Selected Topics of (Quantum) Field Theory

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Kapitel 1

Classical Field Theories

1.1 Introduction and Notation

It is now accepted that *Quantum Chromodynamics* (QCD) is the correct theory of strong interaction. The strong nuclear forces are the 'van der Waals'- forces of the interaction between quarks and gluons. The structure of the interaction (e.g. the Feynman rules) comes from the Lagrangian density, which at the classical level is

$$\mathcal{L}_{QCD} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{f=1}^{N_f}\bar{\psi}^i_f\big(\gamma^\mu D^{ij}_\mu - im_f\delta^{ij}\big)\psi^j_f + \frac{\theta}{16\pi^2}\eta^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta}. \tag{1.1}$$

The implicit sum over the *colour-indices* a, i, j and *Lorentz-indices* μ, ν, α, β is assumed. We have used the totally antisymmetric tensor

$$\eta_{\mu\nu\alpha\beta} = e\,\epsilon_{\mu\nu\alpha\beta} \Longrightarrow \eta_{0123} = e, \quad \eta^{0123} = \frac{\text{sign}(g)}{e}, \quad e = \sqrt{|g|},$$

where g and $\text{sign}(g)$ are the determinant and signature of the metric $g_{\mu\nu}$, e is the determinant of the vierbein and $\epsilon_{\mu\nu\alpha\beta}$ the totally antisymmetric symbol with $\epsilon_{0123} = 1$. We write the expressions such that they also hold in curved space times of an arbitrary signature. This way the transition to Euclidean spacetime is almost evident. The fermionic part is the tricky one.

To find the correct expression for the Euclidean action one may use the fact, that for finite temperature the path integral is automatically the Euclidean one!! Thus the Euclidean Lagrangian is automatically gotten if one (formally) represents the partition function

$$Z = \text{tr}\; e^{-\beta H_{QCD}}$$

by a functional integral[1]

[1] for the following discussion the gauge fixings and ghost-contributions are irrelevant

$$Z = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{-S_E[A,\psi,\psi^\dagger]}. \tag{1.2}$$

The result of the analysis in flat spaces is the following: the Euclidean coordinates, derivative, gamma-matrices and fields are related to the Minkowskian one as follows:

$$\begin{aligned} \Big(x^0, \partial_0, x^i, \partial_i, \gamma^0, \gamma^i\Big)_M &= \Big(-ix^0, i\partial_0, x^i, \partial_i, \gamma^0, i\gamma^i\Big)_E \\ \Big(\psi, \bar\psi, A_0, A_i, F_{0i}, F^{0i}, F_{ij}, F^{ij}\Big)_M &= \Big(\psi, \psi^\dagger, iA_0, A_i, iF_{0i}, -iF^{0i}, F_{ij}, F^{ij}\Big)_E. \end{aligned}$$

As a result of these replacements, the action in Minkowski spacetime, $S_M = \int \eta\, \mathcal{L}$ is to be replaced by iS_E with Euclidean action

$$S_E = \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} - \sum_{f=1}^{N_f} \psi_f^{\dagger i}\big(i\gamma^\mu D^{ij}_\mu - im_f\delta^{ij}\big)\psi^j_f - i\frac{\theta}{16\pi^2}\eta^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta}. \tag{1.3}$$

The indices are raises and lowered with the metric tensor, e.g.

$$F^{\mu\nu}_a = g^{\mu\alpha} g^{\nu\beta} F_{a\alpha\beta}.$$

The gauge- and general covariant derivative of the quark-fields is

$$D_\mu = \partial_\mu + i\omega_\mu - iA_\mu \quad \text{or} \quad D^{ij}_\mu = (\partial_\mu + i\omega_\mu)\delta^{ij} - iA^a_\mu (T_a)^{ij}, \tag{1.4}$$

where ω_μ is the *spin-connection*, which will be discussed below. The commutators of two covariant derivatives yield the components of the Yang-Mills field strength and the 'curvature' of spacetime,

$$[D_\mu, D_\nu] = -iF_{\mu\nu} + IE_{\mu\nu},$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = F^a_{\mu\nu} T_a \\ R_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + i[\omega_\mu, \omega_\nu]. \end{aligned} \tag{1.5}$$

The generators T^a of the colour-symmetry are hermitian, normalized according to $\mathrm{tr}\, T_a T_b = 2\delta_{ab}$ and have real and antisymmetric structure constants,

$$[T^a, T^b] = if^{abc} T_c,$$

such that

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu.$$

We shall discuss the curvature term more carefully later in this chapter.

QCD itself is a theory for six flavored quarks (up, down, strange, charm, bottom and top) in the fundamental representation of the colour group $SU(3)$ that interact strongly with the octet of gluons. The strength of this interaction is described by the dimensionless coupling constant[2] g. Most of the nontrivial properties of QCD result from the three- and four-gluon interaction in F^2 and the, compared to QED, big coupling constant.

In these lectures we shall not confine ourselves to the colour group $SU(3)$ but allow for *other gauge groups* as well. This way we may also include the electroweak theory or GUT-theories in our discussion. For $SU(2)$ we may choose $T^a = \sigma^a$ and for $SU(3)$ the 3×3 Gell-Mann matrices λ^a. Also, the fermions may not necessarily be in the fundamental representation but transform according to a arbitrary representation $U(g), g \in G$ of the gauge group G. The Lagrangian for a general theory reads then

$$\begin{aligned}\mathcal{L}_{gauge} = & - \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(\not{D} - im)\psi + \frac{\theta}{16\pi^2} \eta^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta} \\ & + \text{ Higgs and Yukawa terms,}\end{aligned} \tag{1.6}$$

where

$$\not{D} = \gamma^\mu D_\mu, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad \text{and} \quad D_\mu \psi = \big(\partial_\mu + i\omega_\mu - iU_*(A_\mu)\big)\psi.$$

The θ-Term in (1.1,1.6) is odd under time reversal and thus breaks CP by the CPT-theorem. The *strong CP-problem* in QCD is still a theoretically debated issue. Because of the very small electric dipole moment of the neutron one concludes that the θ-term is negligible, $\theta < 10^{-10}$. The θ-term is a *total derivative* and does not enter the field equations and in particular the Yang-Mills equations. But it has consequences in the quantized theories. Its understanding will lead us to study *instantons*. Since instantons are one of the main topics of these lectures we shall now discuss this term in detail.

To prove, that the θ-term is a total differential, we use the *exterior calculus*. We shall use these calculus in these lectures again and again and thus recall some important formulas. More will come later. Let

$$\alpha = \frac{1}{p!}\, \alpha_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$$

be a p-form. The components $\alpha_{\mu_1 \dots \mu_p}$ of a p-form is an antisymmetric tensor-field. The exterior differential of α is the $p+1$-form

$$d\alpha = \frac{1}{p!} \partial_\nu \alpha_{\mu_1 \dots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$$

and the wedge-product of a p-form and a q-form is a $p+q$-form:

$$\alpha \wedge \beta = \frac{1}{p!q!} \alpha_{\mu_1 \dots \mu_p} \beta_{\nu_1 \dots \nu_q} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_q}.$$

One easily proves, that d is nilpotent, $d^2 = 0$, and that

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \qquad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

[2] unfortunately the symbol g is used for the coupling constant, the determinant of the metric and for elements of the gauge group. The local meaning should follow from the context

$d\alpha$ is the generalization of the rotation of a vector field in 3 dimensions. A form α is called *closed* if $d\alpha = 0$, it is called *exact* if $\alpha = d\beta$. Locally every closed form is exact. Also, there is a generalization of the well-known Stoke-theorem, namely:

Theorem: Let M be a d-dimensional, orientable and differentiable manifold and let $D \subset M$ be a subset of M with smooth boundary ∂D and compact closure $\bar{D}$. For every $d-1$-form α we have

$$\boxed{\int_D d\alpha = \int_{\partial D} \alpha.} \tag{1.7}$$

An integral over an exact form can be converted into a surface integral. We shall need this important result when we discuss the quantization of the instanton-number.

Actually there is a generalization of this theorem to p forms and this generalization is needed when one studies the de Rham cohomology. Is α_{p-1} a $(p-1)$-form and C_p a p-dimensional submanifold (a p-simplex), then the generalization reads

$$\int_{\partial C_p} \alpha_{p-1} = \int_{C_p} d\alpha_{p-1}. \tag{1.8}$$

In particular the gauge-potential and field-strength are Lie algebra-valued 1- and 2-forms, respectively:

$$A = A^a_\mu T_a dx^\mu = A_\mu dx^\mu \quad \text{and} \quad F = \frac{1}{2} F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

Let us see, how the 2-form F is related to the 1-form A. To see that, we calculate

$$\begin{aligned} dA &= \partial_\nu A_\mu dx^\nu \wedge dx^\mu = \frac{1}{2}\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big) dx^\mu \wedge dx^\nu \\ A \wedge A &= A_\mu A_\nu dx^\mu \wedge dx^\nu = \frac{1}{2} A^a_\mu A^b_\nu \,[T_a, T_b] dx^\mu \wedge dx^\nu = \frac{1}{2}[A_\mu, A_\nu] dx^\mu \wedge dx^\nu \end{aligned}$$

from which immediately follows, that

$$dA - iA \wedge A = \frac{1}{2}\Big(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\Big) dx^\mu dx^\nu = F, \tag{1.9}$$

where F is the field strength two-form.

