Non-perturbative Methods in Supersymmetric Theories

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Abstract

These notes are based on Graduate Lectures I gave over the past 5 years. The aim of these notes is to provide a short *introduction* to supersymmetric theories: supersymmetric quantum mechanics, Wess-Zumino models and supersymmetric gauge theories. A particular emphasis is put on the underlying structures and non-perturbative effects in $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=4$ Yang-Mills theories.

Extended version of lectures given at the
TROISIEME CYCLE DE LA PHYSIQUE EN SUISSE ROMANDE.

Contents

Chapter 1

Introduction

Supersymmetric theories are highly symmetric and beautiful. They unify fermions (matter) with bosons (carrier of forces), either in flat or in curved space-time. Supergravity theories with local supersymmetries unify the gravitational with the other interactions. The energy at which gravity and quantum effects become of *comparable strength* can be estimated from the only expression with the dimension of mass that can be formed from the fundamental constants of nature $\hbar, c$ and G: the PLANCK mass

$$m_{\text{Pl}} = \left(\frac{\hbar c}{G}\right)^{1/2} \approx 10^{19}\,\frac{\text{GeV}}{c^2}.$$

For a particle with this mass the SCHWARZSCHILD radius, where its gravitational field becomes strong, is just twice its COMPTON wavelength, which is the minimal length to which it can be localized,

$$r_{\text{S}} = \frac{2Gm_{\text{Pl}}}{c^2} = \frac{\hbar}{2m_{\text{Pl}}c} = 2\lambda_c \qquad \text{for} \quad m = m_{\text{Pl}}.$$

Supersymmetry (susy) transformations relate bosons to fermions,

$$\mathcal{Q}\,|\text{Boson}\rangle \sim |\text{Fermion}\rangle \quad \text{and} \quad \mathcal{Q}\,|\text{Fermion}\rangle \sim |\text{Boson}\rangle,$$

and hence relate particles with different spins. The particles fall into multiplets and the supersymmetry transforms different members of such a supermultiplet into each other. Each supermultiplet must contain at least one boson and one fermion whose spins differ by 1/2 and all states in a multiplet (of unbroken supersymmetry) have the same mass.

So far no experimental observation has revealed particles or forces which manifestly show such a symmetry. Yet supersymmetry has excited great enthusiasm in large parts of the community and more recently in the context of superstring theories. It has even been said of the theory that it

IS SO BEAUTIFUL IT MUST BE TRUE.

The first part of these lectures deals with *supersymmetric quantum mechanics*. There are at least three good reasons to consider such systems,
• they contain the essential structures of susy theories,
• they appear as lattice versions of susy field theories,
• they describe the infrared dynamics of susy field theories in finite volumes.
Some of these topics will be discussed in these lectures.

In the second part of these lectures we review some textbook material, in particular the Coleman-Mandula theorem, supersymmetry algebras, representation theory and simple supersymmetric models.

The third part contains more recent results on supersymmetric gauge theories with one, two and four supercharges, central charges, BPS-states and β-functions.

Notation:

symbols	range	meaning
$i, j, k, \dots$	$1, 2, \dots, d-1$	space indices
$\mu, \nu, \rho, \sigma, \dots$	$0, 1, \dots, d-1$	spacetime indices
$\alpha, \beta, \gamma, \delta \dots$	$1, \dots 2^{[d/2]}$	Dirac-spinor indices
$\alpha, \beta, \dot\alpha, \dot\beta \dots$	$1, \dots 2^{d/2-1}$	Weyl-spinor-indices (d even)
$A^\dagger, A^*, A^T$	A matrix	adjoint, complex conjugate and transpose of A

The (anti)symmetrization of a tensor $A_{\mu_1 \dots \mu_n}$ is

$$A_{(\mu_1 \dots \mu_n)} = \frac{1}{n!} \sum_\sigma A_{\sigma(\mu_1) \dots \sigma(\mu_n)}, \qquad A_{[\mu_1 \dots \mu_n]} = \frac{1}{n!} \sum_\sigma \text{sign}(\sigma) A_{\sigma(\mu_1) \dots \sigma(\mu_n)}.$$

Reading: The introductory books and review articles [1]-[10] I found useful when preparing these lecture notes.

Chapter 2

Supersymmetric QM

In this chapter we examine simple $1+0$-dimensional supersymmetric (susy) field theories. In $1+0$ dimensions the POINCARÉ algebra reduces to time translations generated by the HAMILTONian H and the hermitian field and momentum operators $\phi(t)$ and $\pi(t)$ may be viewed as position and momentum operators of a point particle on the real line in the HEISENBERG-picture. Hence susy field theories in $1+0$ dimensions are particular quantum mechanical systems [11]. There are no technical difficulties hiding the essential structures. Such systems are interesting in their own right since they describe the infrared-dynamics of susy field theories in finite volumes [12]. This observation may be used to improve our understanding of supersymmetric quantum field theories beyond perturbation theory. A susy quantum mechanics with 16 supercharges also emerges in the matrix theory description of M theory [13]. In mathematical physics susy QM has been useful in proving index theorems for physically relevant differential operators [14]. There exist several extensive texts on susy quantum mechanics [15, 16, 17] in which the one-dimensional systems are discussed in detail. But some of the material presented in these notes (in particular on higher-dimensional systems) is not found in reviews.

2.1 Pairing and ground states

The HILBERT space of a supersymmetric system is the sum of its bosonic and fermionic subspaces, $\mathcal{H} = \mathcal{H}_{\mathsf{B}} \oplus \mathcal{H}_{\mathsf{F}}$. Let A be a linear operator $\mathcal{H}_{\mathsf{F}} \to \mathcal{H}_{\mathsf{B}}$. In most cases it is a first order differential operator. We shall use a block notation such that the vectors in $\mathcal{H}_{\mathsf{B}}$ have upper components and those in $\mathcal{H}_{\mathsf{F}}$ lower ones

$$|\psi\rangle = \begin{pmatrix} |\psi_{\mathsf{B}}\rangle \\ |\psi_{\mathsf{F}}\rangle \end{pmatrix}.$$

Then the nilpotent *supercharge* and its adjoint take the form

$$\mathcal{Q} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Q}^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \Longrightarrow \{\mathcal{Q},\mathcal{Q}\} = 0. \tag{2.1}$$

The block-diagonal super-HAMILTONian

$$H \equiv \{\mathcal{Q},\mathcal{Q}^\dagger\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_{\mathsf{B}} & 0 \\ 0 & H_{\mathsf{F}} \end{pmatrix}, \tag{2.2}$$

commutes with the supercharge

$$[\mathcal{Q}, H] = 0. \tag{2.3}$$

It is useful to introduce the (fermion) number operator

$$\mathsf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4}$$

which commutes with H. Bosonic states have number $\mathsf{N} = 0$ and fermionic states $\mathsf{N} = 1$. The supercharge and its adjoint decrease and increase this number by one,

$$[\mathsf{N}, \mathcal{Q}] = -\mathcal{Q} \quad \text{and} \quad [\mathsf{N}, \mathcal{Q}^\dagger] = \mathcal{Q}^\dagger. \tag{2.5}$$

The energies of the partner-HAMILTONians H_{B} and H_{F} in (2.2) are either zero or positive. A bosonic state in $\mathcal{H}_{\mathsf{B}}$ has zero energy if and only if it is annihilated by $A^\dagger$ and a fermionic state in $\mathcal{H}_{\mathsf{F}}$ has zero energy if and only if it is annihilated by A,

$$\begin{aligned} H_{\mathsf{B}}|0_{\mathsf{B}}\rangle = 0 \quad &\Longleftrightarrow \quad A^\dagger|0_{\mathsf{B}}\rangle = 0 \\ H_{\mathsf{F}}|0_{\mathsf{F}}\rangle = 0 \quad &\Longleftrightarrow \quad A\,|0_{\mathsf{F}}\rangle = 0. \end{aligned} \tag{2.6}$$

The states with positive energies come in pairs. Let $|\psi_{\mathsf{F}}\rangle$ be a fermionic eigenstate with *positive energy*,

$$H_{\mathsf{F}}|\psi_{\mathsf{F}}\rangle = A^\dagger A|\psi_{\mathsf{F}}\rangle = E|\psi_{\mathsf{F}}\rangle, \qquad E > 0.$$

It follows, that $A|\psi_{\mathsf{F}}\rangle$ is a bosonic eigenfunction with the same energy,

$$H_{\mathsf{B}}(A|\psi_{\mathsf{F}}\rangle) = (AA^\dagger)A|\psi_{\mathsf{F}}\rangle = A(A^\dagger A)|\psi_{\mathsf{F}}\rangle = AH_{\mathsf{F}}|\psi_{\mathsf{F}}\rangle = E(A|\psi_{\mathsf{F}}\rangle).$$

The fermionic state $|\psi_{\mathsf{F}}\rangle \in \mathcal{H}_{\mathsf{F}}$ and its partner state

$$|\psi_{\mathsf{B}}\rangle = \frac{1}{\sqrt{E}}\,A|\psi_{\mathsf{F}}\rangle \in \mathcal{H}_{\mathsf{B}} \tag{2.7}$$

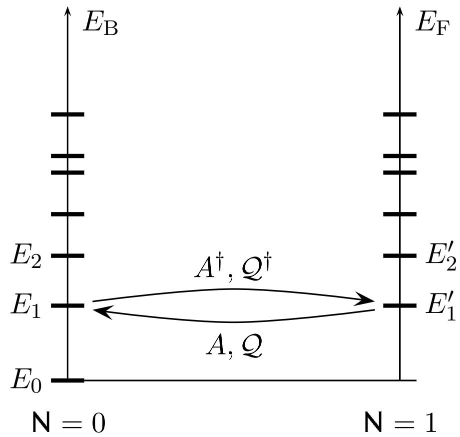
have equal norms,

$$\langle\psi_{\mathsf{B}}|\psi_{\mathsf{B}}\rangle = \frac{1}{E}\langle\psi_{\mathsf{F}}|A^\dagger A|\psi_{\mathsf{F}}\rangle = \langle\psi_{\mathsf{F}}|\psi_{\mathsf{F}}\rangle \tag{2.8}$$

and this proves, that the partner state of any excited state is never the null-vector. Likewise, the nontrivial partner state of any bosonic eigenstate $|\psi_{\mathsf{B}}\rangle \in \mathcal{H}_{\mathsf{B}}$ with positive Energy E is

$$|\psi_{\mathsf{F}}\rangle = \frac{1}{\sqrt{E}}\,A^\dagger|\psi_{\mathsf{B}}\rangle \in \mathcal{H}_{\mathsf{F}}. \tag{2.9}$$

This then proves that the partner HAMILTONians H_{B} and H_{F} have identical spectra, up to possible zero-modes.



The pairing of the non-zero energies and eigenfunctions in supersymmetric quantum mechanics is depicted in the figure on the left. The supercharge $\mathcal{Q}^\dagger$ containing $A^\dagger$ maps bosonic eigenfunctions into fermion ones and $\mathcal{Q}$ containing A maps fermionic eigenfunctions into bosonic ones. For potentials with scattering states there is a corresponding relation between the transmission and reflection coefficients of H_{B} and H_{F}, see below.

In SQM on $\mathbb{R}$ the operator A and its adjoint read in position space

$$A = i\partial_x + iW(x) \quad , \quad A^\dagger = i\partial_x - iW(x) \tag{2.10}$$

and the partner HAMILTONians take the familiar forms

$$\begin{aligned} H_{\mathsf{B}} &= p^2 + V_{\mathsf{B}}, \qquad V_{\mathsf{B}} = W^2 + W' \\ H_{\mathsf{F}} &= p^2 + V_{\mathsf{F}}, \qquad V_{\mathsf{F}} = W^2 - W'. \end{aligned} \tag{2.11}$$

For such simple systems we can find the ground state(s) of the super-HAMILTONian explicitly. With (2.6) we must study the first order differential equations

$$\begin{aligned} (\partial_x - W(x))\, \psi_{\mathsf{B}}(x) &= 0 \\ (\partial_x + W(x))\, \psi_{\mathsf{F}}(x) &= 0. \end{aligned} \tag{2.12}$$

The solutions are

$$\psi_{\mathsf{B}}(x) \propto e^{\chi(x)} \quad \text{and} \quad \psi_{\mathsf{F}}(x) \propto e^{-\chi(x)} \tag{2.13}$$

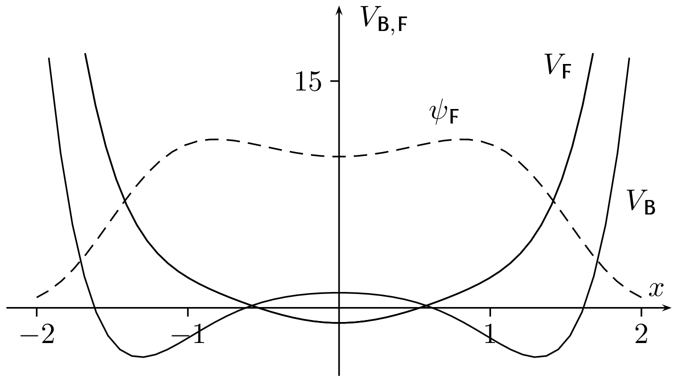
where we have introduced the function

$$\chi(x) = \int^x W(x')dx' \tag{2.14}$$

If at least one of the two solutions is normalizable then susy is *unbroken*. But since $\psi_{\mathsf{B}}(x) \cdot \psi_{\mathsf{F}}(x)$ is constant, at most one of the two solutions can be *normalized*. For example, for

$$W = \lambda x^p + o\left(x^p\right) \quad \text{and} \quad V_{\mathsf{B,F}} = \lambda^2 x^{2p} + o\left(x^{2p}\right)$$

supersymmetry is unbroken for odd p: For positive λ the ground state is fermionic and for negative λ it is bosonic. Below we have depicted the partner potentials and ground state wave function for $W = x(1-x^2)$. The corresponding partner HAMILTON operators have the same positive eigenvalues.



A long time ago, SCHRÖDINGER asked the following question [18]: Given a general non-negative HAMILTONian

$$H = p^2 + V(x) \geq 0 \tag{2.15}$$

in one dimension. Is there always a (in position space) first order differential operator $A = i\partial_x + iW(x)$, such that $H = H_{\mathsf{F}} = A^\dagger A$? This is the so-called *factorization-problem*. In one dimension every non-negative H can be factorized[1].

To construct the factorization we ompare (2.15) with (2.11) which leads to the nonlinear differential equation of RICATTI,

$$V(x) = V_{\mathsf{F}}(x) = W^2(x) - W'(x). \tag{2.16}$$

This equation is solved by the following well-known trick: setting

$$W(x) = -\frac{\psi_0'(x)}{\psi_0(x)}, \tag{2.17}$$

the RICATTI equation transforms into the linear SCHRÖDINGER equation for ψ_0,

$$-\psi_0'' + V\psi_0 = 0. \tag{2.18}$$

Since $H \geq 0$ the solution ψ_0 has no node and W in (2.17) is real and regular, as required. If the ground state energy E_0 of H is zero, then the transformation (2.17) is just the relation (2.13) between the superpotential and the ground state wave function in the fermionic sector. If E_0 is positive, then the solution ψ_0 will not be square integrable.

