^4} = \frac{1}{12\mu^4} \quad \text{and} \quad \int d\Omega = 2\pi^2$$

this yields

$$\int \operatorname{tr} F \wedge F = \pm 8\pi^2$$

as required. The above PBS-instantons have instanton number ± 1. The parameter μ^2 'measures' the scale of the instanton. Indeed, a scale-transformation acting on the instanton solution

$$A_\mu(x,\mu^2) \longrightarrow e^{-\lambda} A_\mu(e^{-\lambda}x,\mu^2) = A_\mu(x, e^{2\lambda}\mu^2)$$

just scales the parameter μ. The translations move the center of the instantons away from the origin and leads to a 5-parameter family of solutions, parametrized by the center of the instanton and the scale parameter. To discuss the other conformal transformations we first discuss how the conformal transformations look like on the complexified Minkowski spacetime.

The complex conformal group: Let $x = (x^{\alpha\beta}),\ \alpha,\beta = 0,1,2,3$ be a skew-symmetric complex matrix with zero determinant. Such a matrix has $6-1=5$ complex entries. If $x^{23} \neq 0$, x is a nonzero complex multiple of

$$x = \begin{pmatrix} s\epsilon & B \\ -B^t & \epsilon \end{pmatrix} \quad \text{with} \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $\det(x) = \Big(s-(z\tilde{z}-w\tilde{w})\Big)^2$ holds true we must demand that $s = z\tilde{z}-w\tilde{w}$ for the determinant of x to vanish. We may identify the points of CM with the complex skew symmetric matrices with vanishing determinant if we identify two matrices who are complex multiples of each other. When $x^{23}=0$, then some or all to the space time coordinates are infinite. These points still belong to the compactification of CM. One easily proves, that

$$\epsilon_{\mu\nu\alpha\beta}dx^{\mu\nu}\otimes dx^{\alpha\beta} = 8(dx^{01}dx^{23} - dx^{02}dx^{13} + dx^{03}dx^{12}) = -8\,ds^2,$$

where $ds^2 = dzd\tilde{z} - dwd\tilde{w}$ is the line element on the complexified Minkowski spacetime CM. It follows, that any transformation

$$x \longrightarrow \tilde{x} = \rho x \rho^t, \quad \text{where} \quad \rho \in GL(4,C)$$

induces a conformal transformation of spacetime.

Real forms: The real forms of the conformal group are obtained by requiring that the transformations should preserve the corresponding real slices:

- Euclidean slice: Because

  $$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} x \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \sim -\bar{x}$$

  the euclidean slice is invariant if

  $$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \rho \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} = -\bar{\rho},$$

  that is, if $\rho \in GL(2,H)$. The minus sign on the right follows from $\epsilon^2 = -1$.

- Minkowski slice: Because $B = B^\dagger$ this slice is invariant, if
$$\rho \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rho^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
that is, if $\rho \in U(2,2)$.
- Ultrahyperbolic slice: Because
$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} x \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \sim \bar{x}$$
the slice is invariant if
$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \rho \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} = \bar{\rho}.$$
In this case the conformal group is isomorphic to $GL(4,R)$.

The infinitesimal conformal transformation $A \in gl(4,C)$ in $\rho = e^A$ fulfill the conditions
$$\begin{cases} (\sigma_0 \otimes \epsilon) A (\sigma_0 \otimes \epsilon) = -\bar{A} & \text{on } E \\ (\epsilon \otimes \sigma_0) A (\epsilon \otimes \sigma_0) = A^\dagger & \text{on } M \\ (\sigma_0 \otimes \sigma_1) A (\sigma_0 \otimes \sigma_1) = \bar{A} & \text{on } U. \end{cases}$$
Tho each A belongs a conformal Killing field K, which can be found by equating δx to a scalar multiple of $(Ax + xA^t)$. If we decompose A into a 2×2 block-form,
$$A = \begin{pmatrix} \lambda & -\tau\epsilon \\ \sigma^t \epsilon & \tilde{\lambda} \end{pmatrix},$$
then
$$\begin{aligned} \delta B &= \tau + s\sigma + \lambda_0 \cdot B + B \cdot \tilde{\lambda}^t \\ \delta s &= \operatorname{tr}(s\lambda) + s \operatorname{tr}(B^{-1}\tau), \quad \delta(1) = \operatorname{tr}(\tilde{\lambda}) + s \operatorname{tr}(B^{-1}\sigma) \end{aligned}$$
where we have used, that
$$\epsilon A + A^t \epsilon = \operatorname{tr}(A)\epsilon \quad \text{and} \quad s \cdot \operatorname{tr} B^{-1}\sigma = -\operatorname{tr}(\sigma^t \epsilon B \epsilon).$$
Hence, the entries of τ generate translations, the entries of σ special conformal transformations, and λ and $\tilde{\lambda}$ infinitesimal rotations and a dilatation. Let us be a bit more precise. We decompose λ and $\tilde{\lambda}$ into their trace-free parts plus multiples of the identity:
$$\lambda = \lambda_T + \frac{1}{2}\operatorname{tr}(\lambda)\sigma_0, \quad \tilde{\lambda} = \tilde{\lambda}_T + \frac{1}{2}\operatorname{tr}(\tilde{\lambda})\sigma_0.$$
Demanding that $\delta(1) = 0$ and setting $\operatorname{tr}(\lambda) = \rho$ and writing again λ for the trace-free λ_T we find
$$\begin{aligned} \delta B &= \tau + s\sigma - \frac{s}{2}\operatorname{tr}(B^{-1}\sigma)B + \frac{\rho}{2}B + \lambda \cdot B + B \cdot \tilde{\lambda}^t \\ \delta s &= s \cdot \rho + s \cdot \operatorname{tr}(B^{-1}\tau), \qquad \operatorname{tr}\lambda = \operatorname{tr}\tilde{\lambda} = 0 \end{aligned}$$

where, of course, the last equation follows from the first.