The θ-term is easily expressed in terms of F:

$$\begin{aligned} \operatorname{tr} F \wedge F &= \frac{1}{4} F^a_{\mu\nu} F^b_{\alpha\beta} \operatorname{tr}(T_a T_b)\, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ &= \frac{1}{2} \operatorname{sign}(g) \eta^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta}\, \eta, \end{aligned}$$

where $\eta = e\, dx^0 \wedge \dots dx^3$ is the volume form and we used

$$\begin{aligned} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta &= \operatorname{sign}(g) \epsilon^{\mu\nu\alpha\beta} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= \operatorname{sign}(g) \eta^{\mu\nu\alpha\beta}\, \eta. \end{aligned}$$

The θ-term in the action

$$S = \int \eta \, \mathcal{L},$$

belonging to the last term in (1.1), reads

$$S_\theta = \frac{\theta}{16\pi^2} \int \eta \, \eta^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta} = \mathrm{sign}(g) \frac{\theta}{8\pi^2} \int \mathrm{tr}\, F \wedge F.$$

Now we claim, that

$$\boxed{\mathrm{tr}\, F \wedge F = d\omega_3, \quad \omega_3 = \mathrm{tr}\, \Big(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A \Big),} \tag{1.10}$$

or that S_θ is a surface term. To prove that, we first note that $\mathrm{tr}\,(A^4)$ vanishes[3]. This follows from

$$\mathrm{tr}\,(AAAA) = -\mathrm{tr}\,(AAAA),$$

where we used the cyclicity of the trace, and that $A^a \wedge A^b = -A^b \wedge A^a$. Now we calculate

$$\begin{aligned} d\omega_3 &= \mathrm{tr}\, \big\{ dAdA - \frac{2i}{3}(dAA^2 - AdAA + A^2 dA) \big\} \\ &= \mathrm{tr}\, \big\{ dAdA - \frac{2i}{3}(dAA^2 + \frac{1}{2} A^2 dA + \frac{1}{2} dAA^2 + A^2 dA) \big\}. \end{aligned}$$

We subtract $\mathrm{tr}\, A^4 = 0$ and end up with

$$d\omega_3 = \mathrm{tr}\, \big(dAdA - idAA^2 - iA^2 dA - A^4 \big) = \mathrm{tr}\, (dA - iA^2)^2 = \mathrm{tr}\, F^2,$$

as was claimed.

To rewrite the action in terms of forms we introduce the dual ${}^*\alpha$ of a p-form α, which is a $(d-p)$-form, by

$$\begin{aligned} {}^*\alpha &= \frac{1}{(d-p)!} {}^*\alpha_{\mu_{p+1}\dots\mu_d} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_d}, \quad \text{where} \\ {}^*\alpha_{\mu_{p+1}\dots\mu_d} &= \frac{1}{p!} \eta_{\mu_1 \dots \mu_d} \, \alpha^{\mu_1 \dots \mu_p}. \end{aligned}$$

The star-operation is, up to a sign, idempotent. For a p-form one shows, that

$${}^*({}^*\alpha) = (-1)^{p(d-p)} \mathrm{sign}(g) \alpha,$$

where in Euclidean spacetimes $\mathrm{sign}(g) = 1$ and in Lorentzian spacetimes -1. We shall need the dual of the field strength, which is

$${}^*F = \frac{1}{2} {}^*F_{\alpha\beta} \, dx^\alpha \wedge dx^\beta, \quad \text{where} \quad {}^*F_{\alpha\beta} = \frac{1}{2} \eta_{\sigma\rho\alpha\beta} \, F^{\sigma\rho} = \frac{1}{2} \eta_{\alpha\beta\sigma\rho} \, F^{\sigma\rho}.$$

[3] we abbreviate $A \wedge A \wedge A \wedge A$ by $AAAA = A^4$.

Note that ${}^*({}^*F) = F$ in Euclidean spacetimes with signatures $(+,+,+,+)$ and ultra-hyperbolic spacetimes with signature $(+,+,-,-,)$, whereas ${}^*({}^*F) = -F$ in Lorentzian spacetimes. The (anti)selfduality conditions

$$ {}^*F = \alpha F \Longrightarrow \pm F = {}^*\,({}^*F) = \alpha\, {}^*F = \alpha^2 F $$

requires, that $\alpha = \pm 1$ for $\mathrm{sign}(g) = 1$ and $\alpha = \pm i$ for $\mathrm{sign}(g) = -1$. There are no real (anti)selfdual field strength on Lorentzian manifolds. But circular polarized light (which maybe described by complex fields) is selfdual.

The product of ${}^*F$ and F is part of the Yang-Mills action:

$$ \begin{aligned} \mathrm{tr}\, {}^*F \wedge F &= \frac{1}{8}\, \eta_{\alpha\beta\sigma\rho} \mathrm{tr}\,(F^{\sigma\rho} F_{\mu\nu}) dx^\alpha dx^\beta dx^\mu dx^\nu \\ &= \mathrm{sign}(g) \frac{1}{8}\, \eta_{\alpha\beta\sigma\rho} \epsilon^{\alpha\beta\mu\nu} \eta \;\mathrm{tr}\, \left(F^{\sigma\rho} F_{\mu\nu}\right). \end{aligned} $$

Using that $\eta_{\alpha\beta\sigma\rho} = e\, \epsilon_{\alpha\beta\sigma\rho}$ and that $\epsilon_{\alpha\beta\sigma\rho}\, \epsilon^{\alpha\beta\mu\nu} = 2\, \mathrm{sign}(g)(\delta^\mu_\sigma \delta^\nu_\rho - \delta^\nu_\sigma \delta^\mu_\rho)$, this yields

$$ \mathrm{tr}\, {}^*F \wedge F = \frac{e}{2} \mathrm{tr}\, \left(F^{\mu\nu} F_{\mu\nu}\right). \tag{1.11} $$

Using this result, the action $S = \int \eta\, \mathcal{L}$ in a Riemannian spacetime acquires the following form, up to Yukawa terms and terms containing a possible Higgs field:

$$ \boxed{S_E = \frac{1}{2g^2} \int {}^*F \wedge F - \int \eta\, \psi^\dagger (i\not{D} - im)\psi - \frac{i\theta}{8\pi^2} \int \mathrm{tr}\, F \wedge F,} \tag{1.12} $$

In Lorentzian manifolds it reads

$$ \boxed{S_M = -\frac{1}{2g^2} \int {}^*F \wedge F + \int \eta\, \bar{\psi} (\not{D} - im)\psi + \frac{\theta}{8\pi^2} \int \mathrm{tr}\, F \wedge F.} \tag{1.13} $$

We have seen, that the θ-term is a surface term,

$$ \int \mathrm{tr}\, F \wedge F = \oint \omega_3, $$

and does not affect the classical dynamics. We recall, that

$$ F = dA - iA \wedge A, \quad \not{D} = \gamma^\mu D_\mu, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad D_\mu = \partial_\mu + i\omega_\mu - iU_*(A_\mu). $$

In Euclidean spacetime the γ^μ are hermitian[4]. Next we discuss the various symmetries of this action or pieces of this action.

[4]In Minkowskian spacetime not all γ^μ can be hermitian. This would contradict the anti-commutation relations.

1.2 Symmetries

The action has several symmetries, namely local gauge symmetry, global chiral invariance if the fermions are massless, local Lorentz-invariance, diffeomorphism-invariance and sometimes Weyl-invariance. We shall now discuss these symmetries and some consequences, like the conformal symmetry, in turn.

1.2.1 Gauge symmetry

Under the local gauge transformations

$$A \longrightarrow A^g = gAg^{-1} + igdg^{-1} \quad \text{and} \quad \psi \longrightarrow \psi^g = U(g)\psi, \quad \psi^\dagger \longrightarrow \psi^\dagger U^\dagger(g)$$

is the action invariant. Indeed, the field strength and covariant derivative transform according to

$$F(A^g) = gF(A)g^{-1} \quad \text{and} \quad d + i\omega - iU_*(A^g) = U\big(d + i\omega - iU_*(A)\big)U^{-1},$$

where $U = U(g)$ is the representation of the gauge group according to which the fermions transform. For example, in QCD the quarks transform according to the fundamental representation and $U(g) = g$. However, the right-handed quarks have vanishing weak isospin and thus do not transform under the weak $SU(2)$. Hence $U(g) = e$ in this case.

The bits entering the action transform as

$$\begin{array}{ccc} F \wedge F \to g(F \wedge F)g^{-1} & , & {}^*F \wedge F \to g({}^*F \wedge F)g^{-1} \\ \psi^\dagger(i\not{D} - im)\psi & \longrightarrow & \psi^\dagger(i\not{D} - im)\psi. \end{array}$$

Because of the trace-operation the Lagrangian and the action are indeed invariant under local gauge transformation.

1.2.2 Chiral symmetry

Since the hermitian $\gamma_5 = c\gamma^0\gamma^1\gamma^2\gamma^3$[5] anti-commutes with the γ^μ and commutes with the spin-connection, and since

$$\gamma_5^2 = 1 \Longrightarrow e^{\alpha\gamma_5} = \cosh(\alpha) + \gamma_5 \sinh(\alpha) \Longrightarrow \gamma^\mu e^{\alpha\gamma_5} = e^{-\alpha\gamma_5}\gamma^\mu$$

the Dirac term in the Euclidean action transforms under a global chiral transformation

$$\psi \to e^{\alpha\gamma_5} \quad \text{and} \quad \psi^\dagger \to \psi^\dagger e^{\alpha\gamma_5}, \quad \alpha \quad \text{reel,} \tag{1.14}$$

as

$$\psi^\dagger(i\not{D} - im)\psi \longrightarrow \psi^\dagger(i\not{D} - ime^{2\alpha\gamma_5})\psi.$$

[5] $c = 1$ in Euclidean and $c = i$ in Minkowskian spacetime, the γ's here are the one in flat spacetime.