As a simple example we consider a constant positive potential $V_{\mathsf{F}} = c^2$. The non-normalizable solution of (2.18) is $\psi_0 = ae^{cx} + be^{-cx}$ and is used for the factorization,

$$c^2 = W^2 - W' \Longrightarrow W = -\frac{d}{dx}\log\psi_0. \tag{2.19}$$

[1]If H is bounded from below but has negative energies, then we add a big enough constant c^2 to H and factorize $H + c^2$.

The corresponding partner potential

$$V_{\mathsf{B}} = W^2 + W' = c^2 - 2\frac{d^2}{dx^2}\log\psi_0 \tag{2.20}$$

has exactly one zero-energy bound state and scattering states with energies bigger than c^2. For $a = b$ it is the reflectionless PÖSCHL-TELLER potential

$$V_{\mathsf{B}} = c^2\left(1 - \frac{2}{\cosh^2 cx}\right) \tag{2.21}$$

In the seminal paper by INFELD and HULL [19] the factorization method for second order differential equations has been worked out in great detail. It was applied to six possible factorization types. These types include the PÖSCHL-TELLER-, MORSE-, ROSEN-MORSE- and radial COULOMB potential.

2.2 SUSY breaking in SQM

The fact that susy has not been observed in nature so far does not imply that there are no practical uses for supersymmetric theories. It could be that every occurring supersymmetry is a broken one. We still would have a supercharge and super-HAMILTONian obeying the super algebra. But the symmetry could be *spontaneously broken*, in which case there is no invariant ground state.

In order for supersymmetry to exist and be unbroken, we require a ground state such that $H_{\mathsf{B}}|0\rangle = H_{\mathsf{F}}|0\rangle = 0|0\rangle$. This means that the ground state is annihilated by the generators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ of supersymmetry. Thus we have

$$\text{susy unbroken} \Longleftrightarrow \text{exist } |0\rangle \in \mathcal{H} \text{ with } \mathcal{Q}|0\rangle = \mathcal{Q}^\dagger|0\rangle = 0.$$

The Witten Index: WITTEN defined an index to determine whether supersymmetry is unbroken. Formally this index is

$$\Delta = \mathrm{Tr}\,(-1)^{\mathsf{N}}, \tag{2.22}$$

where N is the *fermion number*. For simplicity we assume that the spectrum of H is discrete and use the energy eigenfunctions to calculate Δ. Let us first assume that supersymmetry is broken, in which case there is *no* normalizable zero-energy state. Then all eigenstates of H have positive energies and come in pairs: one bosonic state with $\mathsf{N} = 0$ and one fermionic state with $\mathsf{N} = 1$ having the same energy. Their contribution to Δ cancel. Since all states with positive energy are paired we obtain $\Delta = 0$.

If there are n_{B} bosonic and n_{F} fermionic ground states then their contribution to the WITTEN index is $n_{\mathsf{B}} - n_{\mathsf{F}}$. Since the contributions of the excited states cancel pairwise we obtain

$$\Delta = n_{\mathsf{B}} - n_{\mathsf{F}}. \tag{2.23}$$

This yields an efficient method to determine whether susy is broken,

$$\Delta \neq 0 \Longrightarrow \text{supersymmetry is unbroken.} \tag{2.24}$$

The converse need not be true. It could be that susy is unbroken but the number of zero-energy states in the bosonic and fermionic sectors are equal so that Δ vanishes. This does not happen in one-dimensional SQM, so that

$$\Delta \neq 0 \Longleftrightarrow \text{supersymmetry is unbroken in SQM.} \tag{2.25}$$

Already in SQM the operator $(-)^{\mathsf{N}}$ is *in general not trace class* and its trace must be regulated for the WITTEN index to be well defined. A natural definition is

$$\Delta = \lim_{\alpha \searrow 0} \Delta(\alpha), \qquad \Delta(\alpha) = \mathrm{Tr}\left((-1)^{\mathsf{N}} e^{-\alpha H}\right). \tag{2.26}$$

In SQM with discrete spectrum $\Delta(\alpha)$ does not depend on α, since the contribution of all super partners cancel in (2.26). The contribution of the zero-energy states is still $n_{\mathsf{B}} - n_{\mathsf{F}}$. In field theories the excited states should still cancel in $\Delta(\alpha)$ in which case it is α-independent. Since $\Delta(\alpha)$ is constant, it may be evaluated for small α. But for $\alpha \searrow 0$ one can use the asymptotic expansion for the *heat kernel* of $\exp(-\alpha H)$ to actually calculate the WITTEN index.

2.3 Scattering states

Let us see, how supersymmetry relates the *transmission* and *reflection coefficients* of H_{B} and H_{F} for potentials supporting scattering states [20]. Thus we assume that the superpotential tends to constant values for large $|x|$,

$$\lim_{x\to\pm\infty} W(x) = W_\pm, \quad \text{such that} \quad \lim_{x\to\pm\infty} V_{\mathsf{B}}(x) = \lim_{x\to\pm\infty} V_{\mathsf{F}}(x) = W_\pm^2. \tag{2.27}$$

We consider an *incoming* plane wave from the left. The asymptotic form of the wave function for scattering from the one-dimensional potential V_{B} is given by

$$\psi_{\mathsf{B}}(k,x) \longrightarrow \begin{cases} e^{ikx} + R_{\mathsf{B}} e^{-ikx} & x \to -\infty \\ T_{\mathsf{B}}\, e^{ik'x} & x \to +\infty \end{cases}, \tag{2.28}$$

where R_{B} and T_{B} are the reflection and transmission coefficient in the bosonic sector. The properly normalized fermionic partner state has the asymptotic form (cp. 2.9)

$$\psi_{\mathsf{F}}(k,x) = -\frac{1}{k + iW_-} A^\dagger \psi_{\mathsf{B}}(x) \longrightarrow \begin{cases} e^{ikx} + R_{\mathsf{F}} e^{-ikx} & x \to -\infty \\ T_{\mathsf{F}}\, e^{ik'x} & x \to +\infty \end{cases}$$

with the following *reflection* and *transmission coefficients*,

$$R_{\mathsf{F}} = \frac{W_- + ik}{W_- - ik} R_{\mathsf{B}} \quad \text{and} \quad T_{\mathsf{F}} = \frac{W_+ - ik'}{W_- - ik} T_{\mathsf{B}}, \tag{2.29}$$

where k and k' are given by

$$k = (E - W_-^2)^{1/2} \quad \text{and} \quad k' = (E - W_+^2)^{1/2}. \tag{2.30}$$

The scattering data for the supersymmetric partners are not the same but they are related in this simple way. For real k, k' we have

$$|R_{\mathsf{B}}|^2 = |R_{\mathsf{F}}|^2 \quad \text{and} \quad |T_{\mathsf{B}}|^2 = |T_{\mathsf{F}}|^2$$

and the partner systems have identical reflection and transmission probability.
The transmission coefficients have *physical poles* in the upper half of the complex k-plane, their positions $k_j = i\kappa_j$ correspond to energies of bound states

$$E_j = W_-^2 - \kappa_j^2. \tag{2.31}$$

For negative W_- and positive W_+ there is one more zero-energy bound state in $\mathcal{H}_{\mathsf{F}}$ and for positive W_- and negative W_+ one more zero-energy bound state in $\mathcal{H}_{\mathsf{B}}$.
As an example we consider the kink

$$W(x) = -\lambda \tanh(x) \quad \text{with} \quad W_- = -W_+ = \lambda, \tag{2.32}$$

giving rise to the partner potentials

$$V_{\mathsf{B}}(\lambda; x) = \lambda^2 - \frac{\lambda(\lambda+1)}{\cosh^2 x} \quad \text{and} \quad V_{\mathsf{F}}(\lambda; x) = \lambda^2 - \frac{\lambda(\lambda-1)}{\cosh^2 x} \tag{2.33}$$

Supersymmetry, together with the shift-property

$$V_{\mathsf{F}}(\lambda; x) = V_{\mathsf{B}}(\lambda - 1; x) + 2\lambda - 1 \tag{2.34}$$

allows one to find the scattering coefficients for an infinite tower of potentials. Let us assume that we know the coefficients $R_{\mathsf{B}}(\lambda)$ and $T_{\mathsf{B}}(\lambda)$ for a certain value of the parameter λ. It follows that

$$T_{\mathsf{B}}(\lambda - 1) = T_{\mathsf{F}}(\lambda) \overset{(2.29)}{=} -\frac{\lambda + ik}{\lambda - ik} T_{\mathsf{B}}(\lambda) \tag{2.35}$$

and a similar formula for R_{B} and R_{F}. The iteration of this relation yields

$$T_{\mathsf{B}}(\lambda) = (-)^N \prod_{n=0}^{N-1} \frac{\lambda - n - ik}{\lambda - n + ik} T_{\mathsf{B}}(\lambda - N) \tag{2.36}$$

For $\lambda = N$ the transmission coefficient T_{B} on the right hand side is 1 so that

$$T_{\mathsf{B}}(N) = (-)^N \prod_{n=0}^{N-1} \frac{N - n - ik}{N - n + ik} \tag{2.37}$$

is the transmission coefficient for the Pöschl-Teller potentials with $\lambda \in \mathbb{N}_0$,

$$V_{\mathsf{B}} = N^2 - \frac{N(N+1)}{\cosh^2 x}. \tag{2.38}$$

The reflection coefficients for these potentials vanish since $R_{\mathsf{B}}(0) = 0$. The poles $k_n = i(N - n)$ of T_{B} yield the energies of the bound states,

$$E_n = N^2 - (N - n)^2 \qquad n = 0, \dots, N - 1. \tag{2.39}$$

Supersymmetry is unbroken, since the ground state has energy zero.

2.4 Shape invariance

Shape invariance is a property that arises when there is an additional relationship between the partner HAMILTONians H_{B} and H_{F}. Suppose that these HAMILTONians are linked by the condition

$$H_{\mathsf{F}}(\lambda) = A(\lambda)\, A^\dagger(\lambda) = A^\dagger(f(\lambda))\, A(f(\lambda)) + c(\lambda) = H_{\mathsf{B}}(f(\lambda)) + c(\lambda), \qquad (2.40)$$

where f is a mapping from the space of coupling constants into itself and $c(\lambda)$ is a real-valued function. When this condition holds, then the HAMILTON H_{B} is said to be *shape invariant* [21]. For example, the partner potentials (2.33) define a shape invariant system with $f(\lambda) = \lambda - 1$ and $c(\lambda) = 2\lambda - 1$.

One can readily derive recursion relations for the energies and scattering coefficients of a shape-invariant HAMILTONian on $\mathbb{R}$. As indicated in the figure on page 7 we denote the energy levels of H_{B} by E_n and those of H_{F} by E'_n. Then (2.40) implies

$$E'_{n+1}(\lambda) = E_n(f(\lambda)) + c(\lambda) \quad \text{for} \quad n \in \mathbb{N}_0,$$

while supersymmetry implies $E_n(\lambda) = E'_n(\lambda)$ for $n \in \mathbb{N}$. Combining these two properties yields

$$E_{n+1}(\lambda) = E'_{n+1}(\lambda) = E_n(f(\lambda)) + c(\lambda). \qquad (2.41)$$

Iterating this useful relation leads to

$$E_N(\lambda) = E_0\left(f_{\circ N}(\lambda)\right) + \sum_{n=0}^{N-1} c\left(f_{\circ n}(\lambda)\right), \qquad (2.42)$$

where $f_{\circ n}$ is the n-times iterated map f. This result yields an explicit formula for the energies E_N in case $H_{\mathsf{B}}\left(f_{\circ N}(\lambda)\right)$ admits a zero energy bound state.

The shape invariance (2.40) also implies that the *scattering coefficients* of $H_{\mathsf{F}}(\lambda)$ and $H_{\mathsf{B}}(f(\lambda))$ are the same. Together with the supersymmetric relations (2.29) we obtain the recursion relations

$$\begin{aligned} T_{\mathsf{B}}(\lambda) &= \prod_{n=0}^{N-1} \frac{W_-\left(f_{\circ n}(\lambda)\right) - ik}{W_+\left(f_{\circ n}(\lambda)\right) - ik'}\, T_{\mathsf{B}}\left(f_{\circ N}(\lambda)\right) \\ R_{\mathsf{B}}(\lambda) &= \prod_{n=0}^{N-1} \frac{W_-\left(f_{\circ n}(\lambda)\right) - ik}{W_-\left(f_{\circ n}(\lambda)\right) + ik}\, R_{\mathsf{B}}\left(f_{\circ N}(\lambda)\right). \end{aligned} \qquad (2.43)$$

For the kink in (2.32) with $W_- = \lambda$ and $f(\lambda) = \lambda - 1$ this simplifies to the formula (2.36) for the transmission coefficient of the PÖSCHL-TELLER potential.

2.4.1 Hydrogen atom in Einstein universe

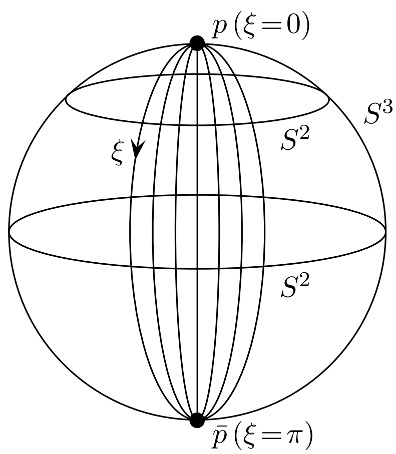
As an application we follow SCHRÖDINGER [22] and consider a hydrogen atom in a closed EINSTEIN universe with spatial line element on a 3-dimensional sphere S^3,

$$ds^2 = R^2\left(d\xi^2 + \sin^2\xi(d\theta^2 + \sin^2\theta\, d\varphi^2)\right).$$

The radial-type coordinate $R\xi$ takes its values in the finite interval $[0,\pi R]$. For a radially symmetric potential $V(\xi)$ the angular momenta on the equatorial 2-spheres commute with the SCHRÖDINGER operator and can be diagonalized. The radial SCHRÖDINGER equation takes the form ($\hbar=1$)

$$-\partial_\xi(\sin^2\xi\partial_\xi f)+\ell(\ell+1)f+\kappa\sin^2\xi\,(V(\xi)-E)\,f=0,\quad \kappa=2mR^2, \tag{2.44}$$

where $\ell\in\mathbb{N}_0$ denotes the angular momentum.



The COULOMB-type potential on the spatial section S^3 of the EINSTEIN universe reads

$$V=-\frac{e^2}{R}\cot\xi$$

and belongs to a proton at $\xi=0$ and an anti-proton at the opposite side of the universe, as depicted in the figure on the left. As for any closed space without boundary there is overall charge neutrality.