We see, that the rotations can be represented by a pair $(\Lambda,\tilde\Lambda)\in SL(2,\mathbb{C})\times SL(2,\mathbb{C})$,

$$B\to\Lambda B\quad\text{and}\quad B\to B\tilde\Lambda^t,\quad \Lambda=e^{\lambda},\ \tilde\Lambda=e^{\tilde\lambda}.$$

The transformations $(\Lambda,\tilde\Lambda)$ and $(-\Lambda,-\tilde\Lambda)$ are identified, and the complex rotation group is

$$\big(SL(2,\mathbb{C})\times SL(2,\mathbb{C})\big)/Z_2=SO(4,C).$$

The generators $X(A)$ of the conformal transformations are calculated via

$$L_{X(A)}\phi=\frac{d}{du}\phi(X+uAx+uxA^t)\Longrightarrow L_{X(A)}\phi\sim(AX)^{ij}\phi_{,ij}\ ,$$

where $\phi_{,ij}$ is the derivative of ϕ with respect to the entry (i,j) of its matrix-valued argument x.
In euclidean space we have

$$\lambda=i\lambda^i\sigma_i\quad\text{with real}\quad\lambda^i$$

and similarly for $\tilde\lambda$, so that Λ and $\tilde\Lambda$ are in $SU(2)$. Thus, under a rotation

$$B\longrightarrow U_1BU_2^t,$$

so that the gauge potential of the BPS-instanton transforms under rotations as

$$A\longrightarrow U_1AU_1^{-1}.$$

This is just a global gauge transformation. We conclude, that the BPS-instanton is invariant under rotations, up to a global gauge transformation.

Null 2-planes

A 2-plane Π in spacetime is *null* if $\eta(X,Y)=0$ for every pair of tangent vectors X,Y. With each Π we associate a tangent bivector $X\wedge Y$ with components $\pi^{ab}=X^{[a}Y^{b]}$, where X and Y are independent tangent vectors, and the corresponding 2-form $\pi=\frac{1}{2}\pi_{ab}dx^a\wedge dx^b$. The tangent-bivector determines the tangent space to the 2-plane, and is determined by it up to a nonzero multiple. Now we have the following

Lemma: If Π is a null-plane, then $\pi_{ab}\pi^{ab}=0$ and π is either selfdual or anti-selfdual.

Proof: Since

$${}^*\pi_{ab}C^b=\frac{1}{2}\Delta\epsilon_{abcd}\pi^{cd}C^b=\frac{1}{2}\Delta\epsilon_{abcd}A^cB^dC^b=0$$

for every C tangent to Π (and hence in the span of A and B) and on the other hand

$$\pi_{ab}C^b=\frac{1}{2}(A_aB_bC^b-B_aA_bC_b)=0$$

we must have $\pi\sim^*\pi$. But * is idempotent which proves, that $^*\pi=\pm\pi$. The second statement in the lemma follows directly from the definition of π.

We call Π an α-plane whenever π is selfdual and a β-plane whenever π is anti-selfdual. In double-null coordinates, the surfaces of constant z, w and the surfaces of constant $\tilde{z}, \tilde{w}$ are α-planes. More generally, since $\pi(\partial_a, \partial_b) = \pi_{ab}$, and a ASD-form has

$$\pi_{zw} = \pi_{\tilde{w}\tilde{z}} = \pi_{z\tilde{z}} - \pi_{w\tilde{w}} = 0$$

it follows, that a 2-form is anti-selfdual if is orthogonal to the selfdual bivectors,

$$\pi(\partial_z, \partial_w) = \pi(\partial_{\tilde{z}}, \partial_{\tilde{w}}) = \pi(\partial_z, \partial_{\tilde{z}}) - \pi(\partial_w, \partial_{\tilde{w}}) = 0,$$

and similarly, a 2-form is selfdual if it is orthogonal to the antiselfdual bivectors. The anti-selfduality condition can be expressed more compactly as the conditions

$$\pi(L, M) = 0, \quad L = \partial_w - \zeta\partial_{\tilde{z}}, \quad M = \partial_z - \zeta\partial_{\tilde{w}},$$

identically in ζ. Later on ζ will be interpreted as spectral parameter. Let us see, how a right rotation

$$\tilde{\Lambda} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts on L, M: Because of

$$\begin{pmatrix} \tilde{z}' & w' \\ \tilde{w}' & z' \end{pmatrix} = \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} \Lambda^t \quad , \quad \begin{pmatrix} \partial_{\tilde{z}'} & \partial_{w'} \\ \partial_{\tilde{w}'} & \partial_{z'} \end{pmatrix} = \begin{pmatrix} \partial_{\tilde{z}} & \partial_w \\ \partial_{\tilde{w}} & \partial_z \end{pmatrix} \Lambda^{-1}$$

we have

$$\begin{aligned} L' &= \partial_{w'} - \zeta\partial_{\tilde{z}'} = (a + \zeta c)\partial_w - (b + d\zeta)\partial_{\tilde{z}} \\ M' &= \partial_{z'} - \zeta\partial_{\tilde{w}'} = (a + \zeta c)\partial_z - (b + d\zeta)\partial_{tw}. \end{aligned} \tag{2.13}$$