It follows, that (1.14) is a global classical symmetry if the fermions are massless. For this reason the limit $m \to 0$ is called the *chiral limit.* We expect that this classical symmetry is spontaneously broken in QCD. We shall come back to this breaking later on.

We remark, that in Minkowskian spacetime the chiral transformations are

$$\psi \to e^{i\alpha\gamma_5} \quad \text{and} \quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}, \quad \alpha \quad \text{reel and} \quad \gamma_5 = \gamma_5^\dagger.$$

This is required by the anti-commutation relations for the Fermi fields or that $\psi^\dagger$ in $\bar{\psi} = \psi^\dagger \gamma^0$ is the adjoint of ψ.

1.2.3 Local Lorentz-invariance

In the second part of the lectures we shall investigate quantum fields in external gravitational fields. As a preparation we discuss the important conformal invariance of gauge theories, we now investigate the coupling of matter fields to gravity. For bosons this is rather easy, at least if we implement the equivalence principle by minimally coupling the bosons to gravity. Let us first recall some important formulas from (pseudo)Riemannian geometry.

The metric tensor in the line-element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

determines the geometry of a (pseudo)Riemannian manifold. The Levi-Civita connection is given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \Big(g_{\beta\mu},_\nu + g_{\beta\nu},_\mu - g_{\mu\nu},_\beta \Big)$$

and enters the covariant derivative of tensor fields

$$T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q;\alpha} = T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q,\alpha} + \Gamma^{\mu_1}_{\alpha\beta} T^{\beta \dots \mu_p}_{\nu_1 \dots \nu_q} + \dots - \Gamma^\beta_{\alpha\nu_1} T^{\mu_1 \dots \mu_p}_{\beta \dots \nu_q} - \dots .$$

The coordinate expression of the Riemann tensor is

$$R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\nu\beta},_\mu - \Gamma^\alpha_{\mu\beta},_\nu + \Gamma^\alpha_{\mu\sigma} \Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma} \Gamma^\sigma_{\mu\beta}.$$

The 'diffeomorphism'-connection 1-form and the curvature 2-form by

$$\Gamma^\alpha_{\ \beta} = \Gamma^\alpha_{\mu\beta} dx^\mu \quad \text{and} \quad \Omega^\alpha_{\ \beta} = \frac{1}{2} R^\alpha_{\beta\mu\nu} dx^\mu \wedge dx^\nu,$$

where

$$\Omega^\alpha_{\ \beta} = d\Gamma^\alpha_{\ \beta} + \Gamma^\alpha_{\ \sigma} \wedge \Gamma^\sigma_{\ \beta}. \tag{1.15}$$

The Ricci tensor and Ricci scalar are

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \quad \text{and} \quad R = g^{\mu\nu} R_{\mu\nu}.$$

For the various symmetry properties of the Riemann tensor I refer you to the extensive literature on general relativity.

For the gauge bosons we have already accomplished the minimal coupling. Let us repeat: the difference to flat space is, that in $F^{\mu\nu}F_{\mu\nu}$ the indices are lowered and raised with the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. The action is the integral over the Lagrangian density, where one integrates with the invariant measure η which contains the determinant g of the metric.

For spin-zero fields the generally covariant derivative is just the ordinary derivative, so that

$$S_\phi = \frac{1}{2}\int \eta\Big(g^{\mu\nu}D_\mu\phi D_\nu\phi \pm V(\phi) + \xi R\phi^2\Big), \tag{1.16}$$

where $D = d - iU_*(A)$ and $V(\phi)$ is the Higgs-potential containing a possible mass term and self-interaction for the scalar field. The last term containing the Ricci-scalar R violates the equivalence principle. It is an additional renormalizable term which is sometimes added to improve the conformal properties of the theory. Note, that after a partial integration this action reads

$$S_\phi = \frac{1}{2}\int \eta\Big(- \phi D^2\phi \pm V(\phi) + \xi R\phi^2\Big),$$

where D^2 is the gauge-covariant d'Alambert operator

$$\begin{aligned} D^2\phi &= g^{\mu\nu}D_\mu D_\nu\phi = g^{\mu\nu}\Big(\big[\partial_\mu - iU_*(A_\mu)\big]D_\nu\phi - \Gamma^\alpha_{\mu\nu}D_\alpha\phi\Big) \\ &= \frac{1}{\sqrt{|g|}}\Big(\partial_\mu - iU_*(A_\mu)\Big)\sqrt{|g|}g^{\mu\nu}\Big(\partial_\nu - iU_*(A_\nu)\Big)\phi. \end{aligned}$$

When proving the last identity, one needs that

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = -\frac{1}{\sqrt{|g|}}\partial_\nu\Big(\sqrt{|g|}g^{\alpha\nu}\Big).$$

For uncharged particles

$$D^2 = \frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Big) = \Box_g$$

is the d'Alambert operator in curved spacetime.

Coupling *fermions* to gravity is a bit more tricky, since in flat space those transform under the 'spin representation' of the Lorentz group. When the tangent space group to a curved spacetime manifold is (pseudo-)orthogonal one can still introduce spinors by referring them to the local orthonormal tangent frame. We assume that this is the case and introduce orthonormal tetrads or 4-beins. A 4-bein is a 'square-root' of the metric:

$$g_{\mu\nu} = \eta_{ab}e^a_{\ \mu}e^b_{\ \nu}, \quad g^{\mu\nu} = \eta^{ab}e_a^{\ \mu}e_b^{\ \nu} \Longrightarrow e_{a\mu}e^{a\nu} = \delta_\mu^{\ \nu},$$

where η_{ab} is the flat metric, i.e. $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ on manifolds with Lorentzian signature and $\eta_{ab} = \mathrm{diag}(1,1,1,1)$ on manifolds with Riemannian signature. Multiplying the last equation with $e^{a\mu}$ and summing over μ yields

$$e^{a\mu}e_{b\mu}e^{b\nu} = e^{a\nu} \Longrightarrow e^{a\mu}e_{b\mu} = \delta^a_b, \quad e^{a\mu}e^b_{\ \mu} = \eta^{ab}.$$

The indices μ, ν are spacetime-indices which are lowered and raised with $g_{\mu\nu}$ and $g^{\mu\nu}$ and a, b are Lorentz-indices which are lowered and raised with η_{ab} and η^{ab}. The $\theta^a = e^a_{\ \mu} dx^\mu$ form a orthonormal basis of tetrads. The price one pays when one introduces 4-beins is an additional symmetry, the local Lorentz invariance. Let $\Lambda(x)$ be a spacetime dependent Lorentz transformation. Then

$$e^a_{\ \mu} \quad \text{and} \quad \tilde{e}^a_{\ \mu} = \Lambda^a_{\ b} e^b_{\ \mu}$$

lead to the same metric

$$\tilde{e}^a_{\ \mu} \eta_{ab} \tilde{e}^b_{\ \nu} = \Lambda^a_{\ c} e^c_{\ \mu} \eta_{ab} \Lambda^b_{\ d} e^d_{\ \nu} = e^a_{\ \mu} \eta_{ab} e^b_{\ \nu} = g_{\mu\nu},$$

since $\Lambda^t \eta \Lambda = \eta$ for Lorentz-transformations. To this local $SO(1,3)$ resp. $SO(4)$ symmetry belongs a covariant derivative. To determine the corresponding connection we observe that with the help of the 4-bein we can convert vectors (tensors) into scalars and vice versa. Let A^μ be the components of a vector field. Its covariant derivative is

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} A^\alpha.$$

We can convert A^μ into into a Lorentz vector and vice versa:

$$A^a = e^a_{\ \mu} A^\mu \quad \text{and} \quad A^\mu = A^a e_a^{\ \mu}$$

which are the entries of a vector under the *local Lorentz transformations*

$$A^a \to \Lambda^a_{\ b} A^b, \qquad \Lambda = \Lambda(x).$$

Since Λ is spacetime-dependent, the ordinary derivative of a Lorentz-tensor is not a Lorentz-tensor. As usual we need to introduce a connection to define a covariant derivative which maps Lorentz-tensors into Lorentz-tensors. In particular there must be a ω, such that

$$\nabla_\mu A^a = \partial_\mu A^a + \omega^a_{\mu b} A^b$$

is a Lorentz-vector. This requirement is fulfilled if it doesn't matter whether we first take the covariant derivative of a vector field and then convert the result into a Lorentz vector, or first convert the spacetime vector into a Lorentz vector and then take the covariant derivative. Thus we demand that

$$e^a_{\ \nu} \nabla_\mu A^\nu = \nabla_\mu (e^a_{\ \nu} A^\nu) = \nabla_\mu A^a.$$

The first equation is equivalent to

$$\partial_\mu e^a_{\ \nu} - \Gamma^\alpha_{\mu\nu} e^a_{\ \alpha} + \omega_{\mu b}^{\ a} e^b_{\ \nu} \equiv \nabla_\mu e^a_{\ \nu} = 0. \tag{1.17}$$

Similarly one finds, that

$$\partial_\mu e_a^{\ \nu} + \Gamma^\nu_{\mu\alpha} e_a^{\ \alpha} + \omega_{\mu ab} e^{b\nu} \equiv \nabla_\mu e_a^{\ \nu} = 0. \tag{1.18}$$