Setting $\psi=\sin\xi\,f$, the radial SCHRÖDINGER equation for ψ becomes

$$-\frac{d^2\psi}{d\xi^2}+V_{\mathsf{F}}\psi=\lambda\psi\,,\quad\text{where}\quad \lambda=1+\kappa E+a(\ell),\quad a(\ell)=\frac{\nu^2}{\ell^2}-\ell^2 \tag{2.45}$$

and we introduced $\nu=mRe^2$. The potential has the form

$$V_{\mathsf{F}}=\frac{\ell(\ell+1)}{\sin^2\xi}-2\nu\cot\xi+a(\ell)=W^2-\frac{dW}{d\xi}\quad\text{with}\quad W=\ell\cot\xi-\frac{\nu}{\ell}. \tag{2.46}$$

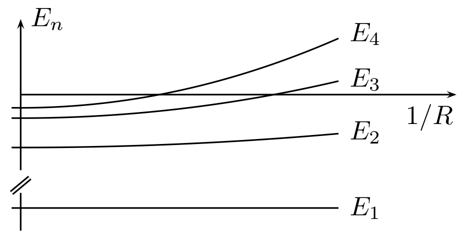
Actually this system is *shape invariant* with intertwining relation

$$V_{\mathsf{F}}(\ell)=V_{\mathsf{B}}(\ell+1)+c(\ell),\quad\text{where}\quad c(\ell)=a(\ell)-a(\ell+1). \tag{2.47}$$

There is one (non-normalizable) fermionic zero-mode, and the general formula (2.42) yields the eigenvalues $\lambda_N=a(\ell)-a(\ell+N)$. Setting $\ell+N=n$ we end up with the following energies for hydrogen in an Einstein universe,

$$E_n=E'_n=\frac{n^2-1}{2mR^2}-\frac{mc^2}{2}\left(\frac{\alpha}{n}\right)^2,\qquad n=1,2,\dots. \tag{2.48}$$

The E_n have no upper limit and all eigenvalues are discrete. With the exception of the ground state energy all energy levels will be shifted as a result of the interaction of the atom with the curvature of space.



The effect differs from the usual gravitational and Doppler shifts in that it perturbs each energy level to a different extend. As expected, for $R \to \infty$ we recover the energy levels of the hydrogen atom in flat space.

2.5 Isospectral deformations

Let us assume that V_{F} supports n bound states. By using supersymmetry one can easily construct an n-parameter family of potentials $V(\lambda; x)$, $\lambda = (\lambda_1, \dots, \lambda_n)$, for which the HAMILTONian has the same energies and scattering coefficients as $H = p^2 + V_{\mathsf{F}}$. The existence of such families of isospectral potentials has been known for a long time from the inverse scattering approach [23] which is technically more involved than the method based on supersymmetry. We show how a *one-parameter* isospectral family of potentials is obtained by first deleting and then re-inserting the ground state of V_{F} using the DARBOUX-procedure [24]. The generalization to an n-parameter family is described in [25].

Suppose that $\psi_{\mathsf{F}}(x)$ is a normalizable zero-energy ground state of the HAMILTONian with potential $V_{\mathsf{F}} = W^2 - W'$. Its explicit form in position space is

$$\psi_{\mathsf{F}}(x) \propto e^{-\chi(x)}, \qquad \chi(x) = \int^x W(x')dx'. \tag{2.49}$$

Suppose further that the partner potential $V_{\mathsf{B}} = W^2 + W'$ is kept fixed. A natural question is whether there are other superpotentials leading to the *same* potential V_{B}. A second solution $\hat{W} = W + \phi$ gives rise to the same V_{B} if

$$(\hat{W}^2 + \hat{W}') - (W^2 + W') = \phi^2 + 2W\phi + \phi' = 0. \tag{2.50}$$

The transformation $\phi = (\log F)'$ leads to a linear differential equation for F'

$$F'' + 2WF' = 0 \tag{2.51}$$

with solution

$$F'(x) = \exp\left(-2\int^x W(x')dx'\right) = \psi_{\mathsf{F}}^2(x). \tag{2.52}$$

The integration constant is just the lower bound of the integral which determines the norm of the fermionic ground state. A further integration yields F and hence ϕ and introduces another constant λ which is identified with the deformation parameter,

$$\phi(x,\lambda) = \frac{d}{dx}\log\left(I(x) + \lambda\right), \qquad I(x) = \int_{-\infty}^{x} \psi_{\mathsf{F}}^2(x')dx' \tag{2.53}$$

In this formula for ϕ we could change the lower integration bound or multiply I with any non-vanishing constant. This is equivalent to a redefinition of the constant λ. By construction W and $\hat{W} = W + \phi$ lead to the same V_{B}. But the corresponding partner potentials are different,

$$\hat{W}^2 - \hat{W}' = W^2 - W' + \phi^2 + 2W\phi - \phi' \overset{(2.50)}{=} V_{\mathsf{F}} - 2\phi'. \tag{2.54}$$

Thus the fermionic HAMILTON operators with superpotentials W and $W + \phi$ are unequal. But since they share the same partner HAMILTONian H_{B} they must have the same spectrum, up to possible zero modes. This then proves that all HAMILTON operators of the one-parameter family

$$H_{\mathsf{F}}(\lambda) = -\frac{d^2}{dx^2} + V_{\mathsf{F}}(\lambda; x), \tag{2.55}$$

$$V_{\mathsf{F}}(\lambda; x) = V_{\mathsf{F}}(x) - 2\frac{d^2}{dx^2}\log\left(I(x) + \lambda\right) \tag{2.56}$$

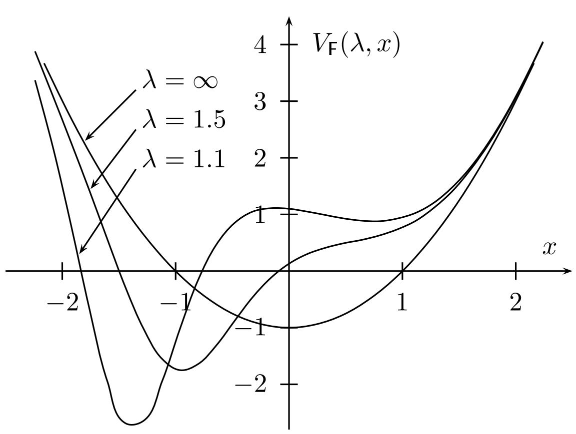
have the same spectrum, up to possible zero modes. The isospectral deformation (2.55) depends via the function $I(x)$ on the ground state wave function of the undeformed operator H_{F}.

Deformation of the harmonic oscillator: Let us see how the deformation looks like for the harmonic oscillator with potential $V_{\mathsf{F}} = \omega^2 x^2 - \omega$ and ground state wave function $\psi_{\mathsf{F}}(x) \propto \exp(-\omega x^2/2)$. We obtain

$$\phi(\lambda, x) = 2\sqrt{\frac{\omega}{\pi}}\,\frac{e^{-\omega x^2}}{\mathrm{erf}(\sqrt{\omega}\,x) + \lambda}, \quad \text{where} \quad \mathrm{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}dt \tag{2.57}$$

is the error function, and this leads to the following isospectral deformation

$$V_{\mathsf{F}}(\lambda; x) = V_{\mathsf{F}}(x) + 4\omega x\,\phi(\lambda, x) + 2\phi^2(\lambda, x) = V_{\mathsf{F}}(-\lambda; -x). \tag{2.58}$$



In the figure on the left we have plotted the potential of the harmonic oscillator and two deformations with parameters $\lambda = 1.5$ and $\lambda = 1.1$. We have set $\omega = 1$. For the deformed potential to be regular we must assume $|\lambda| > 1$. For $\lambda \to \pm\infty$ the potential tends to the potential of the harmonic oscillator. For $|\lambda| \downarrow 1$ the deviation from the oscillator potential become significant near the origin.

Deformation of reflectionless Pöschl-Teller potentials: We deform the reflectionless PÖSCHL-TELLER Potential

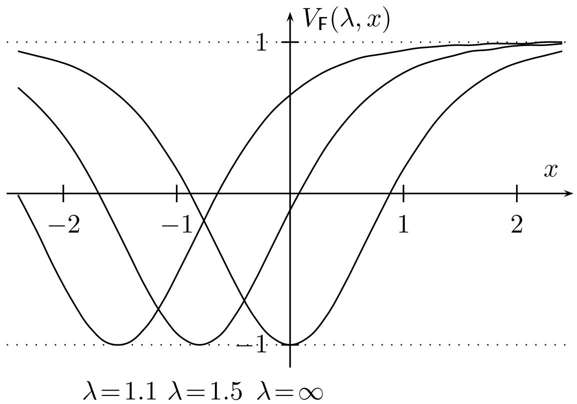
$$V_{\mathsf{F}} = 1 - 2\cosh^{-2} x, \tag{2.59}$$

with just *one supersymmetric bound state*, $\psi_{\mathsf{F}}(x) = 1/\cosh x$. Since $\int^x \psi_{\mathsf{F}}^2 = \tanh x$ we obtain

$$\phi(x) = \frac{1}{(\cosh x)^2} \frac{1}{\tanh x + \lambda} \tag{2.60}$$

giving rise to the following isospectral deformation of V_{F},

$$V_{\mathsf{F}}(\lambda, x) = V_{\mathsf{F}}(x) + 4\tanh(x)\phi(x) + 2\phi^2(x) = V_{\mathsf{F}}(-\lambda, -x). \tag{2.61}$$



In the figure to the left we have plotted the reflectionless PÖSCHL-TELLER potential with one bound state and two of its isospectral deformations with parameters $\lambda = 1.5$ and 1.1. For the deformed potential to be regular we must assume $|\lambda| > 1$. For $\lambda \to \pm\infty$ the potential tends to the PÖSCHL-TELLER potential. For $\lambda \downarrow 1$ the minimum of the potential tends to $-\infty$ and for $\lambda \uparrow -1$ to ∞.

The potential V_{F} may be viewed as a *soliton* with its center at the minimum. For $\lambda = 1$ the soliton is at $x = -\infty$ and moves to the origin for $\lambda \to \infty$. For $\lambda = -1$ the soliton is centered at ∞ and moves with decreasing λ to the left. For $\lambda = -\infty$ it reaches the origin. Actually one can show that for $\lambda = -\coth(4t + c)$ the function $u(t, x) = V_{\mathsf{F}}(\lambda(t), x) - 1$ solves the wellknown KORTEWEG-DEVRIES equation,

$$u_t + u_{xxx} - 6uu_x = 0. \tag{2.62}$$

For a generalization of this construction to n-soliton solutions I refer to the review of F. Cooper et al. [15].

2.6 SQM in higher dimensions

Supersymmetric quantum mechanical systems also exist in higher dimensions [26]. The construction is motivated by the following rewriting of the supercharge (2.1):

$$\mathcal{Q} = \psi \otimes A \quad \text{and} \quad \mathcal{Q}^\dagger = \psi^\dagger \otimes A^\dagger \tag{2.63}$$

containing the *fermionic* operators

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.64}$$

with anti-commutation relations

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0 \quad \text{and} \quad \{\psi, \psi^\dagger\} = \mathbb{1}. \tag{2.65}$$

For the choice $A = i\partial_x + iW$ as in (2.10) the super-HAMILTONian (2.2) reads

$$\begin{aligned} H &= \left(p^2 + W^2\right)\{\psi, \psi^\dagger\} + W'[\psi, \psi^\dagger] \\ &= H_{\mathsf{B}} - 2W'\psi^\dagger\psi = H_{\mathsf{F}} + 2W'\psi\psi^\dagger, \end{aligned} \tag{2.66}$$

where we skipped the tensor product symbol. In [26] this construction has been generalized to $d > 1$ dimensions. Then one has d fermionic annihilation operators ψ_k and d creation operators $\psi_k^\dagger$,

$$\{\psi_k, \psi_\ell\} = \{\psi_k^\dagger, \psi_\ell^\dagger\} = 0 \quad \text{and} \quad \{\psi_k, \psi_\ell^\dagger\} = \delta_{k\ell}, \quad k, \ell = 1, \dots, d. \tag{2.67}$$

For the supercharge we make the ansatz

$$\mathcal{Q} = i\sum \psi_k\left(\partial_k + W_k(\boldsymbol{x})\right) = i\boldsymbol{\psi}\cdot(\nabla + \boldsymbol{W}),$$

where $\boldsymbol{\psi}$ denotes the d-tupel $(\psi_1, \dots, \psi_d)^T$. The supercharge is *nilpotent* if and only if $\partial_k W_\ell - \partial_\ell W_k = 0$ holds true. Locally this integrability condition is equivalent to the existence of a potential $\chi(\boldsymbol{x})$ with $\boldsymbol{W} = \nabla\chi$. Thus we are lead to the following *nilpotent* supercharge and its adjoint,

$$\begin{aligned} \mathcal{Q} &= i\boldsymbol{\psi}\cdot(\nabla + \nabla\chi) = e^{-\chi}\mathcal{Q}_0\, e^{\chi} \quad \text{with} \quad \mathcal{Q}_0 = i\boldsymbol{\psi}\cdot\nabla \\ \mathcal{Q}^\dagger &= i\boldsymbol{\psi}^\dagger\cdot(\nabla - \nabla\chi) = e^{\chi}\mathcal{Q}_0^\dagger\, e^{-\chi} \quad \text{with} \quad \mathcal{Q}_0^\dagger = i\boldsymbol{\psi}^\dagger\cdot\nabla. \end{aligned} \tag{2.68}$$

The super HAMILTONian takes the simple form

$$\begin{aligned} H = \{\mathcal{Q}, \mathcal{Q}^\dagger\} &= H_0 \otimes \mathbb{1}_{2^d} - 2\sum \psi_k^\dagger\psi_\ell\, \partial_k\partial_\ell\chi \\ &= H_d \otimes \mathbb{1}_{2^d} + 2\sum \psi_k\psi_\ell^\dagger\, \partial_k\partial_\ell\chi, \end{aligned} \tag{2.69}$$

where H_0 and H_d are the SCHRÖDINGER operators in the extreme sectors,

$$H_0 = -\triangle + (\nabla\chi, \nabla\chi) + \triangle\chi \quad , \quad H_d = -\triangle + (\nabla\chi, \nabla\chi) - \triangle\chi \tag{2.70}$$

This super HAMILTON generalizes the NICOLAI-WITTEN operator (2.66) in one dimension to d dimensions.

Again there exists a (fermion) *number operator*

$$\mathsf{N} = \sum \psi_k^\dagger\psi_k \tag{2.71}$$

and the ψ_k decrease it by one unit whereas the $\psi_k^\dagger$ increase it by one unit. The same is then true for the supercharge and its adjoint,

$$[\mathsf{N}, \mathcal{Q}] = -\mathcal{Q} \quad \text{and} \quad [\mathsf{N}, \mathcal{Q}^\dagger] = \mathcal{Q}^\dagger. \tag{2.72}$$

A direct way to find a representation for the fermionic operators makes use of the FOCK construction over a 'vacuum'-state $|0\rangle$ which is annihilated by all ψ_k,

$$\psi_k|0\rangle = 0, \quad k = 1, \dots, d. \tag{2.73}$$

Acting with the d raising operators on $|0\rangle$ yields the states

$$|k\rangle = \psi_k^\dagger|0\rangle$$

with $\mathsf{N} = 1$. When counting the states with higher fermion number we should recall that the raising operators anticommute, such that

$$|k_1 \dots k_n\rangle = \psi_{k_1}^\dagger \cdots \psi_{k_n}^\dagger|0\rangle \tag{2.74}$$

is antisymmetric in $k_1, \dots, k_n$. The states and their corresponding eigenvalues of N together with their degeneracies are listed in the following table:

states:	$\vert 0\rangle$	$\vert k\rangle$	$\vert k,\ell\rangle$	$\cdots$	$\vert 1,2,\dots,d\rangle$
N	0	1	2	$\dots$	d
# of states	$\binom{d}{0}=1$	$\binom{d}{1}=d$	$\binom{d}{2}$	$\cdots$	$\binom{d}{d}=1$

The total number of independent states is 2^d and thus we obtain a 2^d-dimensional irreducible representation of the fermionic algebra (2.67). The states with even N are called *bosonic*, those with odd N *fermionic*. The number of bosonic states equals the number of fermionic states.