Hence a right rotation maps Π_ζ to $\Pi_{\zeta'}$, where

$$\zeta' = \frac{b + d\zeta}{a + c\zeta}.$$

They act on the Riemann sphere of α-planes through the origin by Möbius transformations.

2.2.1 Lax Pairs and Yang's equation

In double null coordinates, the field strength

$$F = \frac{1}{2}\big(F_{z\tilde{z}} dz \wedge d\tilde{z} + \dots\big)$$

is antiselfdual, if

$$F_{zw} = F_{\tilde{z}\tilde{w}} = F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \tag{2.14}$$

holds. If we write

$$D_z = \partial_z - iA_z, \quad D_{\tilde{z}} = \partial_{\tilde{z}} - iA_{\tilde{z}}, \quad D_w = \partial_w - iA_w, \quad D_{\tilde{w}} = \partial_{\tilde{w}} - iA_{\tilde{w}},$$

then these conditions read

$$[D_z, D_w] = [D_{\tilde{z}}, D_{\tilde{w}}] = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0. \tag{2.15}$$

An equivalent conditions is that the Lax pair of operators

$$L = D_w - \zeta D_{\tilde{z}} \quad \text{and} \quad M = D_z - \zeta D_{\tilde{w}}$$

should commute for every value of the 'spectral parameter' ζ. This last formulation in terms of a linear system is central to the the theory of integrability.

Yangs equation:

The first two equations in (2.14) are the local integrability conditions for the existence of two matrix-valued functions g and $\tilde{g}$ such that

$$A_z = ig^{-1}\partial_z g, \quad A_w = ig^{-1}\partial_w g, \quad A_{\tilde{z}} = i\tilde{g}\partial_{\tilde{z}}\tilde{g}^{-1}, \quad A_{\tilde{w}} = i\tilde{g}\partial_{\tilde{w}}\tilde{g}^{-1}.$$

This fields are determined uniquely by A up to

$$g \longrightarrow h(\tilde{z}, \tilde{w})g \quad \text{and} \quad \tilde{g} \longrightarrow \tilde{g}\tilde{h}(z, w).$$

If A is replaced by a gauge equivalent potential $U^{-1}AU + iUdU^{-1}$, then g and $\tilde{g}$ can be replaced by gU and $U^{-1}\tilde{g}$. The matrix

$$J = g\tilde{g} \tag{2.16}$$

is *Yang's matrix*. It is determined by A up to the freedom

$$J \longrightarrow h(\tilde{z}, \tilde{w})J\tilde{h}(z, w).$$

Now we can write the remaining ASD-equations in terms of J. Indeed, the field strength component

$$F_{z\tilde{z}} = i\Big(\partial_z(\tilde{g}\partial_{\tilde{z}}\tilde{g}^{-1}) - \partial_{\tilde{z}}(g^{-1}\partial_z g) + [g^{-1}\partial_z g, \tilde{g}\partial_{\tilde{z}}\tilde{g}^{-1}]\Big),$$

is proportional to

$$\tilde{g}\Big(\partial_{\tilde{z}}(J^{-1}\partial_z J)\Big)\tilde{g}^{-1} = \partial_{\tilde{z}}(g^{-1}\partial_z g) - [g^{-1}\partial_z g, \tilde{g}\partial_{\tilde{z}}\tilde{g}^{-1}] + \tilde{g}\partial_{\tilde{z}}(\tilde{g}^{-1}\partial_z\tilde{g})\tilde{g}^{-1}$$

if we use the identity

$$\tilde{g}\partial_{\tilde{z}}(\tilde{g}^{-1}\partial_z\tilde{g})\tilde{g}^{-1} = \partial_z(\partial_{\tilde{z}}\tilde{g}\tilde{g}^{-1})$$

and similarly for $F_{w\tilde{w}}$. Thus we find

$$F_{z\tilde{z}} - F_{w\tilde{w}} = -i\tilde{g}\Big\{\partial_{\tilde{z}}(J^{-1}\partial_z J)\Big) - \partial_{\tilde{w}}(J^{-1}\partial_w J)\Big\}\tilde{g}^{-1}.$$

Thus, the remaining third ASD equation holds if and only if J satisfies the *Yang equation*

$$\boxed{\partial_{\tilde{z}}(J^{-1}\partial_z J) - \partial_{\tilde{w}}(J^{-1}\partial_w J) = 0.} \tag{2.17}$$