These equations allow us to calculate the connection ω_μ from the vierbein as follows: Let

$$\theta^a = e^a_{\ \mu} dx^\mu \quad \text{and} \quad \omega^a_{\ b} = \omega^a_{\mu b} dx^\mu$$

be the tetrad and connection-one-form, respectively. Then

$$d\theta^a + \omega^a_{\ b} \wedge \theta^b = \frac{1}{2}\Big(\partial_\mu e^a_{\ \nu} - \partial_\nu e^a_{\ \mu} + \omega^a_{\mu b}\, e^b_{\ \nu} - \omega^a_{\nu b}\, e^b_{\ \mu}\Big) dx^\mu \wedge dx^\nu = 0,$$

since the term in the brackets is just the anti-symmetrized left hand side of (1.17). The important formula

$$\boxed{d\theta^a + \omega^a_{\ b} \wedge \theta^b = 0} \tag{1.19}$$

is the first *structure equation of Cartan*. To obtain the second structure equation we multiply (1.17) with dx^μ which yields

$$de^a_{\ \nu} - e^a_{\ \alpha}\Gamma^\alpha_{\ \nu} + \omega^a_{\ b} e^b_{\ \nu} = 0, \quad \text{where} \quad \Gamma^\alpha_{\ \mu} = \Gamma^\alpha_{\nu\mu} dx^\nu. \tag{1.20}$$

The curvature tensor can be written as follows:

$$d\Gamma^\alpha_{\ \beta} + \Gamma^\alpha_{\ \sigma} \wedge \Gamma^\sigma_{\ \beta} = \frac{1}{2}\Big(\Gamma^\alpha_{\nu\beta},_\mu - \Gamma^\alpha_{\mu\beta},_\nu + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta}\Big) dx^\mu \wedge dx^\nu \equiv \Omega^\alpha_{\ \beta},$$

where we introduced the curvature 2-form. Now we differentiate (1.20) which results in

$$-de^a_{\ \alpha} \wedge \Gamma^\alpha_{\ \nu} - e^a_{\ \alpha} d\Gamma^\alpha_{\ \nu} + d\omega^a_{\ b} e^b_{\ \nu} - \omega^a_{\ b} \wedge de^b_{\ \nu} = 0. \tag{1.21}$$

Here we insert for $de^a_{\ \alpha}$ the result (1.20) and find the formula

$$d\Gamma^\alpha_{\ \beta} + \Gamma^\alpha_{\ \sigma} \wedge \Gamma^\sigma_{\ \beta} = \Big(d\omega^a_{\ b} + \omega^a_{\ c}\omega^c_{\ b}\Big) e^b_{\ \beta} e_a^{\ \alpha}.$$

Comparison with (1.21) yields the *second structure equation of Cartan*:

$$\boxed{d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b} = \Omega^a_{\ b} \quad \text{where} \quad \Omega^a_{\ b} = e^a_{\ \alpha} e_b^{\ \beta} \Omega^\alpha_{\ \beta}} \tag{1.22}$$

is the curvature 2-form with respect th the orthonormal frame. We now use the structure equation to derive the Schwarzschild solution for a spherically symmetric body. We choose the manifold $M = R \times R_+ \times S^2$. In polar coordinates the metric has the form

$$g = e^{2a(r)} dt^2 - \Big[e^{2b(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big]. \tag{1.23}$$

We now must insert this ansatz into Einstein's field equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.24}$$

This is done most quickly with the help of the Cartan calculus. We choose the following orthonormal tetrad

$$\theta^0 = e^a dt, \quad \theta^1 = e^b dr, \quad \theta^2 = r d\theta, \quad \theta^3 = r \sin\theta d\varphi.$$

According to the first structure equation we need to calculate the derivative of the tetrad:

$$\begin{aligned} d\theta^0 &= a' e^a dr \wedge dt, \qquad d\theta^1 = 0 \\ d\theta^2 &= dr \wedge d\theta, \qquad d\theta^3 = \sin\theta dr \wedge d\varphi + r\cos\theta \; d\theta \wedge d\varphi. \end{aligned}$$

We express the right hand sides in terms of the basis $\theta^\alpha \wedge \theta^\beta$, obtaining

$$\begin{aligned} d\theta^0 &= a' e^{-b} \theta^1 \wedge \theta^0, \quad d\theta^1 = 0 \\ d\theta^2 &= \frac{1}{r} e^{-b} \theta^1 \wedge \theta^2, \quad d\theta^3 = \frac{1}{r}\big(e^{-b}\theta^1 \wedge \theta^3 + \cot\theta \; \theta^2 \wedge \theta^3\big). \end{aligned}$$

When this is compared with the first structure equation one expects the following connection forms:

$$\begin{aligned} \omega^0_{\ 1} &= a' e^{-b} \theta^0, \qquad \omega^0_{\ 2} = \omega^0_{\ 3} = 0, \quad \omega^2_{\ 1} = r^{-1} e^{-b} \theta^2 \\ \omega^3_{\ 1} &= r^{-1} e^{-b} \theta^3, \quad \omega^3_{\ 2} = r^{-1} \cot\theta \, \theta^3. \end{aligned}$$

The other connection forms are determined by $\omega_{ab} = -\omega_{ba}$. This ansatz indeed satisfies the first structure equation. The curvature forms $\Omega^a_{\ b}$ can now be gotten from the second structure equation. The result is

$$\begin{aligned} \Omega^0_{\ 1} &= e^{-2b}(a'b' - a'' - a'^2)\, \theta^0 \wedge \theta^1, \quad \Omega^0_{\ 2} = -\frac{a' e^{-2b}}{r} \theta^0 \wedge \theta^2 \\ \Omega^0_{\ 3} &= -\frac{a' e^{-2b}}{r} \theta^0 \wedge \theta^3, \qquad \Omega^1_{\ 2} = \frac{b' e^{-2b}}{r} \theta^1 \wedge \theta^2 \\ \Omega^1_{\ 3} &= \frac{b' e^{-2b}}{r} \theta^1 \wedge \theta^3, \qquad \Omega^2_{\ 3} = \frac{1 - e^{-2b}}{r^2} \theta^2 \wedge \theta^3. \end{aligned}$$

The other components are gotten from $\Omega_{ab} = -\Omega_{ba}$. For the nonzero components of the Einstein tensor $G^a_{\ b}$ (with Lorentz-indices) one obtains

$$\begin{aligned} G^0_{\ 0} &= \frac{1}{r^2} - e^{-2b}\Big(\frac{1}{r^2} - \frac{2b'}{r}\Big), \quad G^1_{\ 1} = \frac{1}{r^2} - e^{-2b}\Big(\frac{1}{r^2} + \frac{2a'}{r}\Big) \\ G^2_{\ 2} &= G^3_{\ 3} = -e^{-2b}\Big(a'^2 - a'b' + a'' + \frac{a' - b'}{r}\Big). \end{aligned}$$

If we demand asymptotic flatness, then the Einstein equation im vacuum imply $a + b = 0$ and

$$e^{-2b} = 1 - 2m/r$$

so that we obtain the Schwarzschild solution

$$g = \big(1 - \frac{2m}{r}\big) dt^2 - \frac{dr^2}{1 - 2m/r} - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1.25}$$

We shall need this result later, when we discuss quantum fields near black holes.

Let us now determine, how the connection transforms under frame-rotations $\theta \to \tilde\theta = \Lambda\theta$: if (θ, ω) obey the structure equations, then $(\tilde\theta, \tilde\omega)$ must obey them. Since

$$0 = \Lambda^{-1}\Big(d\tilde\theta + \tilde\omega \wedge \tilde\theta\Big) = \Lambda^{-1} d\Lambda \wedge \theta + d\theta + \Lambda^{-1}\tilde\omega\Lambda \wedge \theta$$

this requirement implies the following transformation law for the connection under local frame rotation:

$$\omega^a_{\ b} \longrightarrow \tilde{\omega}^a_{\ b} = \Lambda^a_{\ c}\omega^c_{\ d}(\Lambda^{-1})^d_{\ b} - (d\Lambda\Lambda^{-1})^a_{\ b}. \tag{1.26}$$

$\omega^a_{\ b}$ transforms inhomogeneously under local frame rotations, as expected for a connection.

How do we now couple fermions to gravity? For that we recall that the Lie-algebra of the Lorentz-group consists of 6 matrices $\lambda^a_{\ b}$ which are antisymmetric in the lower indices, $\lambda_{ab} + \lambda_{ba} = 0$. As generators of a representation of the Lorentz-algebra we may choose operators $J_{ab} = -J_{ba}$ which obey the commutation relations

$$[J_{ab}, J_{cd}] = i\Big(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad} - \eta_{bd}J_{ac}\Big).$$

Examples are the generalizations of the orbital angular momentum, spin and total angular momentum (supplemented by the infinitesimal boosts)

$$M_{ab} = i\big(x_a\partial_b - x_b\partial_a\big), \quad \Sigma_{ab} = \frac{1}{4i}[\gamma_a, \gamma_b] \quad \text{and} \quad J_{ab} = M_{ab} + \Sigma_{ab}.$$

Let

$$\Lambda^a_{\ b} = (e^\lambda)^a_{\ b}, \quad \lambda_{ab} = -\lambda_{ba}$$

be a Lorentz-transformation. Spinors transform under Lorentz-transformations with the spin representation

$$\psi(x) \longrightarrow \tilde{\psi}(x) = S\psi(x) = e^{\frac{i}{2}\lambda_{ab}\Sigma^{ab}}\psi \Longrightarrow S^{-1}\gamma^a S = \Lambda(S)^a_{\ b}\gamma^b. \tag{1.27}$$