With the help of (2.67) and (2.73) one may calculate the matrix elements of ψ_k between any two FOCK states (2.74). In *one dimension* there is one bosonic and one fermionic state and for the orthonormal basis

$$e_1 = |0\rangle \quad \text{and} \quad e_2 = \psi^\dagger|0\rangle$$

we recover the annihilation operator ψ in (2.64). In *two dimensions* there are two bosonic and two fermionic states and with respect to the orthonormal basis

$$\{e_1, e_2, e_3, e_4\} = \{|0\rangle, |1\rangle, |2\rangle, |12\rangle\} \tag{2.75}$$

the annihilation operators are represented by

$$\psi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.76}$$

Taking into account the $\boldsymbol{x}$-dependency of the states, the HILBERT space of supersymmetric quantum mechanics in d dimensions is

$$\mathcal{H} = L_2(\mathbb{R}^d) \otimes \mathbb{C}^{2^d} \tag{2.77}$$

and decomposes into sectors with different fermion numbers,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_d \quad \text{with} \quad \mathsf{N}|_{\mathcal{H}_p} = p\mathbb{1}. \tag{2.78}$$

An arbitrary element in $\mathcal{H}$ has the expansion

$$\psi(x) = f(\boldsymbol{x})|0\rangle + f_k(\boldsymbol{x})|k\rangle + \frac{1}{2}f_{k\ell}(\boldsymbol{x})|k\ell\rangle + \frac{1}{3!}f_{k\ell m}(\boldsymbol{x})|k\ell m\rangle + \ldots . \tag{2.79}$$

Recalling that $\mathcal{Q}$ lowers and $\mathcal{Q}^\dagger$ raises N by one unit it follows at once that the super-HAMILTONian (2.69) commutes with N. In a basis adapted to the decomposition (2.78) the number operator is block-diagonal,

$$\mathsf{N} = \begin{pmatrix} 0\cdot\mathbb{1}_{\binom{d}{0}} & & & \\ & 1\cdot\mathbb{1}_{\binom{d}{1}} & & \\ & & \ddots & \\ & & & d\cdot\mathbb{1}_{\binom{d}{d}} \end{pmatrix} \tag{2.80}$$

and so is the super-HAMILTONian

$$H = \begin{pmatrix} H_0 & & & \\ & H_1 & & \\ & & \ddots & \\ & & & H_d \end{pmatrix} \qquad \begin{aligned} H_0 &= p^2 + (\nabla\chi^2) + \triangle\chi \\ H_d &= p^2 + (\nabla\chi^2) - \triangle\chi\,. \end{aligned} \tag{2.81}$$

Note that in the extremal sectors with $\mathsf{N}=0$ and $\mathsf{N}=d$ the super HAMILTONian reduces to ordinary SCHRÖDINGER operators.

SQM in higher dimensions with a nilpotent supercharge defines a *complex* of the following structure:

$$\mathcal{H}_0 \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \mathcal{H}_1 \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \mathcal{H}_2 \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \bullet\;\bullet\;\bullet\;\bullet \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \mathcal{H}_{d-1} \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \mathcal{H}_d$$

For the free supercharge $\mathcal{Q}_0$ this complex is isomorphic to the DE RHAM complex for differential forms. The nilpotent charge $\mathcal{Q}_0^\dagger$ is identified with the *exterior differential* d and $\mathcal{Q}_0$ with the co-differential δ. The super HAMILTONian $H = \{\mathcal{Q}_0, \mathcal{Q}_0^\dagger\}$ corresponds to the LAPLACE-BELTRAMI operator $-\triangle$.

The nilpotent supercharge gives rise to the following HODGE-type decomposition of the HILBERT space,

$$\mathcal{H} = \mathcal{Q}\mathcal{H} \oplus \mathcal{Q}^\dagger\mathcal{H} \oplus \mathcal{H}_0, \qquad \mathcal{H}_0 = \operatorname{Ker} H\ , \tag{2.82}$$

where the finite dimensional subspace $\mathcal{H}_0$ is spanned by the zero-modes of H. Indeed, on the orthogonal complement of $\mathcal{H}_0$ we may invert H and write (using $[H,\mathcal{Q}]=0$),

$$\mathcal{H}_0^\perp = (\mathcal{Q}\mathcal{Q}^\dagger + \mathcal{Q}^\dagger\mathcal{Q})H^{-1}\mathcal{H}_0^\perp = \mathcal{Q}\left(H^{-1}\mathcal{Q}^\dagger\mathcal{H}_0^\perp\right) + \mathcal{Q}^\dagger\left(H^{-1}\mathcal{Q}\mathcal{H}_0^\perp\right),$$

which proves (2.82).

Before we study two relevant systems we rewrite the superalgebra in terms of the hermitian supercharges $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in the decompositions

$$\mathcal{Q} = \tfrac{1}{2}(\mathcal{Q}_1 + i\mathcal{Q}_2) \quad \text{and} \quad \mathcal{Q}^\dagger = \tfrac{1}{2}(\mathcal{Q}_1 - i\mathcal{Q}_2). \tag{2.83}$$

$\mathcal{Q}_1$ and $\mathcal{Q}_2$ anticommute and both are roots of the super-HAMILTONian,

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = 2\delta_{ij}H. \tag{2.84}$$

This means that the systems with super-HAMILTONian (2.69) actually possess an *extended supersymmetry* with two real supercharges. More generally, the non-negative super-HAMILTONian of a SQM with $\mathcal{N}$ supersymmetries can be written as

$$\delta_{ij}H = \tfrac{1}{2}\{\mathcal{Q}_i, \mathcal{Q}_j\}, \qquad i,j = 1,\dots,\mathcal{N}, \tag{2.85}$$

with *hermitian* supercharges $\mathcal{Q}_i$ anticommuting with an involutary operator Γ,

$$\{\mathcal{Q}_i, \Gamma\} = 0, \qquad \Gamma^\dagger = \Gamma, \qquad \Gamma^2 = \mathbb{1}. \tag{2.86}$$

In our case Γ is just the number operator modulo 2.

There exist other definitions for SQM in the literature, for a recent discussion, in particular concerning the role of the grading operator Γ, we refer to [27]. One may also relax the condition for the left-hand side of (2.85), see for example [28], but in these lectures we will not consider such generalizations.

2.6.1 The $2d$ supersymmetric anharmonic oscillator

As an application we consider susy oscillators in $\mathbb{R}^2$ with polar coordinates,

$$z = x_1 + ix_2 = re^{i\varphi}. \tag{2.87}$$

They emerge in the *strong coupling limit* of certain WESS-ZUMINO-models on space lattices and for this reason we are interested in their vacuum structure. We choose the harmonic superpotential,

$$\chi(\boldsymbol{x}) = \frac{\lambda}{p}\, r^p \cos(p\varphi), \tag{2.88}$$

and obtain the following super-HAMILTONian in the basis (2.75),

$$H = \{\mathcal{Q}, \mathcal{Q}^\dagger\} = \begin{pmatrix} H_0 & 0 & 0 \\ 0 & H_1 & 0 \\ 0 & 0 & H_2 \end{pmatrix}, \tag{2.89}$$

with the well-studied anharmonic oscillator in the extremal sectors $\mathcal{H}_0$ and $\mathcal{H}_2$,

$$H_0 = H_2 = -\triangle + (\nabla\chi)^2 = -\triangle + \lambda^2(z\bar{z})^{p-1}. \tag{2.90}$$

and a matrix SCHRÖDINGER operator in $\mathcal{H}_1$,

$$H_1 = H_0 \cdot \mathbb{1}_2 + 2\lambda(p-1)\begin{pmatrix} -\Re z^{p-2} & \Im z^{p-2} \\ \Im z^{p-2} & \Re z^{p-2} \end{pmatrix} \tag{2.91}$$

The ground states of H are known [29, 30, 31], in contrast to the ground state(s) of non-supersymmetric anharmonic oscillator H_0. To construct these states we observe that the 'angular momentum'

$$J = L - s\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L = \frac{1}{i}\partial_\varphi, \quad s = \frac{p-2}{2}, \tag{2.92}$$

is conserved and that the ground states must reside in $\mathcal{H}_1$, since in the other sectors $H = H_0$ is positive. Diagonalizing J on $\mathcal{H}_1$ leads to the ansatz

$$\psi_j(\boldsymbol{x}) = e^{ij\varphi}\left(f_j(r)\, e^{-is\varphi}\sigma_3 + g_j(r)\, e^{is\varphi}\right)\begin{pmatrix} 1 \\ i \end{pmatrix}, \tag{2.93}$$

where the eigenvalues j of J are integers for even p and half-integers for odd p. The zero-energy conditions $\mathcal{Q}\psi_j = \mathcal{Q}^\dagger\psi_j = 0$ yields a coupled system of first order differential equations for the radial functions f_j and g_j. The square integrable solutions are just BESSEL functions,

$$f_j(r) = c\, r^{p-1}\, \mathrm{K}_{\frac{1}{2}+\frac{j}{p}}\left(\tfrac{\lambda}{p}r^p\right) \quad \text{and} \quad g_j(r) = c\, r^{p-1}\, \mathrm{K}_{\frac{1}{2}-\frac{j}{p}}\left(\tfrac{\lambda}{p}r^p\right), \tag{2.94}$$

where $j \in \{-s, -s+1, \dots, s-1, s\}$. The number of supersymmetric ground states of the oscillator with potential $\propto r^{2p-2}$ is just $p-1$. For example, the supersymmetric anharmonic oscillator with r^4 potential has 2 normalizable zero modes.

2.6.2 The supersymmetric hydrogen atom

For a closed system of two non-relativistic point masses interacting via a central force the angular momentum $\boldsymbol{L}$ of the relative motion is conserved and the motion is always in the plane perpendicular to $\boldsymbol{L}$. If the force is derived from the COULOMB potential, there is an additional conserved quantity: the LAPLACE-RUNGE-LENZ[2] vector. This vector is perpendicular to $\boldsymbol{L}$ and points in the direction of the semi-major axis. For the hydrogen atom the corresponding hermitian vector operator has the form

$$\boldsymbol{C} = \frac{1}{2m}(\boldsymbol{p}\times\boldsymbol{L} - \boldsymbol{L}\times\boldsymbol{p}) - \frac{e^2}{r}\boldsymbol{x}. \tag{2.95}$$

[2] A more suitable name for this constant of motion would be HERMANN-BERNOULLI-LAPLACE vector, see [32].

with reduced mass m of the proton-electron system. By exploiting the existence of this conserved vector operator, PAULI calculated the spectrum of the hydrogen atom by purely algebraic means [33]. He noticed that the angular momentum $\boldsymbol{L}$ together with the vector

$$\boldsymbol{K} = \sqrt{-\frac{m}{2H}}\,\boldsymbol{C}\ , \tag{2.96}$$

which is well-defined and hermitian on bound states with negative energies, generate a hidden $SO(4)$ symmetry algebra,

$$[L_i, L_j] = [K_i, K_j] = i\hbar\epsilon_{ijk}L_k\ , \quad [L_i, K_j] = i\hbar\epsilon_{ijk}K_k\ , \tag{2.97}$$

and that the HAMILTONian can be expressed in terms of $\boldsymbol{K}^2 + \boldsymbol{L}^2$, one of the two second-order CASIMIR operators of this algebra, as follows

$$H = -\frac{me^4}{2}\frac{1}{\boldsymbol{K}^2 + \boldsymbol{L}^2 + \hbar^2}\ . \tag{2.98}$$

One also notices that the second CASIMIR operator $\boldsymbol{K}\cdot\boldsymbol{L}$ vanishes and arrives at the bound state energies by purely group theoretical methods. The existence of the conserved vector $\boldsymbol{K}$ also explains the accidental degeneracy of the hydrogen spectrum.

In a recent publication with A. KIRCHBERG and D. LANGE we 'supersymmetrized' this construction and showed that the *supersymmetric hydrogen atom* admits generalizations of the angular momentum and LAPLACE-RUNGE-LENZ vector [34]. Similarly as for the ordinary COULOMB problem the hidden $SO(4)$-symmetry generated by these two vector operators allows for a purely algebraic solution of the supersymmetric system.

To find the supersymmetrized hydrogen atom we choose $\chi = -\lambda r$ in (2.69) and obtain in 3 dimensions the super-HAMILTONian [34]

$$H = (-\triangle + \lambda^2)\mathbb{1}_8 - \frac{2\lambda}{r}B, \qquad B = \mathbb{1} - \mathsf{N} + S^\dagger S, \quad S = \hat{\boldsymbol{x}}\cdot\boldsymbol{\psi} \tag{2.99}$$

on the HILBERT space

$$\mathcal{H} = L_2(\mathbb{R}^3)\times\mathbb{C}^8 = \mathcal{H}_0\oplus\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3. \tag{2.100}$$

We defined the triplet $\boldsymbol{\psi}$ containing the 3 annihilation operators ψ_1, ψ_2, ψ_3. States in $\mathcal{H}_0$ are annihilated by S and states in $\mathcal{H}_3$ by $S^\dagger$. With $\{S^\dagger, S\} = \mathbb{1}$ we obtain the following HAMILTON operators in these subspaces,

$$H_0 = -\triangle + \lambda^2 - \frac{2\lambda}{r} \quad \text{and} \quad H_3 = -\triangle + \lambda^2 + \frac{2\lambda}{r}. \tag{2.101}$$

Hence, the SCHRÖDINGER operators for both the electron-proton H_0 and positron-proton systems H_3 are part of H. The conserved angular momentum contains a spin-type term,

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{x}\wedge\boldsymbol{p} - i\boldsymbol{\psi}^\dagger\wedge\boldsymbol{\psi}, \tag{2.102}$$

and the operators S and B in (2.99) commute with this total angular momentum, since $\boldsymbol{x}$ and $\boldsymbol{\psi}$ are both vector operators. To find the susy extension of the RUNGE-LENZ vector is less simple. It reads [34]

$$\boldsymbol{C} = \boldsymbol{p} \wedge \boldsymbol{J} - \boldsymbol{J} \wedge \boldsymbol{p} - 2\lambda\,\hat{\boldsymbol{x}} B \tag{2.103}$$

with $\boldsymbol{J}$ from (2.102). The properly normalized vector

$$\boldsymbol{K} = \frac{1}{2}\frac{\boldsymbol{C}}{\sqrt{\lambda^2 - H}} \tag{2.104}$$

together with $\boldsymbol{J}$ form an $SO(4)$ symmetry algebra on the subspace of bound states for which $H < \lambda^2$.