Single Instanton: For the BPS-instanton we have

$$A_z = \frac{i}{2f}\begin{pmatrix} -\tilde{z} & 2\tilde{w} \\ 0 & \tilde{z} \end{pmatrix} \qquad A_{\tilde{z}} = \frac{i}{2f}\begin{pmatrix} z & 0 \\ 2w & -z \end{pmatrix}$$
$$A_w = \frac{i}{2f}\begin{pmatrix} -\tilde{w} & 0 \\ -2\tilde{z} & \tilde{w} \end{pmatrix} \qquad A_{\tilde{w}} = \frac{i}{2f}\begin{pmatrix} w & -2z \\ 0 & -w \end{pmatrix},$$

where $f = z\tilde{z} - w\tilde{w} + \mu^2$ and one finds

$$g = \frac{\mu}{\sqrt{f}}\begin{pmatrix} -\sqrt{\frac{\tilde{z}}{\tilde{w}}}\frac{w\tilde{w}}{\mu^2} & \sqrt{\frac{\tilde{w}}{\tilde{z}}}(1+\frac{z\tilde{z}}{\mu^2}) \\ -\sqrt{\frac{\tilde{z}}{\tilde{w}}} & \sqrt{\frac{\tilde{w}}{\tilde{z}}} \end{pmatrix}, \quad \tilde{g} = \frac{\mu}{\sqrt{f}}\begin{pmatrix} \sqrt{\frac{z}{w}} & -\sqrt{\frac{z}{w}}(1-\frac{w\tilde{w}}{\mu^2}) \\ \sqrt{\frac{w}{z}} & \sqrt{\frac{w}{z}}\frac{z\tilde{z}}{\mu^2} \end{pmatrix}.$$

The Yang matrix is

$$J = \frac{\mu^2}{f}\begin{pmatrix} \sqrt{\frac{w\tilde{w}}{z\tilde{z}}} & \Delta(\frac{f}{\mu^2}+1) \\ -\frac{1}{\Delta}(\frac{f}{\mu^2}-1) & \sqrt{\frac{z\tilde{z}}{w\tilde{w}}} \end{pmatrix}, \quad \text{where} \quad \Delta = \frac{\sqrt{z\tilde{z}w\tilde{w}}}{\mu^2}.$$

Note, that $\det J = 1$.

2.2.2 Birkhoff's Factorization Theorem

Let $F(\varphi)$ be a smooth complex-valued function on the unit circle $S^1 = \{\zeta = e^{i\varphi}\}$ in the complex ζ-plane. The Fourier series of F can be split into positive- and negative 'frequency-parts'

$$F = f - \tilde{f}, \quad f = \sum_0^\infty a_j\zeta^j, \quad \tilde{f} = \sum_0^\infty \tilde{a}_j\zeta^{-j}.$$

The positive frequency part f is the limit of a holomorphic function on the disk $|\zeta| < 1$ and the negative frequency part $\tilde{f}$ is the limit of a holomorphic function on the exterior $|\zeta| > 1$, including the point $\zeta = \infty$, where it is regular as a function of $\tilde{\zeta} = 1/\zeta$. This splitting of F into the difference of f and $\tilde{f}$ is unique, apart from the freedom to apportion the constant term in the Fourier series between f and $\tilde{f}$; that is up to $f \to f + c, \tilde{f} \to \tilde{f} + c$ for any complex number c.

Now we want to find the analogous splitting when F takes values in some complex Lie group. In the case of the multiplicative group $\{z|z \neq 0, \infty\}$, the problem is as follows: Given a smooth non-vanishing F on the unit circle, we must find smooth non-vanishing functions f and $\tilde{f}$ on $|\zeta| \leq 1$ and $|\zeta| \geq 1$, respectively, such that f is holomorphic for $|\zeta| < 1$, $\tilde{f}$ holomorphic for $|\zeta| > 1$ (including ∞) and

$$F = \tilde{f}^{-1}f \quad \text{on} \quad S^1.$$

If there is a solution, then

$$q(F) = \oint_{S^1}\frac{dF}{F} = \oint\frac{df}{f} - \oint\frac{d\tilde{f}}{\tilde{f}} = 0,$$

by Chauchy's theorem. We used that f and $\tilde{f}$ possess no zeroes in $|\zeta| \leq 1$ and $|\zeta| \geq 1$, respectively. Thus a factorization can only exist if the winding number q vanishes. In this case $\log F$ is single valued and we can split its Fourier series and then exponentiate.

For any $q(F) \in Z$ the function $\zeta^{-q}F$ has zero winding number and can therefore be factorized. Thus a non-vanishing smooth function on the circle can always be written as

$$F = \tilde{f}^{-1}\zeta^q f, \quad \text{where} \quad q = q(F)$$

and $f, \tilde{f}$ are non-vanishing holomorphic functions inside and outside of the unit circle.

This Birkhoff theorem has been extended to other Lie groups by Pressley and Segal (1986). Let us discuss the generalization to $GL(n, C)$. We use the following definitions:

The *loop group* $LGL(n, C)$ of $GL(n, C)$ is the group of smooth maps or loops

$$F : S^1 \longrightarrow GL(n, C)$$

under pointwise multiplication. The subsets of loops that are boundary values of holomorphic maps on

$$\{|\zeta| \leq 1\} \quad \text{and} \quad \{|\zeta| \geq 1\} \cup \{\infty\},$$

respectively, will be denoted by $LGL_+(n, C)$ and $LGL_-(n, C)$. $LGL(n, C)$ is an infinite-dimensional Lie group.