The map $S \longrightarrow \Lambda(S)$ is a representation of the spin group by Lorentz transformations

$$\Lambda(S_1S_2) = \Lambda(S_1)\Lambda(S_2), \quad \Lambda(1) = 1. \tag{1.28}$$

The covariant derivative $D_\mu = \partial_\mu + i\omega_\mu$ acting on spinors must be compatible with the local spin-rotation

$$\tilde{D}_\mu\tilde{\psi} = \big(\partial_\mu + i\tilde{\omega}_\mu\big)\tilde{\psi} = S\big(\partial_\mu + i\omega_\mu\big)\psi. \tag{1.29}$$

This implies, that

$$\tilde{\omega}_\mu = S\omega_\mu S^{-1} + i\partial_\mu S S^{-1} \tag{1.30}$$

must hold. The spin connection must be in the Lie-algebra of the spin group, i.e. a linear combination of its generators,

$$\omega_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}. \tag{1.31}$$

As indicated by the notation, we claim that the $\omega_{\mu ab}$ is the connection of the frame rotations. We must show, that this spin-connection indeed transforms as in (1.30) under spin rotations. Using the transformation (1.26) for the connection we find

$$\tilde{\omega}_\mu = \frac{1}{2}\tilde{\omega}_{\mu cd}\Sigma^{cd} = \frac{1}{2}\Big(\Lambda_c^{\ a}\omega_{\mu ab}(\Lambda^{-1})^b_{\ d} - (\partial_\mu \Lambda\Lambda^{-1})_{cd}\Big)\Sigma^{cd}. \tag{1.32}$$

On the other hand, using the last identity in (1.27) and $(\Lambda^{-1})^a_{\ b} = \Lambda_b^{\ a}$ we find

$$S\omega_\mu S^{-1} = \frac{1}{2}\omega_{\mu ab}S\Sigma^{ab}S^{-1} = \frac{1}{2}\Lambda_c^{\ a}\omega_{\mu ab}(\Lambda^{-1})^b_{\ d}\Sigma^{cd}, \tag{1.33}$$

that is, the homogeneous term in (1.32). To see that the inhomogeneous term in (1.30) coincides with that in (1.32) we note, that the last identity in (1.27) implies

$$[\gamma^a, dSS^{-1}] = (d\Lambda\Lambda^{-1})^a_{\ b}\gamma^b$$

Now we multiply this equation with γ_a from the left and sum over a. Using that[6]

$$\gamma_a\gamma^a = 4 \quad \text{and} \quad \gamma_a dSS^{-1}\gamma^a = 0$$

we end up with

$$4dSS^{-1} = (d\Lambda\Lambda^{-1})_{ab}\gamma^a\gamma^b,$$

we find, that

$$idSS^{-1} = -\frac{1}{2}(d\Lambda\Lambda^{-1})_{ab}\Sigma^{ab}$$

which proves that also the inhomogeneous term in (1.30) coincides with that in (1.32).

So summarize: The covariant derivative of spinors is

$$D_\mu\psi = \partial_\mu\psi + i\omega_\mu\psi, \quad \text{where} \quad \omega_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}.$$

$D_\mu\psi$ transforms under frame-rotations the same way as ψ does.

Spinor fields are scalars with respect to general coordinate transformations or diffeomorphism. Is the spinor field charged, then

$$D_\mu\psi = \partial_\mu\psi + i\omega_\mu\psi - iU_*(A_\mu)\psi. \tag{1.34}$$

One should keep in mind that the local spin rotation are implemented differently in Riemannian and pseudo-Riemannian manifolds. In Riemannian manifolds

$$\psi \to S\psi, \quad \psi^\dagger \to \psi^\dagger S^\dagger = \psi^\dagger S^{-1} \quad \text{and} \quad \omega \to S\omega S^{-1} + idSS^{-1} \tag{1.35}$$

and in pseudo-Riemannian manifolds

[6] to get the second of the following identities, one uses that $[\Sigma_{ab}, \gamma_c] = i(\eta_{ac}\gamma_b - \eta_{bc}\gamma_a)$.

$$\psi \to S\psi, \quad \psi^\dagger \to \psi^\dagger S^\dagger, \quad \bar{\psi} \to \bar{\psi} S^{-1} \quad \text{and} \quad \omega \to S\omega S^{-1} + idSS^{-1} \tag{1.36}$$

In the Riemannian case $S^\dagger = S^{-1}$. The corresponding relation in the pseudo-Riemannian case is $\gamma^0 S^\dagger \gamma^0 = S^{-1}$. The reason for this difference is that the spin groups are the coverings of $SO(4)$ and $SO(1,3)$, respectively. Now we can show, that the fermionic part of the action is local Lorentz-invariant. From our previous results it follows immediately, that

$$\int \tilde{\psi}^\dagger (i\tilde{\gamma}^\mu \tilde{D}_\mu - im)\tilde{\psi} = \int \psi^\dagger S^\dagger (i\tilde{\gamma}^\mu \tilde{D}_\mu - im) S\psi = \int \psi^\dagger (iS^\dagger \tilde{\gamma}^\mu S D_\mu - im)\psi,$$

where we have used (1.29). It remains to be shown, that

$$S^\dagger \tilde{\gamma}^\mu S = \gamma^\mu. \tag{1.37}$$

The γ^μ are spacetime-dependent gamma-matrices which must transform the spacetime-vector D_μ into a Lorentz vector. Their anti-commutators are

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

One convinces one-selves, that they are related to the numerical gamma-matrices γ^a on flat spacetime according to

$$\gamma^\mu = \gamma^a e_a^{\ \mu}. \tag{1.38}$$

Now we calculate

$$S^\dagger \tilde{\gamma}^\mu S = S^\dagger \gamma^a S \tilde{e}_a^{\ \mu} = \Lambda_a^{\ c} \Lambda^a_{\ b} \gamma^b e_a^{\ \mu} = \gamma^\mu$$

and see, that (1.37) indeed holds. Thus the action is local Lorentz-invariant.

1.2.4 Diffeomorphism and Lie derivative

The general coordinate invariance is built into our action since only spacetime scalars enter the Lagrangian density. Now we interpret a coordinate transformation actively as a point transformation, rather than passively as one usually does. Consider a congruence of curves $x^\mu(u)$ and define the tangent vector field dx^μ/du along the curve. We do that for every curve in the congruence and end up with a vector field X^μ. Conversely, given a non-vanishing vector field $X^\mu(x)$ defined over the manifold, then this can be used to define a congruence of curves in the manifold called the orbits or trajectories of X^μ. This curves are the integral curves on the vector field and are obtained by solving

$$\frac{dx^\mu}{du} = X^\mu\big(x(u)\big)\big).$$

We suppose that X^μ has been given and the corresponding congruence of curves has been constructed. We want to differentiate a tensor field $T^{\mu_1 \cdots}_{\nu_1 \cdots}$. For that we drag the tensor field at some point

p along the curve passing through p to some neighboring point q and compare the dragged-along tensor with the tensor already there. We subtract the two tensors at q and define the derivative by some limiting process as $q \to p$.

Consider the transformation

$$y^\mu = x^\mu + \epsilon X^\mu(x),$$

where ϵ is small. This point transformation sends a point p with coordinates x to a point q which lies on the curve of the congruence through p and has coordinates $x + \epsilon X$ (in the same coordinate system). Under a point transformation a tensor $T^{\mu\nu}$ is mapped according to

$$T^{\mu\nu}(x) \longrightarrow \tilde{T}^{\mu\nu}(y) = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}T^{\alpha\beta}(x). \tag{1.39}$$

Since

$$\frac{\partial y^\mu}{\partial x^\alpha} = \delta^\mu_{\ \alpha} + \epsilon\partial_\alpha X^\mu$$

we have

$$\begin{aligned}\tilde{T}^{\mu\nu}(y) &= \big(\delta^\mu_{\ \alpha} + \epsilon\partial_\alpha X^\mu\big)\big(\delta^\nu_{\ \beta} + \epsilon\partial_\beta X^\nu\big)T^{\alpha\beta}(x)\\ &= T^{\mu\nu}(x) + \epsilon\Big(\partial_\alpha X^\mu T^{\alpha\nu}(x) + \partial_\beta X^\nu T^{\mu\beta}(x)\Big) + O(\epsilon^2).\end{aligned}$$

Applying Taylor's theorem to first order, we also get

$$T^{\mu\nu}(y) = T^{\mu\nu}(x + \epsilon X(x)) = T^{\mu\nu}(x) + \epsilon X^\alpha\partial_\alpha T^{\mu\nu}(x).$$

Now we are ready to define the Lie derivative of a tensor by

$$L_X T^{\mu\nu} = \lim_{\epsilon\to 0}\frac{T^{\mu\nu}(y) - \tilde{T}^{\mu\nu}(y)}{\epsilon}. \tag{1.40}$$

The Lie derivative compares the tensor $T^{\mu\nu}(y)$ at the point q with $\tilde{T}^{\mu\nu}(y)$, the dragged-along tensor at q. We find

$$L_X T^{\mu\nu} = X^\alpha\partial_\alpha T^{\mu\nu} - T^{\alpha\nu}\partial_\alpha X^\mu - T^{\mu\alpha}\partial_\alpha X^\nu.$$

The Lie derivative maps (p,q)-tensors into (p,q) tensors, is linear, fulfills the Leibniz rule and commutes with contractions. For general tensor field is it

$$L_X T^{\mu_1\mu_2\dots}_{\nu_1\nu_2\dots} = X^\alpha\partial_\alpha T^{\mu_1\mu_2\dots}_{\nu_1\nu_2\dots} - T^{\alpha\mu_2\dots}_{\nu_1\nu_2\dots}\partial_\alpha X^{\mu_1} - \dots + T^{\mu_1\mu_2\dots}_{\alpha\nu_2\dots}\partial_{\nu_1}X^\alpha + \dots. \tag{1.41}$$