Finally we would like to find a relation similar to (2.98) to solve for the spectrum. However, one soon realizes that there is no algebraic relation between the conserved operators $\mathbb{1}, \mathsf{N}, \boldsymbol{J}^2, \boldsymbol{K}^2$ and H. However, we can prove the equation

$$\lambda^2 \mathcal{C}_{(2)} = \boldsymbol{K}^2 H + \left(\boldsymbol{J}^2 + (1-\mathsf{N})^2\right)\mathcal{Q}\mathcal{Q}^\dagger + \left(\boldsymbol{J}^2 + (2-\mathrm{N})^2\right)\mathcal{Q}^\dagger\mathcal{Q}\,, \tag{2.105}$$

where $\mathcal{C}_{(2)}$ is the second-order CASIMIR of the dynamical symmetry group $SO(4)$,

$$\mathcal{C}_{(2)} = \boldsymbol{J}^2 + \boldsymbol{K}^2 \tag{2.106}$$

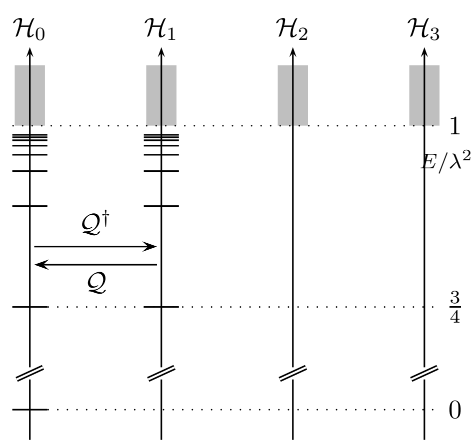
and this relation is sufficient to obtain the energies of the supersymmetric H-atom. Each of the three subspaces in the decomposition (2.82) is left invariant by H and thus we may diagonalize it on each subspace separately. Since $H|_{\mathcal{Q}\mathcal{H}} = \mathcal{Q}\mathcal{Q}^\dagger$ and $H|_{\mathcal{Q}^\dagger\mathcal{H}} = \mathcal{Q}^\dagger\mathcal{Q}$ we can solve (2.105) for H in both subspaces,

$$H|_{\mathcal{Q}\mathcal{H}} = \lambda^2\,\frac{\mathcal{C}_{(2)}}{(1-\mathsf{N})^2 + \mathcal{C}_{(2)}} \quad \text{and} \quad H|_{\mathcal{Q}^\dagger\mathcal{H}} = \lambda^2\,\frac{\mathcal{C}_{(2)}}{(2-\mathsf{N})^2 + \mathcal{C}_{(2)}}. \tag{2.107}$$

The states with zero energy are annihilated by both $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, and according to (2.105) the second-order CASIMIR must vanish on these modes, such that

$$\mathcal{C}_{(2)}|_{\mathrm{Ker}\,H} = 0\,.$$

We conclude that every supersymmetric ground state of H is an $SO(4)$ singlet.



In the figure on the left we have plotted the spectrum of the supersymmetric H-atom. The bound states are in the sectors with fermion number 0 and 1. The allowed $SO(4)$-representations, energies, degeneracies and wave functions have been calculated in [34] with group theoretical methods. Actually the HAMILTONian (2.69) with $\chi(\boldsymbol{x}) = -\lambda/r$ can be diagonalized in arbitrary dimensions. Again there exist a generalized angular momentum and RUNGE-LENZ vector which generate a symmetry $SO(d+1)$.

Much work went into investigating supersymmetric QM since the pioneering papers of Infeld and Hull [19], Witten [11] and Gendenshtein [21]. It is impossible to cover all topics in one chapter and I presented those which I personally find most interesting and/or to which we contributed. For example, I omitted the formulation of SQM in superspace and the interesting interrelation with index theorems. I omitted the applications of SQM to atomic, nuclear and statistical physics. Also there is a close relation between 'shape-invariant susy' and group theory. Most of these aspects are covered in the reviews [15, 16, 17]. Very recently the following question has been answered: Given a Dirac operator $i\not{D}$ for charged particles in a (Euclidean) curved space. What are the conditions on the background gauge field A_μ and metric $g_{\mu\nu}$ such that $-\not{D}^2$ possesses $\mathcal{N} > 1$ first order hermitian differential operator $\mathcal{Q}_i$ as square roots and that these roots form an extended superalgebra

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = 2\delta_{ij} H, \qquad i, j = 1, \ldots, \mathcal{N}. \tag{2.108}$$

For example, one finds $\mathcal{N} = 2$ hermitian supercharges in Kähler spaces and for particular gauge fields and $\mathcal{N} = 4$ supercharges in hyper-Kähler spaces and (in 4 dimensions) selfdual or anti-selfdual gauge fields. The solutions for arbitrary $\mathcal{N}$ and space dimensions can be found in [35].

Chapter 3

Symmetries and Spinors

Some time ago the idea came up that perhaps the approximate $SU(3)$ symmetry of strong interaction is part of a larger $SU(6)$ symmetry and that mesons (or baryons) with different spins belong to *one multiplet* of this bigger symmetry group. Various attempts were made to generalize this symmetry of the non-relativistic quark model to a fully relativistic QFT. These attempts failed, and several authors proved no-go theorems showing that in fact this is impossible. Well-known is the COLEMAN-MANDULA theorem [36] which states that in a theory with *nontrivial scattering* in more than $1+1$ dimensions, the only possible conserved quantities that transform as tensors under the LORENTZ group (i.e. without spinor indexes) are the 4-momentum P_μ generating spacetime translations, the generalized angular momenta $J_{\mu\nu}$ generating LORENTZ transformations and possible 'internal' symmetry charges B_k which commute with P_μ and $J_{\mu\nu}$. The $(P_\mu, J_{\mu\nu})$ generate the POINCARÉ *algebra* $\mathcal{P}$,

$$
\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}) && (3.1)\\
[J_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) && (3.2)\\
[P_\mu, P_\nu] &= 0 && (3.3)
\end{aligned}
$$

with $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, \ldots, -1)$. It is the symmetry algebra of any relativistic field theory. There exists an extension of the COLEMAN-MANDULA theorem for *massless particles* which allows for the generators of conformal transformations.

3.1 Coleman-Mandula theorem

In order to understand the COLEMAN-MANDULA theorem better we consider the theory for two free real scalar fields with LAGRANGEan density

$$\mathcal{L}_0' = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 \tag{3.4}$$

and linear EULER-LAGRANGE equations

$$\Box\phi_i = 0, \qquad i = 1, 2. \tag{3.5}$$

Besides the conserved energy-momentum and 'angular momentum' tensors

$$T_{\mu\nu} = \sum_i \partial_\mu\phi_i\partial_\nu\phi_i - \eta_{\mu\nu}\mathcal{L}_0' \quad \text{and} \quad J_{\mu\rho\sigma} = \tfrac{1}{2}x_\sigma T_{\mu\rho} - \tfrac{1}{2}x_\rho T_{\mu\sigma} \tag{3.6}$$

such a model has infinitely many conserved currents. For example it follows immediately that the series of currents

$$J_{\mu\rho_1\dots\rho_n} = \phi_1\partial_\mu\left(\partial_{\rho_1}\dots\partial_{\rho_n}\phi_2\right) - \partial_\mu\phi_1\left(\partial_{\rho_1}\dots\partial_{\rho_n}\phi_2\right) \tag{3.7}$$

are conserved for solutions of the field equations. One says they are conserved *on shell*, since for solutions the 4-momentum lies on the mass shell. The corresponding conserved charges

$$Q_{\rho_1\dots\rho_n} = \int d\boldsymbol{x}\, J_{0\rho_1\dots\rho_n}, \tag{3.8}$$

where one integrates with $d\boldsymbol{x}$ over space, are tensorial charges of higher rank. According to the Coleman-Mandula theorem these conservation laws can not be extended to the interacting case in $d > 2$ dimensions [36, 37], since all additional conserved charges must be Lorentz scalars.
The theorem does not apply to spinorial charges, though. Let us add a Majorana spinor in 4 dimensions to the above system and consider the LagrangeEan

$$\mathcal{L}_0 = \mathcal{L}_0' + \tfrac{i}{2}\bar\psi\partial\!\!\!/\psi \tag{3.9}$$

The equations of motion are (3.5) supplemented by the free Dirac equation $\partial\!\!\!/\psi = 0$. Now there is an infinite number of conserved currents with *spinor indexes*, e.g.

$$\begin{aligned} S_{\mu\alpha} &= \partial_\rho(\phi_1 - i\phi_2)(\gamma^\rho\gamma_\mu\psi)_\alpha \\ S_{\mu\nu\alpha} &= \partial_\rho(\phi_1 - i\phi_2)(\gamma^\rho\gamma_\mu\partial_\nu\psi)_\alpha. \end{aligned} \tag{3.10}$$

For example, the first current is conserved since in

$$\partial_\mu S^\mu_{\ \alpha} = \partial_\mu\partial_\rho(\phi_1 - i\phi_2)(\gamma^\rho\gamma^\mu\psi)_\alpha + \partial_\rho(\phi_1 - i\phi_2)(\gamma^\rho\partial\!\!\!/\psi)_\alpha.$$

the first term is proportional to $\Box(\phi_1 - i\phi_2)$ and the second term to $\partial\!\!\!/\psi$. Adding the interaction

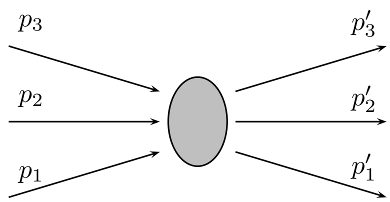
$$\mathcal{L}_1 = -g\bar\psi\left(\phi_1 - i\gamma_5\phi_2\right)\psi - \tfrac{1}{2}g^2\left(\phi_1^2 + \phi_2^2\right)^2 \tag{3.11}$$

to the LagrangeEan $\mathcal{L}_0$ of the free model, $S_{\mu\alpha}$ (with corrections proportional to g) remains conserved. But a current with more indexes can never be deformed to a conserved current of the interacting theory.
After these preliminaries we formulate the theorem more precisely. The Hilbert space of scattering theory, $\mathcal{H}$ is the infinite sum of $n-$particle subspaces

$$\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)} \oplus \dots, \tag{3.12}$$

where $\mathcal{H}^{(n)}$ is the (properly symmetrized) subspace of the tensor product of one-particle HILBERT spaces. The scattering matrix is then a unitary operator on $\mathcal{H}$ describing all possible scattering processes of the underlying theory. It is usually written as

$$S = \mathbb{1} - i(2\pi)^4\delta^4(P - P')T, \tag{3.13}$$



where the delta distribution ensures the conservation of the total energy and momentum during the scattering process. In (3.13) $P = \sum p$ denotes the sum of the ingoing 4-momenta and $P' = \sum p'$ the sum of the outgoing momenta.

A unitary operator U on $\mathcal{H}$ is a *symmetry transformation* of the S-matrix if

- it maps 1-particle states into 1-particle states;
- it acts on many-particle states as if they were tensor products of one-particle states;
- U commutes with S.

Internal symmetries of the scattering matrix are symmetries which do not act on space-time coordinates, prominent examples being gauge symmetries. The question arises whether the POINCARÉ group $\mathcal{P}$ can be combined in a nontrivial way with internal symmetries of the S-matrix.
In what follows we restrict ourselves to theories for which the scattering states are in positive mass representations of $\mathcal{P}$. Such representations are characterized by the mass M and the spin. We shall further assume that for any finite M there is only a finite number of particle types with masses less than M.

Lemma 1 (The Coleman-Mandula-Theorem [36]) *Let a* LIE *group G be a symmetry group of the S-matrix which contains the* POINCARÉ *group and which connects a finite number of particles in a multiplet. Assume furthermore that*

- *Elastic-scattering amplitudes are analytic functions of center-of-mass energy squared s and invariant momentum transfer squared t, in some neighborhood of the physical region, except at normal thresholds.*
- *For $|p, q\rangle$ one has $S|p, q\rangle \neq |p, q\rangle$ for almost all s.*

Then G is locally isomorphic to the direct product of an internal symmetry group and the POINCARÉ *group $\mathcal{P}$.*

The theorem of COLEMAN and MANDULA implies that the most general symmetry algebra of the S-matrix contains the 4-momentum P_μ, the LORENTZ generators $J_{\mu\nu}$ and a finite number of LORENTZ scalars B_k,

$$[P_\mu, B_k] = 0 \quad , \quad [J_{\mu\nu}, B_k] = 0, \tag{3.14}$$

where the B_k constitute a LIE-algebra with structure constants ${c_{k\ell}}^m$:

$$[B_k, B_\ell] = i{c_{k\ell}}^m B_m. \tag{3.15}$$

It follows that the CASIMIR operators of the POINCARÉ algebra, P^2 and W^2, where

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}, \qquad \epsilon_{0123} = 1. \tag{3.16}$$

is the PAULI-LJUBANSKI polarization vector, commute with *all* generators of $\mathcal{G}$, and in particular with the generators of the internal symmetry group,

$$[B_\ell, P^2] = 0 \quad \text{and} \quad [B_\ell, W^2] = 0. \tag{3.17}$$

The first equation says that all members of an irreducible multiplet of the internal symmetry group have the same mass. This is known as O'RAIFEARTAIGH'S *theorem* [38]. The second equation says that they must also have the same spin. For *massless states* with discrete helicities we have

$$W_\mu = \lambda P_\mu, \qquad \lambda \in \left\{0, \tfrac{1}{2}, 1, \dots\right\} \tag{3.18}$$

and no generator B_ℓ can change the helicity since $[B_\ell, \lambda] = 0$.
To see the arguments leading to the COLEMAN-MANDULA theorem consider a forbidden tensorial charge $\mathcal{Q}_{\mu\nu}$ which for simplicity we shall assume to be traceless, $\mathcal{Q}^\mu_\mu = 0$. Assume a scalar particle of mass m, carrying the charge $\mathcal{Q}_{\mu\nu}$ appears in the theory and let $|p\rangle$ be a corresponding one-particle state. Then

$$\langle p|\mathcal{Q}_{\mu\nu}|p\rangle = \left(p_\mu p_\nu - \frac{p^2}{d}\eta_{\mu\nu}\right)C, \quad C \neq 0. \tag{3.19}$$

We consider a $2 \to 2$ scattering process. The incoming particles with momenta p_1, p_2 scatter and then go out with final momenta p_1' and p_2'. The conservation law of $\mathcal{Q}$ applied between asymptotic incoming and outgoing states requires

$$C\left(p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu} - \frac{1}{d}\eta_{\mu\nu}(m^2 + m^2)\right) = C\left(p_{1\mu}, p_{2\mu} \to p'_{1\mu}, p'_{2\mu}\right). \tag{3.20}$$

If $C \neq 0$, these equations imply that the scattering must proceed either in the forward or backward direction whereas in all other directions there is no scattering. This conflicts the analyticity properties of scattering amplitudes in more then 2 dimensions. No interacting theory can carry the charge $\mathcal{Q}_{\mu\nu}$. Similar arguments can be used for amplitudes of non-identical particles to prove this 'no-go' theorem.
The COLEMAN-MANDULA theorem shows the impossibility of nontrivial symmetries that connect particles of different spins, if all the particles have integer spin or if all the particles have half-odd-integer spins.