Birkhoff's Theorem: Any loop $F \in LGL(n, C)$ can be factorized

$$F = \tilde{f}^{-1}\Delta f$$

where $f \in LGL_+(n, C), \tilde{f} \in LGL_-(n, C)$ and $\Delta = \text{diag}(\zeta^{q_1}, \ldots \zeta^{q_n})$ for some integers q_i. These integers are unique up to permutations. For loops with $\Delta = 1$ the factorization is unique up to $f \to cf$ and $\tilde{f} \to c\tilde{f}$ for some constant $c \in GL(n, C)$.

The theorem holds true if we replace $GL(n, C)$ by $SL(n, C)$ (in which case $f, \tilde{f}$ and Δ are in $SL(n, C)$ and in particular $\sum q_i = 0$) and for polynomials in ζ and ζ^{-1} or rational functions of ζ instead of holomorphic functions.

Example 1: Let $w \in C$ and put

$$F = \begin{pmatrix} \zeta & w \\ 0 & \zeta^{-1} \end{pmatrix} \in SL(2, \mathbb{C}).$$

Then, whenever $w \neq 0$, we have the Birkhoff factorization

$$F = \tilde{f}^{-1} f \quad \text{where} \quad \tilde{f} = \begin{pmatrix} \zeta^{-1} & -w \\ w^{-1} & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 0 \\ w^{-1}\zeta & 1 \end{pmatrix}.$$

However, for $w = 0$ the factorization is $F = \tilde{f}^{-1}\Delta f$ with $\tilde{f} = f = 1$ and $\Delta = \text{diag}(\zeta, \zeta^{-1})$.

Example 2: Suppose that $F = CR$, where $C : C \to GL(n, C)$ is entire and R is a rational matrix-valued function of ζ. We shall consider the case where all poles of R and all zeros of $r = \det R$ lie inside of the unit circle. Then, in general, one can construct the factorization with $\Delta = 1$ explicitly.

Since $\Delta = 1$, the winding number of $\det F$ must vanish. Thus r must have an equal number of poles and zeros in the unit disk. So we assume that

$$r(\zeta) = \prod_1^q \frac{\zeta - \alpha_i}{\zeta - \beta_i},$$

where $|\alpha_i| < 1$ and $|\beta_i| < 1$, and that R is holomorphic except at the points β_i. Furthermore, we assume that, for each i

- $A_i = R(\alpha_i)$ has rank $n - 1$
- $B_i = \lim_{\zeta \to \beta_i} (\zeta - \beta_i) R(\beta_i)$ exists and has rank 1.

These holds for almost all choices of R. For each i, we choose $a_i, b_i \in C^n$ such that

$$a_i^t A_i = 0 \quad \text{and} \quad b_i \in \quad \text{Image of} \quad B_i.$$

The factorization is constructed by taking $\tilde{f}$ to be of the form

$$\tilde{f} = 1 + \sum_1^q \frac{x_i y_i^t}{\zeta - \alpha_i},$$

where $x_i, y_i \in C^n$. We must choose x_i and y_i so that $f = \tilde{f} C R$ is holomorphic everywhere inside the unit circle. For that we must have for each j that

$$y_j^t C(\alpha_j) A_j = 0, \qquad \Big(1 - \sum_{i=1}^q \frac{x_i y_i^t}{\beta_j - \alpha_i}\Big) C(\beta_j) B_j = 0.$$

These we can satisfy by putting $y_j^t = \alpha_j^t C^{-1}(\alpha_j)$ and by choosing the x_i so that

$$C(\beta_j) b_j + \sum_{i=1}^q x_i M_{ij} = 0,$$

where M is the $q \times q$ matrix

$$M_{ij} = \frac{\alpha_i^t C^{-1}(\alpha_i) C(\beta_j) b_j}{\beta_j - \alpha_i}.$$

We must make the further assumption that M is nonsingular. We have the freedom of rescaling the a_i and b_i; but this leaves $\tilde{f}$ unaltered. Thus f is uniquely determined by C and by the data consisting of the points α_i, β_i together with the one-dimensional subspaces of C^n spanned by the vectors a_i, b_i.

2.2.3 The zero-curvature condition revisited

The zero-curvature condition $[L, M]$ implies that the linear system

$$L\psi = 0, \quad M\psi = 0$$

can be integrated for each value of the spectral parameter ζ in $L = D_w - \zeta D_{\tilde{z}}$ and $M = D_z - \zeta D_{\tilde{w}}$. We can put together the n independent solutions to form the columns of a $n \times n$ matrix fundamental solutions f. The equations satisfied by the fundamental solution are

$$\begin{aligned} (\partial_w - iA_w)f - \zeta(\partial_{\tilde{z}} - iA_{\tilde{z}})f &= 0 \\ (\partial_z - iA_z)f - \zeta(\partial_{\tilde{w}} - iA_{\tilde{w}})f &= 0. \end{aligned} \tag{2.18}$$