In particular, the Lie-derivative of the metric is

$$L_X g_{\mu\nu} = X^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\nu}X^\alpha{}_{,\mu} + g_{\mu\alpha}X^\alpha{}_{,\nu}\,.$$

Using the metricity of the connection,

$$\nabla_\alpha g_{\mu\nu} - \Gamma^\beta_{\ \alpha\mu}g_{\beta\nu} - \Gamma^\beta_{\ \alpha\nu}g_{\mu\beta} = 0,$$

this Lie-derivative can be rewritten as

$$L_X g_{\mu\nu} = X^\alpha\big(\Gamma^\beta{}_{\alpha\mu} g_{\beta\nu} + \Gamma^\beta{}_{\alpha\nu} g_{\mu\beta}\big) + g_{\beta\nu} X^\beta{}_{,\mu} + g_{\mu\beta} X^\beta{}_{,\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu.$$

If there is a vector field such that $L_X g_{\mu\nu} = 0$, such a field is called *Killing vector field*, then the metric is dragged into itself by the flux generated by X. In other word, the flux is an *isometry.* A Killing vector field generates an infinitesimal isometry and obeys the Killing equation

$$L_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0. \tag{1.42}$$

Let us see more explicitly that the flux generated by a Killing field is an isometry. The diffeomorphism generated by $X(x)$ maps the curve $x(v)$ connecting the points A and B to the curve $y(x(v))$ connecting A' and B'.

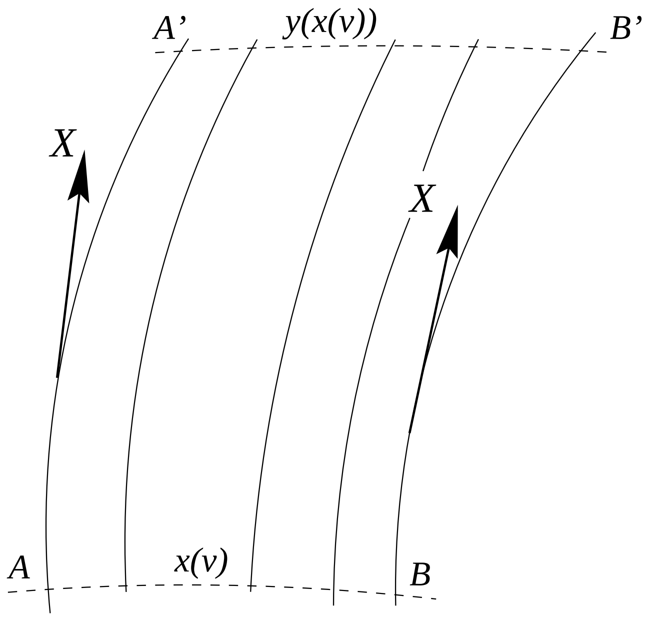

Abbildung 1.1: *The isometry generated by a Killing field*

Let us see, that the two curves have the same length:

$$\begin{aligned} d(A', B') &= \int \sqrt{g_{\mu\nu}\big(y(v)\big)\dot{y}^\mu \dot{y}^\nu}\,dv = \int \sqrt{g_{\mu\nu}\big(y[x(v)]\big)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}\dot{x}^\alpha \dot{x}^\beta}\,dv \\ &= \int \sqrt{\frac{\partial x^\sigma}{\partial y^\mu}\frac{\partial x^\rho}{\partial y^\nu} g_{\mu\nu}\big(x(v)\big)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}\dot{x}^\alpha \dot{x}^\beta}\,dv = d(A, B), \end{aligned}$$

where in the second to last equation we used, that the metric at y coincides with the dragged along metric if $X(x)$ is a Killing field.

The isometries of Minkowski spacetime are the d translations with constant Killing fields $X^\mu(x) = c^\mu$, and the $\frac{1}{2}d(d-1)$ Lorentz transformations with Killing fields $X^\mu(x) = \omega^\mu_{\ \nu} x^\nu$, $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$. There is another spacetime which admits the same maximal number of $\frac{1}{2}d(d+1)$ Killing fields, namely the (anti) de Sitter spacetimes.

A spacetime is *stationary* if there exists a special coordinate system in which the metric is time-independent, i.e.

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0, \tag{1.43}$$

where x^0 is a time-like coordinate. In an arbitrary coordinate system the metric will probably depend explicitly on all the coordinates; so we need to make the statement (1.43) coordinate-independent. If we define a vector field

$$X^\mu = \delta^\mu_0 \tag{1.44}$$

in the special coordinate system, then

$$L_X g_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu X^\alpha + g_{\mu\alpha}\partial_\nu X^\alpha = \delta^\alpha_0 \partial_\alpha g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^0} = 0.$$

Since $L_X g_{\mu\nu}$ is a tensor it vanishes in all coordinate systems and hence X^μ is a Killing vector field. Conversely, given a time-like Killing field X^μ, then there always exist coordinates adapted to the Killing field, that is, in which (1.44) holds. In this coordinate system the metric is time independent. Thus, a spacetime is stationary if and only if it admits a time-like Killing vector field.

A *static spacetime* admits a hypersurface-orthogonal time-like Killing field. To see what this means let

$$f(x) = \mu \tag{1.45}$$

define a family of hypersurfaces. Different members of the family correspond to different values of μ. If two neighboring point with coordinates x and $x + dx$ lie on the same surface, then

$$\frac{\partial f}{\partial x^\mu} dx^\mu \equiv n_\mu dx^\mu = 0.$$

Since dx^μ lies in a surface S defined by (1.45) it follows by construction that n^μ is orthogonal to S. A vector field X^μ is called hypersurface-orthogonal if it is everywhere orthogonal to the family of hypersurfaces, in which case it must be proportional to n^μ,

$$X^\mu = \lambda(x) n^\mu = \lambda \partial_\mu f.$$

This conditions imply

$$X_\mu \partial_\nu X_\alpha = \lambda f_{,\mu}\, \lambda_{,\nu}\, f_{,\alpha} + \lambda^2 f_\mu f_{,\nu\alpha}\,.$$

Taking the total antisymmetric part of this equation we find

$$X_{[\mu}\partial_\nu X_{\alpha]} = 0 \Leftrightarrow X \wedge dX = 0, \quad X = X_\mu dx^\mu.$$

This equation is unchanged, if we replace the ordinary derivative by a covariant derivative, namely

$$X_{[\mu}\nabla_\nu X_{\alpha]} = 0. \tag{1.46}$$

Any hypersurface-orthogonal vector field satisfies this Frobenius condition. The converse is also true: any non-null Killing vector field ($X^\mu X_\mu \neq 0$) satisfying the Frobenius condition is necessarily hypersurface-orthogonal. Indeed, one can show that then

$$X_{,\mu} = X^2 f_{,\mu} \quad \text{for some function} \quad f.$$

Given a hypersurface-orthogonal time-like Killing field one can introduce adapted coordinates along the congruence and in one hypersurface (see figure (1.2)) such that the metric is time-independent

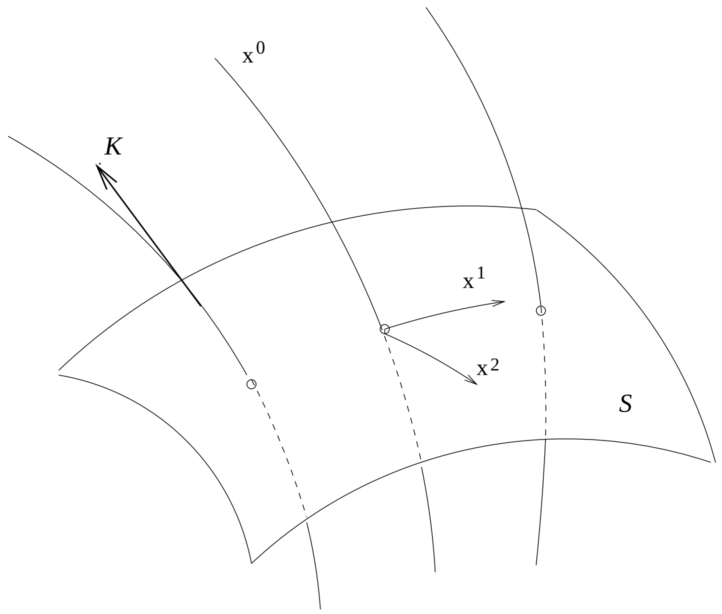

Abbildung 1.2: *Adapted coordinates in a static spacetime*

and no cross terms appear in the line element involving the time, i.e. the shift vector g_{0i} vanishes.