3.2 Noether theorem

For LAGRANGEan field theories the field equations are the EULER-LAGRANGE equation of a POINCARÉ invariant action integral S. In a local field theory the action is the space-time integral of a local LAGRANGEan density $\mathcal{L}(x)$,

$$S=\int d^dx\;\mathcal{L}(x)\equiv\int dtd\boldsymbol{x}\mathcal{L}(t,\boldsymbol{x}), \tag{3.21}$$

where $\mathcal{L}$ depends on the fields and their derivatives. Here we consider theories for which $\mathcal{L}$ does not depend on second or higher derivatives, $\mathcal{L}=\mathcal{L}(\phi,\partial_\mu\phi)$. The dimension of spacetime is left open. The volume element of space is denoted by $d\boldsymbol{x}$. The action is assumed to be invariant under POINCARÉ *transformations*,

$$\tilde{x}=\Lambda x+a,\qquad \Lambda\in L_+^\uparrow,\; a\in\mathbb{R}^d, \tag{3.22}$$

and the fields transform covariantly under the POINCARÉ transformations:

$$\phi'(x)=S(A)\phi\left(\Lambda^{-1}(A)x+a\right)\sim\phi(x)+\delta_\xi\phi(x), \tag{3.23}$$

where A is from the the universal covering group (the *spin group*) of the restricted LORENTZ group. Here $A\to S(A)$ a finite-dimensional representations of the spin group and $\Lambda(A)$ the LORENTZ transformation belonging to A. For simplicity, in the following the 'label' A in $\Lambda(A)$ will not always be spelled out.
In addition the action may be invariant under *global gauge transformations*

$$\phi'(x)=U\phi(x)\sim\phi+\delta_\xi\phi. \tag{3.24}$$

They are called global since the transformation matrix U it the same for all space-time points.

3.2.1 Noether theorem for internal symmetries

According to the first theorem of EMMY NOETHER, to each parameter of the symmetry group there corresponds a conserved current. The global gauge transformations (3.24) leave the LAGRANGEan density invariant so that

$$0=\delta_\xi\mathcal{L}=\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\left(\delta_\xi\phi\right)+\frac{\delta\mathcal{L}}{\delta\phi}\,\delta_\xi\phi=\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_\xi\phi\right), \tag{3.25}$$

where we used the EULER-LAGRANGE equation (field equation, equation of motion)

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)-\frac{\delta\mathcal{L}}{\delta\phi}=0 \tag{3.26}$$

in the last step. Thus the conserved NOETHER *current* for an internal symmetry takes the form

$$J_\xi^\mu=\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_\xi\phi,\qquad \partial_\mu J_\xi^\mu=0. \tag{3.27}$$

Integrating the last equation over the space-time region $[t_0, t] \times \mathbb{R}^{d-1}$ and converting the volume- into a surface integral, shows that the NOETHER *charge*

$$Q_\xi = \int_{x^0} d\boldsymbol{x} J^0_\xi = \int_{x^0} d\boldsymbol{x}\, \pi(x)\, \delta_\xi \phi(x), \tag{3.28}$$

is time-independent. We introduced the momentum density conjugate to ϕ,

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot\phi(x)}. \tag{3.29}$$

To every internal symmetry there is one conserved NOETHER charge. The dimension of the symmetry group equals the number of independent vector fields ξ in (3.28) and hence equals the number of independent NOETHER charges.
The fundamental POISSON bracket between field and momentum density is

$$\{\phi(x), \pi(y)\}_{x^0=y^0} = \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{3.30}$$

and can be used to calculate the POISSON brackets between the conserved charges and the field,

$$\{\phi(x), Q_\xi\} = \int d\boldsymbol{y}\, \{\phi(x), \pi(y)\, \delta_\xi \phi(y)\} = \delta_\xi \phi(x), \tag{3.31}$$

where we assumed that $\delta_\xi \phi$ contains no time-derivatives of the field. We used that Q_ξ is conserved and set $y^0 = x^0$ when calculating (3.31). For the quantized field the corresponding result reads

$$[iQ_\xi, \phi_(x)] = \delta_\xi \phi(x). \tag{3.32}$$

NOETHER charges generates the symmetries from which they have been derived!

3.2.2 Noether theorem for space-time symmetries

Under space-time translations a field changes into $\phi(x+a) \sim \phi(x) + \delta_a \phi(x)$ and the infinitesimal variation is a total divergence,

$$\delta_a \phi = \partial_\mu V^\mu_a \quad \text{with} \quad V^\mu_a = a^\mu \phi(x). \tag{3.33}$$

Under LORENTZ transformations a general field transforms into

$$S(A)\, \phi\left(\Lambda^{-1}(A)x\right) = e^{\frac{i}{2}\,\omega^{\mu\nu} S_{\mu\nu}} \phi\left(e^{-\omega} x\right) \sim \phi(x) + \delta_\omega \phi(x) \tag{3.34}$$

where the $S_{\mu\nu}$ form a representation of the LORENTZ algebra (3.1) and

$$\delta_\omega \phi = \tfrac{i}{2} \omega^{\mu\nu} \left(L_{\mu\nu} + S_{\mu\nu}\right) \phi, \qquad L_{\mu\nu} = \tfrac{1}{i}(x_\mu \partial_\nu - x_\nu \partial_\mu). \tag{3.35}$$

For a scalar field $S_{\mu\nu}$ is absent and the variation becomes a total divergence[1]

$$\delta_\omega \phi = \partial_\mu V^\mu_\omega, \qquad V^\mu_\omega = -\omega^{\mu\rho} x_\rho \phi(x) \quad (\phi \text{ scalar}). \tag{3.36}$$

[1] Exercise: prove this

The LAGRANGEan density of a POINCARÉ invariant theory is a scalar and its variation under *all* small spacetime symmetries is a total divergence, $\delta\mathcal{L} = \partial_\mu V_\xi^\mu$, where ξ characterizes the type of transformation. The action stays invariant and this is sufficient for the field equations to be covariant.
Since the LAGRANGEan transforms into a total divergence, the zero on the left in (3.25) is replaced by $\partial_\mu V_\xi^\mu$ and we obtain the conserved NOETHER current

$$J_\xi^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_\xi\phi - V_\xi^\mu. \tag{3.37}$$

The corresponding conserved charge takes the form

$$Q_\xi = \int d\boldsymbol{x}\, \big(\pi(x)\,\delta_\xi\phi(x) - V_\xi^0(x)\big) \tag{3.38}$$

Now we discuss the currents for translations and LORENTZ transformations in turn.

Translations: The NOETHER current belonging to the translations defines the *canonical energy-momentum* tensor

$$J_a^\mu = a^\nu T^\mu_{\ \nu}, \qquad T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\,\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}. \tag{3.39}$$

The conserved charges are the total momentum of the field,

$$P^\mu = \int_{x^0} d\boldsymbol{x}\, T^{0\mu}, \qquad \dot{P}^\mu = 0. \tag{3.40}$$

Its components, the *energy* and *momentum*, have the explicit forms

$$P^0 \equiv H = \int_{x^0} d\boldsymbol{x}\,(\pi\dot{\phi} - \mathcal{L}) \quad , \quad P^i = \int_{x^0} d\boldsymbol{x}\,\pi\,\partial^i\phi, \tag{3.41}$$

and they generate infinitesimal spacetime translations,

$$\{\phi, P_\mu\} = \partial_\mu\phi \quad \text{or after quantization} \quad [iP_\mu, \phi] = \partial_\mu\phi. \tag{3.42}$$

Lorentz transformations: Under infinitesimal LORENTZ transformations

$$\delta_\omega\mathcal{L} = \partial_\mu V_\omega^\mu \quad \text{with} \quad V_\omega^\mu = -\omega^{\mu\rho}x_\rho\mathcal{L}. \tag{3.43}$$

Inserting $\delta_\xi\phi$ from (3.35) in the general formula (3.37) and subtracting V_ω^μ yields the following NOETHER current for LORENTZ transformations,

$$J_\omega^\mu = \omega_{\rho\sigma}M^{\mu\rho\sigma}, \quad M^{\mu\rho\sigma} = \tfrac{1}{2}x^\rho T^{\mu\sigma} - \tfrac{1}{2}x^\sigma T^{\mu\rho} + \tfrac{i}{2}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,S^{\rho\sigma}\phi. \tag{3.44}$$

The corresponding conserved NOETHER charges read

$$J^{\rho\sigma} = -J^{\sigma\rho} = -2i\int_{x^0} d\boldsymbol{x}\, M^{0\rho\sigma}, \tag{3.45}$$

and they generate infinitesimal LORENTZ transformations,

$$\{\phi, J^{\mu\nu}\} = (L_{\mu\nu} + S_{\mu\nu})\,\phi \quad \text{or} \quad [iJ^{\mu\nu}, \phi] = (L_{\mu\nu} + S_{\mu\nu})\,\phi \tag{3.46}$$

In passing we note, that for theories with non-scalar fields things may become tricky. The canonical energy-momentum tensor is generically not symmetric and must be improved. It is possible to correct the non-symmetric $T_{\mu\nu}$ through the BELINFANTE symmetrization procedure [39].
For bosonic fields the most efficient way to do this is to couple the fields to gravity and vary the resulting LAGRANGEan with respect to the metric[2]. However, when coupling fermions to gravity one needs a vielbein. When one varies the action with respect to the vielbein one again gets a conserved but not necessarily symmetric $T_{\mu\nu}$ which needs further improvement.

3.3 Spinors

In this section we study the transformation property of spinors under 'LORENTZ'-transformations and the properties of MAJORANA-, WEYL- and DIRAC spinors.

3.3.1 Clifford algebras

The CLIFFORD algebra is the free algebra generated by the d elements $\gamma_0, \dots, \gamma_{d-1}$, modulo the quadratic relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \equiv \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\,. \tag{3.47}$$

For even d there exists *one* irreducible representation of dimension $2^{d/2}$ and for odd d *two* inequivalent irreducible representations of dimension $2^{(d-1)/2}$. Thus

$$\gamma^\mu \in \mathrm{GL}\left(2^{[d/2]}, \mathbb{C}\right), \tag{3.48}$$

where $[a]$ is the biggest integer less or equal to a.
For applications the following observations are relevant:

- In *even dimensions* a complete set of $2^{d/2}$-dimensional matrices is provided by the antisymmetrized products of gamma-matrices,

  $$\gamma_{\mu_1\dots\mu_n} \equiv \gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_n]} \tag{3.49}$$

  The antisymmetrized product of all γ's is proportional to

  $$\gamma_* = i^{1+d/2}\gamma_0\dots\gamma_{d-1} \quad \text{with} \quad \gamma_*\gamma_* = 1. \tag{3.50}$$

  In addition, we have

  $$\gamma_{\mu_1\dots\mu_n} = i^{1+d/2}\frac{1}{(d-n)!}\epsilon_{\mu_1\dots\mu_d}\,\gamma_*\gamma^{\mu_d\dots\mu_{n+1}}, \quad \epsilon_{01\dots d-1} = 1. \tag{3.51}$$

  γ_* anti-commutes with all γ_μ and thus can be viewed as γ_d in $d+1$ dimensions.

[2] One may even allow for a non-minimal coupling to gravity to obtain an improved $T_{\mu\nu}$.

- In *odd dimensions* the product of all γ-matrices is a multiple of the identity. A basis for the $2^{[d/2]}$-dimensional matrices is formed by antisymmetrized products with $n = 0, 1, \dots, \frac{1}{2}(d-1)$.

The formula for the expansion of an arbitrary $\Delta = 2^{[d/2]}$-dimensional matrix M is

$$M = \frac{1}{\Delta} \sum_{n=0}^{D} \frac{1}{n!} (-)^{n(n-1)/2} \gamma_{\mu_1 \dots \mu_n} \operatorname{Tr} \left(\gamma^{\mu_1 \dots \mu_n} M \right) , \tag{3.52}$$

where $D = d$ in even dimensions and $D = \frac{1}{2}(d-1)$ in odd dimensions.

3.3.2 Spin transformations

To study the transformation property of spinors under 'Lorentz'-transformations we introduce the $d(d-1)/2$ matrices

$$\Sigma^{\mu\nu} = -\Sigma^{\nu\mu} = \tfrac{1}{2i} \gamma^{\mu\nu} = \tfrac{1}{4i} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu). \tag{3.53}$$

They furnish a $2^{[d/2]}$-dimensional *representation* of the Lorentz algebra (3.1) and generate *spin-transformations*,

$$S = e^{\frac{i}{2}(\omega, \Sigma)} = \mathbb{1} + \tfrac{i}{2}(\omega, \Sigma) + \dots . \tag{3.54}$$

The mapping $S \to \Lambda(S)$ defined by

$$S^{-1} \gamma^\rho S = \Lambda^\rho{}_\sigma \gamma^\sigma \quad \text{with} \quad \Lambda = e^\omega \tag{3.55}$$

defines a representation of the spin group as Lorentz transformations. The infinitesimal Lorentz transformations are $(\delta^\mu{}_\nu + \omega^\mu{}_\nu)$ with antisymmetric $\omega_{\mu\nu}$. A Dirac spinor transforms with S in (3.54) such that

$$\delta_\omega \psi(x) = \tfrac{i}{2} \omega_{\mu\nu} \left(L^{\mu\nu} + \Sigma^{\mu\nu} \right) \psi(x), \tag{3.56}$$

where the $\Sigma_{\mu\nu}$ generate spin rotations and the $L_{\mu\nu}$ orbital transformations. Both satisfy the commutation relations of the Lorentz algebra and so does their sum. To construct bilinear *tensor fields* we use that γ^0 conjugates the γ and Σ-matrices into their adjoints. It follows that γ^0 conjugates the adjoint of S into the inverse,

$$\gamma^0 S^\dagger \gamma^0 = S^{-1}, \tag{3.57}$$

such that the *conjugate spinor*, $\bar\psi \equiv \psi^\dagger \gamma^0$ transforms with the inverse spin rotation,

$$\bar\psi(x) \longrightarrow \bar\psi \left(\Lambda^{-1} x \right) S^{-1}. \tag{3.58}$$

With the help of (3.55) it is now easy to prove that the bilinear objects

$$A^{\mu_1 \dots \mu_n} = \bar\psi \gamma^{\mu_1 \dots \mu_n} \psi \tag{3.59}$$

are *antisymmetric tensor fields*. The transformation of these objects follow from that of ψ and $\bar\psi$,

$$A^{\mu_1\dots\mu_n}(x) \longrightarrow \Lambda^{\mu_1}_{\ \nu_1}\cdots\Lambda^{\mu_n}_{\ \nu_n}\,A^{\nu_1\dots\nu_n}\left(\Lambda^{-1}x\right).$$

In 4 dimensions there are 5 tensor fields

$$\bar\psi\psi \quad , \quad \bar\psi\gamma_*\psi \quad , \quad \bar\psi\gamma^\mu\psi, \quad , \quad \bar\psi\gamma_*\gamma^\mu\psi \quad , \quad \bar\psi\gamma^{\mu\nu}\psi,$$

a scalar, pseudo-scalar, vector, pseudo-vector and antisymmetric 2-tensor field.