The fundamental solution cannot, however, be regular (holomorphic with non-vanishing determinant) on the whole ζ-plane. If f were regular for all ζ, including $\zeta = \infty$, then, by Liouville's theorem, it would be independent of ζ. In that case (2.18) would imply, that

$$D_w f = D_z f = D_{\tilde{z}} f = D_{\tilde{w}} f = 0 \Longrightarrow F_{zw} f = \ldots = 0,$$

so that the connection would be flat. If f is a fundamental solution, then fH is one, if $M(fH) = (Mf)H + f(\partial_w - \zeta\partial_{\tilde{z}})H = 0$, and similarly for L, that is, if H is a regular solution of

$$\partial_w H - \zeta \partial_{\tilde{z}} H = 0, \qquad \partial_z H - \zeta \partial_{\tilde{w}} H = 0.$$

That is, H can be expressed as a function of

$$\lambda = \zeta w + \tilde{z}, \quad \mu = \zeta z + \tilde{w} \quad \text{and} \quad \zeta.$$

When the connection is not flat, then it is impossible to choose f so that it is regular at $\zeta = \infty$ as well as for finite values of ζ. We can, however, find another fundamental solution $\tilde{f}$ which is holomorphic in ζ on the whole Riemann sphere, except at $\zeta = 0$, by setting $\tilde{\zeta} = 1/\zeta$ and solving the linear system in the form

$$\begin{aligned} \tilde{\zeta}(\partial_w - iA_w)f - (\partial_{\tilde{z}} - iA_{\tilde{z}})f &= 0 \\ \tilde{\zeta}(\partial_z - iA_z)f - (\partial_{\tilde{w}} - iA_{\tilde{w}})f &= 0. \end{aligned} \tag{2.19}$$

The solution is unique, up to

$$\tilde{F} \longrightarrow \tilde{F}\tilde{H} \quad \text{where} \quad \tilde{H} = \tilde{H}(w + \tilde{\zeta}\tilde{z}, z + \tilde{\zeta}\tilde{w}, \tilde{\zeta}).$$

The patching matrix:

We shall denote by $V, \tilde{V}$ a two-set cover of the Riemann sphere, such that V is contained in the complement of $\zeta = \infty$ and $\tilde{V}$ is contained in the complement of $\zeta = 0$. In the overlap $V \cap \tilde{V}$ of the domains of f and $\tilde{f}$ we have

$$f = \tilde{f}F,$$

where F satisfies

$$\partial_w F - \zeta \partial_{\tilde{z}} F = 0, \qquad \partial_z F - \zeta \partial_{\tilde{w}} F = 0. \tag{2.20}$$

F is the patching matrix associated with A. It is determined by A up to the equivalence

$$F \sim \tilde{H}^{-1} F H$$

where $\tilde{H}$ is regular on $\tilde{V}$ and H regular on V. The equivalence classes are the patching data of A. When F is in the class of the identity function, that is when F can be factorized in the form

$$F = \tilde{H}^{-1} H,$$

with $H, \tilde{H}$ regular in $V, \tilde{V}$, respectively, we have a fundamental solution $fH = \tilde{f}\tilde{H}$ which is global in ζ. Then the curvature vanishes. When such a factorization does not exist, the curvature is nonzero. Actually, the ASDYM-field can be recovered from F. The map that assigns the patching data to an $ASDYM$ field is the *forward Penrose transform.*

The reverse Penrose transform.

For each fixed $(z, w, \tilde{z}, \tilde{w})$ we have the Birkhoff factorization

$$F(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{f}^{-1} f.$$

From $Lf = L\tilde{f} = 0$ we have

$$A_w - \zeta A_{\tilde{z}} = -i\big(\partial_w f - \zeta \partial_{\tilde{z}} f\big) f^{-1} = -i\big(\partial_w \tilde{f} - \zeta \partial_{\tilde{z}} \tilde{f}\big) \tilde{f}^{-1},$$

together with a analogous formula following from $Mf = M\tilde{f} = 0$. By the uniqueness statement of the Birkhoff theorem, any other factorization must be given by

$$f' = gf \quad \text{and} \quad \tilde{f}' = g\tilde{f}$$

with ζ-independent g. The potentials belonging to $f', \tilde{f}'$ are just the gauge transform of A. Thus F determines the gauge potential up to a gauge transformation.

Now we start with a given $F(\lambda, \mu, \zeta)$ on the annulus $V \cap \tilde{V}$. Applying Birkhoffs theorem for each space-time point, we can factorize F in the form

$$F(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{f}^{-1} \Delta f,$$

where $f(z, w, \tilde{z}, \tilde{w}, \zeta)$ is regular for $|\zeta| \leq 1$, $\tilde{f}$ is regular for $|\zeta| \geq 1$ and $\Delta = \mathrm{diag}(\zeta^{q_1}, \ldots, \zeta^{q_N})$ for some integers q_i which may jump at sub-manifolds of spacetime.