1.2.5 Weyl-transformations

Under a local Weyl-transformation

$$ds^2 \longrightarrow e^{2\sigma} ds^2$$

transform the metric and vielbein according to

$$g_{\mu\nu} \longrightarrow \hat{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu}, \qquad e^a_{\ \mu} \longrightarrow \hat{e}^a_{\ a} = e^\sigma e^a_{\ \mu} \Longrightarrow \hat{e} = e^{d\sigma} e.$$

The transformed tetrad $\hat\theta^a = e^\sigma\theta^a$ must obey the first structure equation with transformed connection $\hat\omega^a_{\ b}$:

$$d\hat\theta^a + \hat\omega^a_{\ b}\hat\theta^b = e^\sigma\left(d\theta^a + \hat\omega^a_{\ b}\theta^b + d\sigma\theta^a\right) = 0.$$

The right hand is zero, iff

$$\hat\omega^a_{\ b}\theta^b = \omega^a_{\ b}\theta^b - d\sigma\theta^a$$

holds. Together with the antisymmetry of the connection forms this fixes the Weyl-transformed connection as

$$\hat\omega^a_{\ b} = \omega^a_{\ b} + \theta^a\partial_b\sigma - \theta_b\partial^a\sigma, \quad \text{where} \quad \partial_b\sigma = e_b^{\ \mu}\partial_\mu\sigma.$$

The Christoffel-symbols transform as

$$\hat\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \Big(\delta^\alpha_\mu\sigma_{,\nu} + \delta^\alpha_\nu\sigma_{,\mu} - g_{\mu\nu}\sigma^{,\alpha}\Big).$$

Finally we need the transformation of the curvature: From

$$\hat\Omega_{ab} = \Omega_{ab} - \theta_a\theta_c\big(\nabla^c\sigma_{,b} - \sigma^{,c}\sigma_{,b}\big) + \theta_b\theta_c\big(\nabla^c\sigma_{,a} - \sigma^{,c}\sigma_{,a}\big) - \theta_a\theta_b(\nabla\sigma)^2.$$

One derives, that

$$\begin{aligned}\hat R_{abcd} \quad = \quad & e^{-2\sigma}\Big[R_{abcd} + \eta_{bc}\big(\sigma_{;ad} - \sigma_{,a}\sigma_{,d}\big) - \eta_{ac}\big(\sigma_{;bd} - \sigma_{,b}\sigma_{,d}\big) - \eta_{ac}\eta_{bd}(\nabla\sigma)^2 \\ & -\eta_{bd}\big(\sigma_{;ac} - \sigma_{,a}\sigma_{,c}\big) + \eta_{ad}\big(\sigma_{;bc} - \sigma_{,b}\sigma_{,c}\big) + \eta_{ad}\eta_{bc}(\nabla\sigma)^2\Big].\end{aligned}$$

Contractions yield the following transformation laws for the Ricci tensor and Ricci scalar:

$$\begin{aligned}\hat R_{ab} &= e^{-2\sigma}\Big[R_{ab} + (2-d)\Big(\sigma_{;ab} - \sigma_{,a}\sigma_{,b} + \eta_{ab}(\nabla\sigma)^2\Big) - \eta_{ab}\triangle\sigma\Big] \\ \hat R &= e^{-2\sigma}\Big[R - 2(d-1)\triangle\sigma - (d-1)(d-2)(\nabla\sigma)^2\Big].\end{aligned}$$

The Weyl tensor, which is the traceless part of the curvature tensor,

$$\begin{aligned}C_{abcd} = R_{abcd} \quad + \quad & \frac{1}{d-2}\Big(\eta_{ad}R_{cb} + \eta_{bc}R_{da} - \eta_{ac}R_{db} - \eta_{bd}R_{ca}\Big) \\ & + \frac{1}{(d-1)(d-2)}\Big(\eta_{ac}\eta_{db} - \eta_{ad}\eta_{cb}\Big)R\end{aligned}$$

is Weyl-invariant:

$$\hat C^\alpha_{\ \beta\mu\nu} = C^\alpha_{\ \beta\mu\nu}.$$

Hence, if a spacetime can be related to flat spacetime by a Weyl-transformation (is conformally flat) then the Weyl tensor vanishes. In $d \geq 4$ dimensions the converse is also true: a spacetime is conformally flat

$$g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}, \quad \text{up to coordinate transformations,}$$

if and only if the Weyl-tensor vanishes. In 3 dimensions the Weyl tensor vanishes identically and another tensor (see Eisenhart) can be used to determine if a space is conformally flat. Every 1 and 2-dimensional spacetime is conformally flat (up to moduli parameter).

Transformation of wave-operators

The Laplacian or d'Alambertian

$$\triangle = \frac{1}{e}\partial_\mu\big(eg^{\mu\nu}\partial_\nu\big), \qquad e = \det(e^a_{\ \mu})$$

transforms under Weyl transformations as

$$\hat{\triangle} = e^{-\frac{d+2}{2}\sigma}\Big(\triangle - \frac{d-2}{2}\triangle\sigma + (\frac{d-2}{2})^2(\nabla\sigma)^2\Big)e^{\frac{d-2}{2}\sigma}.$$

When rewriting the transformation of the Ricci scalar as

$$\hat{R} = e^{-\frac{d+2}{2}\sigma}\Big(R - 2(d-1)\triangle\sigma - (d-1)(d-2)(\nabla\sigma)^2\Big)e^{\frac{d-2}{2}\sigma}$$

then one sees immediately, that the wave operator

$$\triangle_c = \triangle - \frac{d-2}{4(d-1)}R$$

transforms homogeneously under Weyl-transformations,

$$\hat{\triangle}_c = e^{-\frac{d+2}{2}\sigma}\triangle_c e^{\frac{d-2}{2}\sigma}. \tag{1.47}$$

Using the transformation property of the connection,

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} + \big(e_{a\mu}e_b^{\ \nu} - e_{b\mu}e_a^{\ \nu}\big)\partial_\nu\sigma \quad \text{and} \quad \gamma_a\Sigma^{ab} = \frac{d-1}{2i}\gamma^b$$

one easily finds, that

$$\hat{\not{D}} = e^{-\sigma}\gamma^\mu\big(\partial_\mu + i\omega_\mu + \frac{d-1}{2}\partial_\mu\sigma\big)$$

which is a homogeneous transformation

$$\hat{\not{D}} = e^{-\frac{d+1}{2}\sigma}\not{D}e^{\frac{d-1}{2}\sigma}. \tag{1.48}$$

Now we turn to the Yang-Mills equations. From the very definition of the star-operator, one finds

$$\hat{{}^*\alpha} = e^{(d-2p)\sigma}{}^*\hat{\alpha} \quad \text{for a p-form} \quad \alpha.$$

We see, that the Yang-Mills action and theta terms are both invariant under Weyl-transformations in 4 dimensions if

$$\hat{F} = F \Longrightarrow \hat{A} = A.$$

Weyl-invariant actions

The term $\int \eta\phi\triangle_c\phi$ is Weyl-invariant, if the scalar field has *Weyl-weight* $-(d-2)/2$, i.e. transforms according to

$$\hat{\phi} = e^{\alpha\sigma}\phi \quad \text{with weight} \quad \alpha = -\frac{1}{2}(d-2)$$

under Weyl transformations, since then

$$\hat{\phi}\hat{\triangle}_c\hat{\phi} = e^{-d\sigma}\phi\triangle_c\phi$$

as it must be for the action to be invariant[7]. Also, an interaction term

$$\lambda\int \eta\phi^{d_c}, \quad \text{where} \quad d_c = \frac{2d}{d-2}$$

is Weyl invariant. The general Weyl-invariant action for a scalar field reads

$$S_\phi = \int \eta\Big(-\frac{1}{2}(\phi,\triangle_c\phi) + \lambda\phi^{d_c}.\Big).$$

The action for massless spin-$\frac{1}{2}$ particles,

$$S_\psi = \int \eta\psi^\dagger i\not{D}\psi$$

is Weyl invariant in any dimensions if we assign the weight $\alpha = -(d-1)/2$ to a spinor field in d dimensions. The Yang-Mills action is Weyl invariant in 4 dimensions and the gauge potential has weight zero.

A Yukawa term transforms as

$$\hat{\psi}^\dagger\hat{\phi}\hat{\psi} = e^{(2-\frac{3d}{2})\sigma}\psi^\dagger\phi\psi$$

and hence $\int \eta\psi^\dagger\phi\psi$ is Weyl invariant in 4 dimensions.

Now we wish to combine the diffeomorphism- and Weyl transformations. For Weyl-invariant theories these are symmetry transformations. Give a vector field X and its corresponding flux, the dragged along metric is

$$\tilde{g}_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}g_{\alpha\beta}(x).$$

Now we assume, that the $\tilde{g}_{\mu\nu}(y)$ coincides with the metric present at y, up to a conformal factor:

$$\tilde{g}_{\mu\nu}(y) = e^{2\sigma(y)}g_{\mu\nu}(y).$$

As above, we investigate the infinitesimal form of this condition. Setting $y = x + \epsilon X$ we find

$$\big(1 + 2\epsilon\sigma(x)\big)\big(1 + \epsilon X^\alpha\partial_\alpha\big)g_{\mu\nu}(x) = g_{\mu\nu}(x) - \epsilon\big(\partial_\mu X^\alpha g_{\alpha\nu} + \partial_\nu X^\alpha g_{\mu\alpha}\big) + O(\epsilon^2).$$

[7] Recall, that $\hat{\eta} = e^{d\sigma}\eta$.

We see, that the vector field must obey the equation

$$2\sigma g_{\mu\nu} + L_X g_{\mu\nu} = 0.$$

When using $g^{\mu\nu}\partial_\alpha g_{\mu\nu} = 2\Gamma^\mu{}_{\mu\alpha}$ the contraction of this equation determines the leading contribution to σ as

$$\sigma = -\frac{1}{d}\nabla_\mu X^\mu = -\frac{1}{d}\,\mathrm{div}X.$$

Thus the field X must obey the *conformal Killing equation*

$$L_X g_{\mu\nu} = \frac{2}{d}\,\mathrm{div}X\; g_{\mu\nu}. \tag{1.49}$$

Minkowski spacetime allows for $\frac{1}{2}(d+1)(d+2)$ conformal Killing fields. Beside the Killing fields belonging to the translations and Lorentz transformations these are the dilatations and d special conformal transformations with conformal Killing fields

$$X^\mu = \lambda x^\mu \quad \text{and} \quad X^\mu = 2(c\cdot x)x^\mu - x^2 c^\mu.$$

The dimension of the conformal group is $\frac{1}{2}(d+1)(d+2)$ and is just the group $SO(d,2)$ (resp. $SO(d+1,1)$ in Euclidean spacetime).