3.3.3 Charge conjugation

Assume that there exists a *charge conjugation matrix* $\mathcal{C}$ which fulfills

$$\mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = \eta\,\gamma^\mu, \quad \text{with} \quad \eta = \pm 1, \tag{3.60}$$

in which case we define the *charge conjugated spinor*

$$\psi_{\rm c} = \mathcal{C}\bar\psi^T = \mathcal{C}\gamma_0^T\psi^*. \tag{3.61}$$

Now we multiply the Dirac equation

$$i\gamma^\mu D_\mu(e)\psi - m\psi = 0, \qquad D_\mu(e) = \partial_\mu - ieA_\mu \tag{3.62}$$

with γ^0, complex conjugate and multiply with $\mathcal{C}$ from the left and obtain

$$i\gamma^\mu D_\mu(-e)\psi_{\rm c} + \eta\bar m\psi_{\rm c} = 0. \tag{3.63}$$

For a vanishing mass $\psi_{\rm c}$ fulfills the Dirac equation with reversed electric charge and this justifies its name charge conjugated spinor[3]. If there exists a Majorana representation with *real* or *imaginary* γ's then we may choose in $\mathcal{C} = \gamma_0^T$ in (3.60),

$$\gamma_\mu^* = \eta\gamma_\mu: \quad \mathcal{C} = \gamma_0^T, \quad \psi_{\rm c} = \psi^* \tag{3.64}$$

A *spinor* invariant under charge conjugations is called Majorana spinor. In a Majorana representation such spinors are real.
Now one can prove that there always exist a *symmetric* or *antisymmetric* charge conjugation matrix $\mathcal{C}$ [40, 41]. In the following table we have summarized the results. The symbols S and A denote symmetric and antisymmetric matrices. For example, in 3 dimensions there exists an antisymmetric solution with $\eta = -1$ but no solution with $\eta = 1$. Since the results are identical in d and $d + 8n$ dimensions, it is sufficient to give the results for $d = 1, \dots, 8$:

d	1	2	3	4	5	6	7	8
$\eta=+1$	S	S		A	A	A		S
$\eta=-1$		A	A	A		S	S	S

(3.65)

[3] In the presence of a real mass we would want $\eta = -1$.

3.3.4 Irreducible spinors

We do not know whether the spinor representations (3.54) are *irreducible*. In general they are not and there are two types of projections onto invariant subspaces that one can envisage:

• The first exists in even dimensions only, where one has a γ_* anti-commuting with all γ-matrices. γ_* can be used to define *left* and *right*-handed (chiral) spinors,

$$\psi_{\mathsf{L}} = \tfrac{1}{2}\,(1-\gamma_*)\,\psi \equiv P_{\mathsf{L}}\psi\,, \qquad \psi_{\mathsf{R}} = \tfrac{1}{2}\,(1+\gamma_*)\,\psi \equiv P_{\mathsf{R}}\psi\,. \tag{3.66}$$

Since γ_* commutes with the generators $\Sigma_{\mu\nu}$ a left(right)-handed spinors is left(right)-handed in any inertial system.

• The second possible projection is a Majorana *reality condition.*

$$\psi = \psi_{\rm c} = \mathcal{C}\bar\psi^T. \tag{3.67}$$

This is a Lorentz invariant condition since $\psi_{\rm c}$ transforms the same way as ψ, as follows from $S\mathcal{C}S^T = \mathcal{C}$. Since $\psi^{**} = \psi$ we must demand

$$(\mathcal{C}\gamma_0^T)^*(\mathcal{C}\gamma_0^T) = \mathcal{C}^*\gamma_0\mathcal{C}\gamma_0^T \overset{(3.60)}{=} \eta\,\mathcal{C}^*\mathcal{C} \overset{!}{=} \mathbb{1} \quad \text{or} \quad \mathcal{C}^* = \eta\,\mathcal{C}^{-1} = \eta\,\mathcal{C}^\dagger. \tag{3.68}$$

Since $\mathcal{C}$ is unitary this condition is equivalent to $\mathcal{C}^T = \eta\mathcal{C}$. Comparing with the above table leads to the solutions given in the following table:

d	1	2	3	4	5	6	7	8
$\eta=+1$	S	S						S
$\eta=-1$		A	A	A				
		MW				SM		

(3.69)

In cases there exist no Majorana spinor one may still try do define *symplectic* Majorana *spinors* in theories with extended supersymmetry. They obey

$$\psi_i = \mathcal{C}\gamma_0^T\,\Omega_{ij}\psi_j^*\,, \tag{3.70}$$

where Ω is some antisymmetric matrix, with $\Omega\Omega^* = -1$. Symplectic Majorana spinors exist in $6, 14, \dots$ dimensions, as indicated in the above table.

Having two projections, to chiral and to Majorana spinors, one may ask whether one can define a reality condition respecting the chiral projection. This is indeed possible in $2+8n$ dimensions, where such Majorana-Weyl fermions exist.

3.3.5 Fierz identities

We take two spinors ψ and χ whose components anticommute and choose

$$M = \psi\bar\chi, \quad \text{such that} \quad \mathrm{Tr}\,(\gamma^{\mu_1\cdots\mu_n}M) = -\bar\chi\gamma^{\mu_1\cdots\mu_n}\psi, \tag{3.71}$$

where the minus sign originates from the anticommuting nature of the spinor components. The expansion (3.52) becomes the general FIERZ-identity

$$\psi\bar{\chi} = -\frac{1}{\Delta}\sum_n \frac{1}{n!}\,(-)^{n(n-1)/2}\,\gamma_{\mu_1\dots\mu_n}\,(\bar{\chi}\gamma^{\mu_1\dots\mu_n}\psi), \tag{3.72}$$

which allows us to write the matrix $\psi\bar{\chi}$ as linear combination the antisymmetrized products of γ-matrices. For example, in 4 dimensions the general identity reads

$$4\psi\bar{\chi} = -(\bar{\chi}\psi) - \gamma_\mu(\bar{\chi}\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar{\chi}\gamma_5\gamma^\mu\psi) - \gamma_5(\bar{\chi}\gamma_5\psi), \tag{3.73}$$

where $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$.

The DIRAC-conjugate of the charge conjugated spinor ψ_c is $\bar{\psi}_c = \eta\psi^T\mathcal{C}^{-1}$ and is used to compute the bilinears for the charge conjugated fields,

$$\bar{\psi}_c\gamma_{\mu_1\dots\mu_n}\chi_c = -\eta^{1+n}(-1)^{n(n-1)/2}\,\bar{\chi}\gamma_{\mu_1\dots\mu_n}\psi \overset{\text{Majorana}}{=} \bar{\psi}\gamma_{\mu_1\dots\mu_n}\chi. \tag{3.74}$$

In 4 dimensions with $\eta = -1$ we find for MAJORANA spinors

$$\bar{\psi}M\chi = \pm\bar{\chi}M\psi, \quad +:\ M\in\{\mathbb{1},\gamma_5,\gamma_5\gamma_\mu\}, \quad -:\ M\in\{\gamma_\mu,\gamma_{\mu\nu}\} \tag{3.75}$$

such that $\bar{\psi}\gamma_\mu\psi = \bar{\psi}\gamma_{\mu\nu}\psi = 0$. All FIERZ identities you find in the literature can be derived from the general identity (3.72) and the symmetry relations (3.74). For example, setting $\chi = \psi$ in (3.73) we obtain for MAJORANA spinors the identity

$$\left(\psi\bar{\psi} + \gamma_5\psi\bar{\psi}\gamma_5\right)\psi = \psi(\bar{\psi}\psi) + \gamma_5\psi(\bar{\psi}\gamma_5\psi) = 0. \tag{3.76}$$

3.3.6 Hermitian conjugation

We *define* the hermitian conjugate as if the spinor-components are operators in a HILBERT-space (which they are in quantum field theory). For example

$$(\bar{\psi}\chi)^\dagger \equiv \chi^\dagger\bar{\psi}^\dagger. \tag{3.77}$$

One can show that for fermionic bilinears

$$(\bar{\psi}M\chi)^\dagger = -\eta\,\bar{\psi}_c M_c\chi_c \quad \text{with} \quad M_c = \mathcal{C}\gamma_0^T M^*\gamma_0^T\mathcal{C}^{-1}. \tag{3.78}$$

In a MAJORANA representation with $\mathcal{C} = \gamma_0^T$ this simplifies to

$$(\bar{\psi}M\chi)^\dagger = -\eta\,\bar{\psi}_c M^*\chi_c \tag{3.79}$$

and with $\gamma_\mu^* = \eta\gamma_\mu$ (see (3.64)) we obtain for MAJORANA spinors

$$\left(\bar{\psi}\gamma_{\mu_1\dots\mu_n}\chi\right)^\dagger = (-1)^n\bar{\psi}\gamma_{\mu_1\dots\mu_n}\chi. \tag{3.80}$$

This property is useful when one enumerates all possible terms in a real action or in supersymmetry transformations.

3.3.7 Chiral spinors in 4 dimensions

In theories with extended supersymmetry one often uses left- and right-handed spinors. Then it is convenient to use the *chiral representation*

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde\sigma_\mu & 0 \end{pmatrix} \quad \text{with} \quad \sigma_\mu = (\sigma_0, -\tau_i) \quad \text{and} \quad \tilde\sigma_\mu = (\sigma_0, \tau_i), \tag{3.81}$$

where $\sigma_0 = \mathbb{1}_2$ and τ_1, τ_2, τ_3 are the Pauli matrices. Since $\gamma_5 = \mathrm{diag}(-\sigma_0, \sigma_0)$ the chiral projectors take the simple form

$$P_{\mathsf{L}} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_{\mathsf{R}} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \tag{3.82}$$

such that a lefthanded spinor has upper and a right-handed spinor lower components. The infinitesimal spin-rotations are block-diagonal,

$$\gamma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde\sigma_{\mu\nu} \end{pmatrix}, \quad \sigma_{\mu\nu} = \tfrac{1}{2}(\sigma_\mu\tilde\sigma_\nu - \sigma_\nu\tilde\sigma_\mu), \quad \tilde\sigma_{\mu\nu} = -\sigma^\dagger_{\mu\nu} \tag{3.83}$$

and so are the spin rotations generated by them,

$$S = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger-1} \end{pmatrix}, \quad A = e^{\frac{1}{4}\omega^{\mu\nu}\sigma_{\mu\nu}} \in SL(2,\mathbb{C}). \tag{3.84}$$

The left- and right-handed parts of a Dirac spinor

$$\psi = \begin{pmatrix} \varphi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \tag{3.85}$$

transform with the two inequivalent irreducible representations A and $A^{\dagger-1}$ of $SL(2,\mathbb{C})$,

$$\varphi_\alpha \longrightarrow A_\alpha^{\ \beta}\varphi_\beta \quad , \quad \bar\chi^{\dot\alpha} \longrightarrow (A^{\dagger-1})^{\dot\alpha}_{\ \dot\beta}\,\bar\chi^{\dot\beta} \tag{3.86}$$

Next we define the ε-tensors

$$(\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot\alpha\dot\beta}) = -(\varepsilon^{\alpha\beta}) = -(\varepsilon^{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \varepsilon, \tag{3.87}$$

which obey the relations,

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta^\gamma_\alpha \quad , \quad \varepsilon_{\alpha\beta}\varepsilon^{\delta\gamma} = \delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta. \tag{3.88}$$

Because $A^T\varepsilon A = \varepsilon$ for any matrix A with determinant one, the bilinears

$$\varphi\chi = \varphi^\alpha\chi_\alpha \quad \text{and} \quad \bar\varphi\bar\chi = \bar\varphi_{\dot\alpha}\bar\chi^{\dot\alpha} \tag{3.89}$$

are Lorentz invariant, where the raising and lowering of indexes are done with ε,

$$\varphi^\alpha = \varepsilon^{\alpha\beta}\,\varphi_\beta, \quad \varphi_\alpha = \varepsilon_{\alpha\beta}\varphi^\beta \quad \text{and} \quad \bar\varphi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\varphi_{\dot\beta}, \quad \bar\varphi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\varphi^{\dot\beta}. \tag{3.90}$$

The condition $A^T\varepsilon A=\varepsilon$ translates into

$$(A^{T-1})^{\alpha}{}_{\beta}=\varepsilon^{\alpha\rho}A_{\rho}{}^{\sigma}\varepsilon_{\sigma\beta}\quad\text{and}\quad(A^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}=\varepsilon^{\dot\alpha\dot\rho}\bar A_{\dot\rho}{}^{\dot\sigma}\varepsilon_{\dot\sigma\dot\beta}\tag{3.91}$$

which means, that A can be conjugated into A^{T-1} and $\bar A$ into $A^{\dagger-1}$. Using these formulas one can show that the spin-transformation in (3.86) are equivalent to

$$\varphi^{\alpha}\longrightarrow(A^{T-1})^{\alpha}{}_{\beta}\varphi^{\beta}\quad,\quad\bar\chi_{\dot\alpha}\longrightarrow\bar A_{\dot\alpha}{}^{\dot\beta}\bar\chi_{\dot\beta}.\tag{3.92}$$

We conclude that the components φ^{α} transform as the complex conjugate of the components $\bar\chi^{\dot\alpha}$ and the components $\bar\chi_{\dot\alpha}$ as the complex conjugate of the components φ_{α}. The index structure of the Dirac-conjugate spinor is

$$\bar\psi=(\chi^{\alpha},\bar\varphi_{\dot\alpha})\quad\text{such that}\quad\bar\psi\psi=\chi\varphi+\bar\varphi\bar\chi.\tag{3.93}$$

There is no mass term for a left- or for a right-handed spinor. Since σ_{μ} maps right- into left-handed spinors and $\tilde\sigma_{\mu}$ does the opposite, they have the index structure

$$(\sigma_{\mu})_{\alpha\dot\beta}\quad\text{and}\quad(\tilde\sigma_{\mu})^{\dot\alpha\beta}.\tag{3.94}$$

The generator $\sigma_{\mu\nu}$ and $\tilde\sigma_{\mu\nu}$ preserve chirality and have the index structure

$$(\sigma_{\mu\nu})_{\alpha}{}^{\beta}\quad\text{and}\quad(\tilde\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}.\tag{3.95}$$

Spinor components are Grassmann variables such that

$$\begin{aligned}\varphi\chi=\varphi^{\alpha}\chi_{\alpha}=-\chi_{\alpha}\varphi^{\alpha}=\chi^{\alpha}\varphi_{\alpha}=\chi\varphi\\ \bar\varphi\bar\chi=\bar\varphi_{\dot\alpha}\bar\chi^{\dot\alpha}=-\bar\chi^{\dot\alpha}\bar\varphi_{\dot\alpha}=\bar\chi_{\dot\alpha}\varphi^{\dot\alpha}=\bar\chi\bar\varphi.\end{aligned}\tag{3.96}$$

The vector current, on the other hand, can be written as

$$\bar\psi\gamma^{\mu}\psi=\bar\varphi_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\beta}\varphi_{\alpha}+\chi^{\alpha}\sigma^{\mu}_{\alpha\dot\beta}\bar\chi^{\dot\beta}=\bar\varphi\,\tilde\sigma^{\mu}\varphi+\chi\sigma^{\mu}\bar\chi,\tag{3.97}$$

so that the 'kinetic term' for fermions take the form

$$\bar\psi\partial\!\!\!/\psi=\bar\varphi\,\tilde\sigma^{\mu}\partial_{\mu}\varphi+\chi\sigma^{\mu}\partial_{\mu}\bar\chi.\tag{3.98}$$

In particular in 2 and 4 dimensions the Fierz identities take a simpler form when one uses chiral spinors. The explicit formulas can be found in text books on supersymmetry, for examples the ones cited in the introduction.

Chapter 4

The Wess-Zumino Model

In a supersymmetric model we expect an equal number of bosonic and fermionic states of equal mass. For example, a Majorana fermion has 2 polarization states and could be accompanied by two neutral scalar particles. Here we study the most simple of these models in four spacetime dimensions. It has been constructed by Wess and Zumino [42] when they extended the 2-dimensional supersymmetric string-model of Gervais and Sakita[1] [43] to four dimensions. The model contains a super-multiplet with

- a single Majorana field ψ
- a pair of real scalar and pseudo-scalar bosonic fields ϕ_1 and ϕ_2

In the off-shell formulations it also contains a pair of real scalar and pseudoscalar bosonic auxiliary fields $\mathcal{F}_1$ and $\mathcal{F}_2$. In passing we note that there was some earlier work on supersymmetric field theories in four dimensions. A version of supersymmetric quantum electrodynamics (QED) was found by Golfand and Likhtman in 1970 and published in 1971 [46]. This massive super-QED contains a massive photon and photino, a charged Dirac spinor and two charged scalars. Subsequently Akulov and Volkov tried to associate the massless fermion - appearing due to spontaneous supersymmetry breaking - with the neutrino [47].