Let us assume, that F is chosen such that $\Delta = 1$ at some point of space time. Then $\Delta = 1$ in an open set U containing this point. Now we show, that such a F is a patching matrix associated with some solution of the ASDYM-equations on U. Since $(\partial_w - \zeta \partial_{\tilde{z}}) F = 0$ we have

$$\tilde{f}^{-1}(\partial_w - \zeta \partial_{\tilde{z}}) f) - \tilde{f}^{-1}\Big((\partial_w - \zeta \partial_{\tilde{z}}) \tilde{f}\Big) \tilde{f}^{-1} f = 0$$

or that

$$(\partial_w f - \zeta \partial_{\tilde{z}} f) f^{-1} = (\partial_w \tilde{f} - \zeta \partial_{\tilde{z}} \tilde{f}) \tilde{f}^{-1}$$

at each point in U, for all ζ in some neighborhood of the unit circle. The left hand side is holomorphic inside and the right hand side holomorphic outside, except for a simple pole at infinity. It follows from the Liouville theorem, that both sides must be of the form

$$(\partial_w f - \zeta \partial_{\tilde{z}} f) f^{-1} = (\partial_w \tilde{f} - \zeta \partial_{\tilde{z}} \tilde{f}) \tilde{f}^{-1} = i(A_w - \zeta A_{\tilde{z}}).$$

Similarly, one concludes, that

$$(\partial_z f - \zeta \partial_{\tilde{w}} f) f^{-1} = (\partial_z \tilde{f} - \zeta \partial_{\tilde{w}} \tilde{f}) \tilde{f}^{-1} = i(A_z - \zeta A_{\tilde{w}}).$$

We then have

$$D_w f - \zeta D_{\tilde{z}} f = 0 \quad \text{and} \quad D_z f - \zeta D_{\tilde{w}} f = 0.$$

It follows, that the linear system associated with D is integrable and hence A is ASD. A can be recovered from the patching matrix F and is a solution of the ASDYM equation on an open subset of space-time.

Lemma: The gauge potential is given in terms of f and $\tilde{f}$ by

$$\boxed{iA = \partial f(0) f^{-1}(0) + \tilde{\partial} \tilde{f}(\infty) \tilde{f}^{-1}(\infty),} \tag{2.21}$$

where $f(0) = f(\zeta = 0)$ etc. The proof is simple. Just set $\zeta = 0$ in (2.18) and $\tilde{\zeta} = 0$ in (2.19). Comparing (2.21) with

$$A = ig^{-1}\partial g + i\tilde{g}\tilde{\partial}\tilde{g}^{-1}$$

which leads to the Yang-equation, we see, that we may identify

$$g = f^{-1}(0), \quad \tilde{g} = \tilde{f}(\infty) \Longrightarrow J = f^{-1}(0)\tilde{f}(\infty).$$

The Atiyah-Ward ansatz: Consider the patching matrix

$$F = \begin{pmatrix} \zeta & \gamma \\ 0 & \zeta^{-1} \end{pmatrix}$$

where γ is a holomorphic function on the annulus. Again we put $\lambda = \zeta w + \tilde{z}$ and $\mu = \zeta z + \tilde{w}$ and expand γ in a Laurent series in ζ:

$$\gamma = \sum_{-\infty}^{\infty} \gamma_i \zeta^i = \gamma_+ + \phi + \gamma_-$$

where we have split γ into a positive frequency part, a ζ-independent part and a negative frequency part. Now (2.20) implies the recursion relation

$$\partial_w \gamma_i = \partial_{\tilde{z}} \gamma_{i-1} \quad \text{and} \quad \partial_z \gamma_i = \partial_{\tilde{w}} \gamma_{i-1}.$$

Taking the $\tilde{w}$-derivative of the first equation, exchanging the two derivatives and using the second equation, one obtains

$$\partial_{\tilde{w}} \partial_w \gamma_i = \partial_{\tilde{w}} \partial_{\tilde{z}} \gamma_{i-1} = \partial_z \partial_{\tilde{z}} \gamma_i,$$

so that each γ_i obeys the scalar wave equation. The Birkhoff factorization is $F = \tilde{f}^{-1} f$ where

$$f = \frac{1}{\sqrt{\phi}} \begin{pmatrix} \zeta & \phi + \gamma_+ \\ -1 & -\zeta^{-1}\gamma_+ \end{pmatrix}, \quad \tilde{f} = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 1 & -\zeta\gamma_- \\ -\zeta^{-1} & \phi + \gamma_- \end{pmatrix}.$$

The factorization is non-degenerate whenever $\phi \neq 0$ (when $\phi = 0$ then we must take $\Delta = \text{diag}(\zeta, \zeta^{-1})$) From the above lemma and the recursions relations, we have

$$A = \frac{i}{2\phi} \begin{pmatrix} \tilde{\partial}\phi - \partial\phi & 2(\phi_z d\tilde{w} + \phi_w d\tilde{z}) \\ 2(\phi_{\tilde{z}} dw + \phi_{\tilde{w}} dz) & \partial\phi - \tilde{\partial}\phi \end{pmatrix}.$$

With

$$f(0) = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 0 & \phi \\ -1 & -\gamma_1 \end{pmatrix} \quad \text{and} \quad \tilde{f}(\infty) = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 1 & -\gamma_{-1} \\ 0 & \phi \end{pmatrix}$$

one finds the Yang-matrix

$$J = \frac{1}{\phi} \begin{pmatrix} -\gamma_1 & \gamma_1\gamma_{-1} - \phi^2 \\ 1 & -\gamma_{-1} \end{pmatrix}.$$