1.2.6 Conformal transformations in Minkowski spacetime

If a spacetime possesses Killing fields, then a diffeomorphism invariant theory possesses symmetries. However, a diffeomorphism- and Weyl invariant theory has additional symmetries. They can be combined such that the metric is invariant, as it is the case for isometries. In Minkowski spacetime such a theory is not only invariant under translations and Lorentz boost but under all conformal transformations.

So let us assume, that X is a conformal Killing field with corresponding flux $x \longrightarrow y(x)$. The metric and matter fields are dragged along according to

$$\begin{aligned} g_{\mu\nu}(x) &\longrightarrow \tilde g_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}(x) = e^{2\sigma(y)} g_{\mu\nu}(y) \\ T^{\mu_1\dots}_{\nu_1\dots}(x) &\longrightarrow \tilde T^{\mu_1\dots}_{\nu_1\dots}(y) = \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x^{\beta_1}}{\partial y^{\nu_1}}\cdots T^{\alpha_1\dots}_{\beta_1\dots}(x). \end{aligned}$$

Next we perform a compensating Weyl transformation with conformal factor

$$e^{2\sigma(y)} = \frac{1}{d}\, g^{\mu\nu}(y)\tilde g_{\mu\nu}(y).$$

which results in

$$\begin{aligned} \tilde g_{\mu\nu}(y) &\longrightarrow e^{-2\sigma(y)}\tilde g_{\mu\nu}(y) \\ \tilde T^{\mu_1\dots}_{\nu_1\dots}(y) &\longrightarrow e^{-\alpha\sigma(y)}\tilde T^{\mu_1\dots}_{\nu_1\dots}(y) \end{aligned}$$

The composition a diffeomorphism generated by a conformal Killing field and a compensating Weyl transformation leaves the metric tensor invariant

$$g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(y) \tag{1.50}$$

and changes a matter field with Weyl-weight α according to

$$T^{\mu_1\dots}_{\nu_1\dots}(x) \longrightarrow \bar{T}^{\mu_1\dots}_{\nu_1\dots}(y) = e^{-\alpha\sigma(y)}\frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x^{\beta_1}}{\partial y^{\nu_1}}\cdots T^{\alpha_1\dots}_{\beta_1\dots}(x). \tag{1.51}$$

The infinitesimal form of these transformations is

$$\delta_X T^{\mu_1\dots}_{\nu_1\dots} = \Big(L_X - \frac{2\alpha}{d}\,\mathrm{div}X\Big)T^{\mu_1\dots}_{\nu_1\dots} \tag{1.52}$$

For a diffeomorphism- and Weyl invariant theory the transformation (1.51) or its infinitesimal form (1.52) are symmetries. The important point is, that if $x \to y$ is a diffeomorphism generated by a conformal Killing field, then the *metric remains unchanged* under this transformation.

Let us apply these general results to Euclidean and Minkowski[8] spacetime. In the following we shall need these symmetry transformations for the conformal Killing field for scalar fields in arbitrary dimensions and gauge potentials in 4 dimensions:

$$\delta_X\phi = L_X\phi + \frac{d-2}{d}\partial_\alpha X^\alpha\,\phi \quad \text{and} \quad \delta_X A_\mu = L_X A_\mu.$$

Inserting the explicit expressions for the the conformal Killing fields we end up with the following infinitesimal conformal symmetries:

X^α	$\delta_X\phi$	$\delta_X A_\mu$
a^α	$X^\alpha\partial_\alpha\phi$	$X^\alpha\partial_\alpha A_\mu$
$\omega^\alpha_{\ \beta}x^\beta$	$X^\alpha\partial_\alpha\phi$	$X^\alpha\partial_\alpha A_\mu + \omega^{\ \alpha}_{\mu}A_\alpha$
λx^α	$\Big(X^\alpha\partial_\alpha + \lambda(d-2)\Big)\phi$	$\Big(X^\alpha\partial_\alpha + \lambda\Big)A_\mu$
$x^2c^\alpha - 2(c,x)x^\alpha$	$\Big(X^\alpha\partial_\alpha - 2(d-2)(c,x)\Big)\phi$	$\Big(X^\alpha\partial_\alpha + 2[(c,A)x_\mu - (x,A)c_\mu - (c,x)A_\mu]\Big)A_\mu$

For an arbitrary tensor field of weight α, the infinitesimal dilatations read

$$\delta_{X_D}T^{\alpha\beta\dots}_{\mu\nu\dots} = \lambda\Big(x^\mu + s - 2\alpha\Big)T^{\alpha\beta\dots}_{\mu\nu\dots},$$

where s is the number of covariant minus the number contravariant indices of $T^{\alpha\beta\dots}_{\mu\nu\dots}$. The number $\Delta = s - 2\alpha$ is the *conformal weight* of $T^{\alpha\beta\dots}_{\mu\nu\dots}$.

For completeness we recall the form of the conformal transformations:

[8] actually compactified Minkowski spacetime, see below

Translations	$y^\mu = x^\mu + a^\mu$
Lorentz transformations	$y^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \Lambda = e^\omega$
Dilatations	$y^\mu = e^\lambda x^\mu$
Special conformal transformations	$y^\mu = \big(1 + 2(c,x) + c^2x^2\big)^{-1}\big(x^\mu + c^\mu x^2\big)$

Let us assume, that $S[g_{\mu\nu}, \Phi]$ is Weyl-invariant, where Φ denotes all matter fields of the theory. Now we define the energy-momentum tensor according to

$$T_{\mu\nu} = \frac{2}{e}\frac{\delta}{\delta g^{\mu\nu}} S[g_{\mu\nu}, \Phi].$$

The Weyl-invariance implies

$$\frac{\delta S}{\delta\sigma} = 0 = \frac{\delta S}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta\sigma} + \frac{\delta S}{\delta\Phi}\frac{\delta\Phi}{\delta\sigma} = -eT^\mu_{\ \mu} + \alpha\frac{\delta S}{\delta\Phi}\Phi. \tag{1.53}$$

We see, that the $T^{\mu\nu}$ is traceless off-shell if the weights of all matter fields vanish. For example, using $\delta e = -\frac{1}{2}e\, g_{\mu\nu}\delta g^{\mu\nu}$ the variation of the Yang-Mills action under a change of the metric becomes

$$\delta\int\sqrt{|g|}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} = 2\int\sqrt{|g|}\delta g^{\mu\nu}\Big(F_{\mu\sigma}F_\nu^{\ \sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big)$$

and we can read off the *traceless energy-momentum tensor* in spaces with Minkowskian resp. Euclidean signatures

$$T^M_{\mu\nu} = -\frac{1}{g^2}\Big(F_{\mu\alpha}F_\nu^{\ \alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big) \quad T^E_{\mu\nu} = \frac{1}{g^2}\Big(F_{\mu\alpha}F_\nu^{\ \alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big). \tag{1.54}$$

To write this in terms of the chromoelectric and chromomagnetic fields, we insert

$$F^{\mu\nu}_{M,E} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{1.55}$$

$$F^E_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad , \quad F^M_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

It follows in particular, that

$$T^M_{00} = \frac{1}{2g^2}\Big(\vec{E}^{\,2} + \vec{B}^{\,2}\Big), \qquad T^E_{00} = \frac{1}{2g^2}\Big(\vec{E}^{\,2} - \vec{B}^{\,2}\Big).$$

For a scalar field in 4 dimensions the *improved* energy-momentum tensor, which is gotten by varying

$$S_\phi = \int\eta\Big(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \frac{d-2}{4(d-1)}R\phi^2\Big)$$

is traceless only on shell, since $\alpha \neq 0$.

Finally we note, that to each conformal Killing field there belongs a conserved current. Let $T^{\mu\nu}$ be the traceless energy-momentum tensor and X^μ a conformal Killing field. We define the *Bessel-Hagen current* belonging to X by

$$J_X^\mu = T^{\mu\nu} X_\nu.$$

Using the conservation of $T^{\mu\nu}$ we find

$$\nabla_\mu J_X^\mu = T^{\mu\nu} \nabla_\mu X_\nu = T^{\mu\nu} \frac{1}{2} \big(\nabla_\mu X_\nu + \nabla_\nu X_\mu \big) = \frac{1}{d} \mathrm{div}\, X \, T^{\mu\nu} g_{\mu\nu} = 0,$$

where we used the symmetry of $T^{\mu\nu}$, the conformal Killing equation $L_X g_{\mu\nu} \sim g_{\mu\nu} \mathrm{div}\, X$ and that $T^\mu_{\ \mu} = 0$. These conserved currents lead to conserved charges. Since

$$\frac{1}{e} \partial_\mu \big(e J_X^\mu \big) = \partial_\mu J_X^\mu + \frac{1}{e} (\partial_\mu e) J_X^\mu = \nabla_\mu J_X^\mu = 0,$$

these charges read

$$Q_X = \int_\Sigma \hat{e}\, J^0.$$

They are conserved if the fields fall off fast enough at spatial infinity. In this formula Σ is a spacelike hypersurface and $\hat{e}$ denotes the induced volume form on this hypersurface. In particular, in Minkowski spacetime there are $\frac{1}{2}(d+1)(d+2)$ conserved Bessel-Hagen currents. The corresponding charges are the d momenta, the angular momenta, dilatonic charge and d additional charges.