The Yang equation reads

$$\begin{aligned} 0 &= \partial_{\tilde{z}}\left(\frac{\partial_z \phi}{\phi}\right) - \frac{\partial_{\tilde{z}}\gamma_{-1}\partial_z\gamma_1}{\phi^2} - \partial_{\tilde{w}}\left(\frac{\partial_w \phi}{\phi}\right) + \frac{\partial_{\tilde{w}}\gamma_{-1}\partial_w\gamma_1}{\phi^2} \\ 0 &= \partial_{\tilde{z}}\left(\frac{\partial_z\gamma_1}{\phi^2}\right) - \partial_{\tilde{w}}\left(\frac{\partial_w\gamma_1}{\phi^2}\right) \\ 0 &= \partial_{\tilde{z}}\partial_z\gamma_{-1} - \partial_{\tilde{w}}\partial_w\gamma_{-1}. \end{aligned}$$

The last equation is just the wave equation for γ_{-1} Using

$$\partial_z\gamma_1 = \partial_{\tilde{w}}\phi \quad \text{and} \quad \partial_w\gamma_1 = \partial_{\tilde{z}}\phi \tag{2.22}$$

the middle equation becomes

$$(\partial_{\tilde{z}}\partial_{\tilde{w}} - \partial_{\tilde{w}}\partial_{\tilde{z}}) \log\phi = 0$$

which is also fulfilled. The last equation reads

$$\frac{\Box\phi}{\phi} - \frac{1}{\phi^2}\Big(\partial_{\tilde{z}}\phi\partial_z\phi + \partial_{\tilde{z}}\gamma_{-1}\partial_z\gamma_1 - \partial_{\tilde{w}}\phi\partial_w\phi - \partial_{\tilde{w}}\gamma_{-1}\partial_w\gamma_1\Big) = 0.$$

Using (2.22) together with

$$\partial_{\tilde{z}}\gamma_{-1} = \partial_w\phi \quad \text{and} \quad \partial_{\tilde{w}}\gamma_{-1} = \partial_z\phi \tag{2.23}$$

this equation is also fulfilled, since ϕ must obey the wave equation. Thus we have explicitly checked, that the Atiyah-Ward ansatz for the patching matrix and the corresponding f and $\tilde{f}$ given by the Birkhoff theorem yield a selfdual solution of the Yang-Mills-equation with charge -1.

2.2.4 Instantons on the euclidean torus

The Abelian gauge potentials

$$A_z = \frac{\pi}{2iV_{01}}(z-\bar{z})H_1, \qquad A_w = \frac{\pi}{2iV_{23}}(w-\bar{w})H_3 \tag{2.24}$$

are anti-selfdual on the euclidean torus, if

$$\frac{H_1}{V_{01}} + \frac{H_3}{V_{23}} = 0$$

holds. In this case

$$\begin{aligned} D_z &= \partial_z - \frac{\pi}{2V_{01}}(z-\bar{z})H_1 \quad , \quad D_w = \partial_w + \frac{\pi}{2V_{01}}(w-\bar{w})H_1 \\ D_{\bar{z}} &= \partial_{\bar{z}} - \frac{\pi}{2V_{01}}(z-\bar{z})H_1 \quad , \quad D_{\bar{w}} = \partial_{\bar{w}} + \frac{\pi}{2V_{01}}(w-\bar{w})H_1. \end{aligned}$$

Two solutions of $Lf = Mf = 0$ with the correct holomorphic properties are

$$\begin{aligned} f &= e^{A(z^2-w^2+\bar{w}^2-\bar{z}^2)}e^{-2A(\lambda z+\mu w)} \\ \tilde{f} &= e^{A(z^2-w^2+\bar{w}^2-\bar{z}^2)}e^{-2A(\lambda\bar{w}-\mu\bar{z})/\zeta}, \end{aligned}$$

where $A = \pi H_1/4V_{01}$. Recall, that $\mu = \zeta z - \bar{w}$ and $\lambda = \zeta w + \bar{z}$. Clearly, f is holomorphic in a neighborhood of $\zeta = 0$ and $\tilde{f}$ in a neighborhood of $\zeta = \infty$. The two regions have an overlap which contains the unit circle $|\zeta| = 1$. The patching matrix has the simple form

$$F = \tilde{f}^{-1}f = e^{-A\mu\lambda/\zeta} \tag{2.25}$$

and is a function of λ, μ and ζ and is holomorphic on the annulus containing the unit circle, as required by the general theory. Using

$$f(0) = e^{A(z^2-2z\bar{z}-\bar{z})}e^{A(\bar{w}^2+2w\bar{w}-w^2)} \quad , \quad \tilde{f}(\infty) = e^{A(z^2+2z\bar{z}-\bar{z}^2)}e^{A(\bar{w}^2-2w\bar{w}-w^2)}$$

in (2.21) one immediately reconstructs the Abelian gauge potential A in (2.24). Actually, there is a simpler factorization of (2.25), namely by

$$f = e^{-2A(\lambda z+\mu w)} \quad \text{and} \quad \tilde{f} = e^{-2A(\lambda\bar{w}-\mu\bar{z})/\zeta}.$$

The corresponding gauge potential is

$$A = \frac{\pi H_1}{2iV_{01}}\Big(-\bar{z}dz + zd\bar{z} + \bar{w}dw - wd\bar{w}\Big)$$

and is gauge equivalent to A in (2.24) by the non-periodic gauge transformation

$$e^{A(z^2-w^2+\bar{w}^2-\bar{z}^2)}.$$