4.1 The free massless Wess-Zumino model

Without interaction the Lagrangean of the massless model takes the form (3.9)

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial_\mu\phi_1)^2 + \tfrac{1}{2}(\partial_\mu\phi_2)^2 + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi. \tag{4.1}$$

Besides the well-known space-time symmetries the action $S_0 = \int d^4x\, \mathcal{L}_0$ is invariant under the following supersymmetry transformations

$$\delta_\varepsilon\phi_1 = \bar{\varepsilon}\psi \quad , \quad \delta_\varepsilon\phi_2 = i\,\bar{\varepsilon}\gamma_5\psi \tag{4.2}$$

$$\delta_\varepsilon\psi = -i\partial\!\!\!/\phi\varepsilon \quad , \quad \phi = \phi_1 + i\gamma_5\phi_2 \tag{4.3}$$

[1] which was based on earlier work of Ramond [44] and Neveu and Schwartz [45].

where ε is an arbitrary constant anticommuting MAJORANA parameter with dimensions $[\varepsilon] = L^{1/2}$. Clearly, these transformations map bosons into fermions and vice versa. They are very restricted by the following requirements:

- LORENTZ covariance,
- dimensions of fields: $[\delta_\varepsilon \phi_i] = L^{-1}, \quad [\delta_\varepsilon \psi] = L^{-3/2}$,
- hermiticity: ψ, ϕ_i real.

For example, with $\bar{\varepsilon}\psi$ the variation $\delta_\varepsilon \phi_1$ must be a real scalar field and with $i\,\bar{\varepsilon}\gamma_5\psi$ the variation $\delta_\varepsilon \phi_2$ must be a real pseudo-scalar field. We conclude that ϕ_1 is a scalar and ϕ_2 a pseudo-scalar field. Using $\gamma^0\gamma_\mu\gamma^0 = \gamma_\mu^\dagger$ the variation of the $\bar{\psi}$ reads

$$\delta_\varepsilon \bar{\psi} = (\delta_\varepsilon \psi)^\dagger \gamma^0 = i\,\bar{\varepsilon}\partial_\mu \phi \gamma^\mu. \tag{4.4}$$

Not unexpected the LAGRANGEan density is invariant only up to a total divergence. For newcomers to supersymmetry the proof is enlightening and since it is simple for the model (4.1) we shall give it. Clearly,

$$\begin{aligned} \tfrac{1}{2}\delta_\varepsilon \left(\sum \partial_\mu \phi_i \partial^\mu \phi_i\right) &= \bar{\varepsilon}\partial^\mu \phi \partial_\mu \psi \\ \tfrac{i}{2}\delta_\varepsilon (\bar{\psi}\partial\!\!\!/\psi) &= -\tfrac{1}{2}\,\bar{\varepsilon}\partial\!\!\!/\gamma^\mu \phi \partial_\mu \psi + \tfrac{1}{2}\bar{\psi}\Box\phi\,\varepsilon, \end{aligned} \tag{4.5}$$

where we may interchange ψ and ε in the last term since $\bar{\psi}\varepsilon = \bar{\varepsilon}\psi$ and $\bar{\psi}\gamma_5\varepsilon = \bar{\varepsilon}\gamma_5\psi$ hold true. The sum of the two terms is a divergence

$$\delta_\varepsilon \mathcal{L}_0 = \partial_\mu(\,\bar{\varepsilon}V_0^\mu), \quad V_0^\mu = \tfrac{1}{2}\gamma^\mu\gamma^\nu\partial_\nu\phi\,\psi \tag{4.6}$$

which already proves the invariance of S_0. To find the NOETHER current we calculate

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi = \,\bar{\varepsilon}K^\mu, \qquad K^\mu = \left(\tfrac{1}{2}\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\nu\phi\,\psi. \tag{4.7}$$

and subtract V_0^μ in (4.6). This yields the *conserved current*

$$\bar{\varepsilon}J^\mu = \,\bar{\varepsilon}\partial_\nu\phi\gamma^\nu\gamma^\mu\psi = -i\delta_\varepsilon\bar{\psi}\gamma^\mu\psi = i\bar{\psi}\gamma^\mu\delta_\varepsilon\psi \tag{4.8}$$

and conserved NOETHER charge $\bar{\varepsilon}\mathcal{Q} = \int d\boldsymbol{x}\,\bar{\varepsilon}J^0$ with *spinorial supercharge*

$$\mathcal{Q} = \int_{x^0} d\boldsymbol{x}\,\left(\pi_1 + i\gamma_5\pi_2 - \alpha^i\partial\phi\right)\psi, \quad \alpha^i = \gamma^0\gamma^i, \tag{4.9}$$

where π_i is the momentum field conjugate to ϕ_i and $\phi = \phi_1 + i\gamma_5\phi_2$. The fundamental equal-time (anti)commutators of the field operators $\{\phi_i(\boldsymbol{x}), \pi_i(\boldsymbol{x}), \psi(\boldsymbol{x})\}$ are

$$[\phi_i(x), \pi_j(y)]_{x^0=y^0} = i\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \;\; , \;\; \{\psi_\alpha(x), \psi_\beta(y)\}_{x^0=y^0} = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.10}$$

where the anti-commutators holds in a MAJORANA representation. In an arbitrary representation $\delta_{\alpha\beta}$ is replaced by $-(\gamma^0\mathcal{C})_{\alpha\beta}$. Now its not difficult to prove that the supercharge generate the supersymmetry,

$$i[\,\bar{\varepsilon}\mathcal{Q}, \phi_i] = \delta_\varepsilon \phi_i \quad \text{and} \quad i[\,\bar{\varepsilon}\mathcal{Q}, \psi] = \delta_\varepsilon \psi, \tag{4.11}$$

similarly as P_μ and $J_{\mu\nu}$ generate translations and LORENTZ transformations.

4.1.1 Superalgebra

The commutator of two successive symmetry transformations must itself be a symmetry transformation. This way we can identify the algebra of group generators. Let us see what the commutator of two susy transformations looks like. For example,

$$\begin{aligned}[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\phi_1 &= \delta_{\varepsilon_1}(\bar\varepsilon_2\psi)-\delta_{\varepsilon_2}(\bar\varepsilon_1\psi) = -i\,\bar\varepsilon_2\partial\!\!\!/\phi\varepsilon_1-(1\leftrightarrow 2)\\ &= -2i\,\bar\varepsilon_2\gamma^\mu\varepsilon_1\partial_\mu\phi_1.\end{aligned} \tag{4.12}$$

Similarly one finds for the pseudo-scalar field

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\phi_2=-2i\,\bar\varepsilon_2\gamma^\mu\varepsilon_1\partial_\mu\phi_2,$$

Calculating the commutator on ψ is more involved. Upon using the FIERZ rearrangement formula

$$\varepsilon_2\,\bar\varepsilon_1-\varepsilon_1\,\bar\varepsilon_2=-\tfrac{1}{2}\gamma_\rho(\bar\varepsilon_1\gamma^\rho\varepsilon_2)+\gamma_{\rho\sigma}(\bar\varepsilon_1\gamma^{\rho\sigma}\varepsilon_2) \tag{4.13}$$

one obtains

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi=-2(\bar\varepsilon_2\gamma^\rho\varepsilon_1)\partial_\rho\psi+i\gamma_\rho(\bar\varepsilon_2\gamma^\rho\varepsilon_1)\partial\!\!\!/\psi.$$

Only if we impose the *field equation* for ψ do we get the expected commutator,

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi=-2i(\bar\varepsilon_2\gamma^\mu\varepsilon_1)\partial_\mu\psi. \tag{4.14}$$

On *all fields* the commutator of two transformations yield a translation such that

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]=-2i(\bar\varepsilon_2\gamma^\mu\varepsilon_1)\partial_\mu. \tag{4.15}$$

Since $\bar\varepsilon\mathcal{Q}$ generates the supersymmetry transformations δ_ε, the result (4.15) means that the commutator of two $\bar\varepsilon\mathcal{Q}$ generates infinitesimal translations. Using

$$[\bar\varepsilon_1\mathcal{Q},\,\bar\varepsilon_2\mathcal{Q}]=[\bar{\mathcal{Q}}\,\varepsilon_1,\,\bar\varepsilon_2\mathcal{Q}]=\varepsilon_{1\beta}\,\bar\varepsilon_2^\alpha\{\mathcal{Q}_\alpha,\bar{\mathcal{Q}}^\beta\} \tag{4.16}$$

and that the translations are generated by P_μ, see (3.42), we obtain

$$\bullet\ \ \{\mathcal{Q}_\alpha,\bar{\mathcal{Q}}^\beta\}=2(\gamma^\mu)_\alpha^{\ \beta}P_\mu \tag{4.17}$$

which may be rewritten as

$$\{\mathcal{Q}_\alpha,\mathcal{Q}_\beta\}=-2(\gamma^\mu\mathcal{C})_{\alpha\beta}P_\mu. \tag{4.18}$$

On the other hand the supersymmetry commutes with the translations,

$$[\delta_\varepsilon,\delta_a]=0,\quad\text{where}\quad \delta_a=a^\mu\partial_\mu, \tag{4.19}$$

and we conclude that the supercharge commutes with the momentum,

$$\bullet\ \ [\mathcal{Q}_\alpha,P_\mu]=0. \tag{4.20}$$

Let us finally calculate the commutator of LORENTZ and susy transformations. For the scalar field ϕ_1 we find

$$[\delta_\varepsilon, \delta_\omega]\phi_1 = \tfrac{i}{2}\delta_\varepsilon(\omega, L)\phi_1 - \delta_\omega\,\bar\varepsilon\psi \overset{(3.56)}{=} -\tfrac{i}{2}\omega_{\mu\nu}\,\bar\varepsilon\,\Sigma^{\mu\nu}\psi, \tag{4.21}$$

and similarly for the pseudo-scalar and spinor. Finally, using

$$[\delta_\varepsilon, \delta_\omega]\phi_1 = -\tfrac{i}{2}\,[\,\bar\varepsilon\mathcal{Q}, [\omega^{\mu\nu}J_{\mu\nu}, \phi_1]] \quad \text{and} \quad i[\,\bar\varepsilon\mathcal{Q}, \phi_1] = \bar\varepsilon\psi$$

we conclude, that on ϕ_1 we have $[\mathcal{Q}_\alpha, J_{\mu\nu}] = -(\Sigma_{\mu\nu}\mathcal{Q})_\alpha$. The same holds true for the other fields, such that

$$\bullet\;\; [J_{\mu\nu}, \mathcal{Q}_\alpha] = (\Sigma_{\mu\nu}\mathcal{Q})_\alpha. \tag{4.22}$$

As expected, the spinorial supercharge transforms as spin-$\frac{1}{2}$ object.

4.2 The off-shell formulation with interaction

We continue and allow for mass terms and/or interactions between the fermions and scalars. If we proceed as we did for the free massless model, then we would find that the supersymmetry transformations (4.2,4.3) were deformed and would

- only close if ψ fulfills the field equation (closes on-shell),
- become non-linear in the fields,
- depend on the masses and coupling constants of the model.

However, if we introduce *auxiliary* LAGRANGEan multiplier fields, then the transformations become linear, close off-shell and are independent of the details of the model. Off-shell a MAJORANA spinor contains 4 real fields, whereas ϕ only contains 2. Thus we need *two additional scalars*, denoted by $\mathcal{F}_1$ and $\mathcal{F}_2$, for the degrees of freedom to match off-shell. Similarly as for ϕ_1, ϕ_2 we introduce the field $\mathcal{F} = \mathcal{F}_1 + i\gamma_5\mathcal{F}_2$.
The idea is to deform the susy transformation of the FERMI field in (4.3)

$$\delta_\varepsilon\psi = -i\partial\!\!\!/\phi + \mathcal{F}\varepsilon \tag{4.23}$$

such that the algebra closes off-shell,

$$\begin{aligned}[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\psi &= -2(\,\bar\varepsilon_2\gamma^\rho\varepsilon_1)\partial_\rho\psi + i\gamma_\rho(\,\bar\varepsilon_2\gamma^\rho\varepsilon_1)\partial\!\!\!/\psi + (\delta_{\varepsilon_1}\mathcal{F})\varepsilon_2 - (\delta_{\varepsilon_2}\mathcal{F})\varepsilon_1 \\ &\overset{!}{=} -2(\,\bar\varepsilon_2\gamma^\rho\varepsilon_1)\partial_\rho\psi \end{aligned} \tag{4.24}$$

The transformation law (4.23) reveals that $\mathcal{F}_1$ is a scalar and $\mathcal{F}_2$ a pseudoscalar and that both fields have dimensions L^{-2}. Hence their variations can only be proportional to the hermitian objects $i\,\bar\varepsilon\partial\!\!\!/\psi$ and $\bar\varepsilon\gamma_5\partial\!\!\!/\psi$, respectively. Indeed, setting

$$\delta_\varepsilon\mathcal{F}_1 = -i\,\bar\varepsilon\partial\!\!\!/\psi \quad \text{and} \quad \delta_\varepsilon\mathcal{F}_2 = \bar\varepsilon\gamma_5\partial\!\!\!/\psi \tag{4.25}$$

we find with $M=\varepsilon_1\,\bar\varepsilon_2-\,\bar\varepsilon_2\varepsilon_1$ the result

$$(\delta_{\varepsilon_1}\mathcal{F})\varepsilon_2-(\delta_{\varepsilon_2}\mathcal{F})\varepsilon_1=i\,(M-\gamma_5M\gamma_5)\,\partial\!\!\!/\phi\psi=-i\gamma_\rho(\,\bar\varepsilon_2\gamma^\rho\varepsilon_1)\partial\!\!\!/\phi\psi,$$

where in the last step we used a FIERZ identity. In a perturbatively renormalizable local field theory the LAGRANGEan density must not contain any derivatives of the multiplier fields and their elimination should yield the on-shell density (4.1). This way one is lead to the following ansatz,

$$\mathcal{L}_0=\tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1+\tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2+\tfrac{i}{2}\bar\psi\partial\!\!\!/\psi+\tfrac{1}{2}\left(\mathcal{F}_1^2+\mathcal{F}_2^2\right)\tag{4.26}$$

and as required, this hermitian density transforms into a divergence,

$$\delta_\varepsilon\mathcal{L}_0=\partial_\mu(\,\bar\varepsilon V_0^\mu),\qquad V_0^\mu=\tfrac{1}{2}\left(\gamma^\mu\gamma^\nu\partial_\nu\phi-i\mathcal{F}\gamma^\mu\right)\psi.\tag{4.27}$$

Note that the field-equations for the auxiliary fields are simply $\mathcal{F}_1=\mathcal{F}_2=0$.
We can add a *mass term*

$$\mathcal{L}_m=m\left(\mathcal{F}_1\phi_1+\mathcal{F}_2\phi_2-\tfrac{1}{2}\bar\psi\psi\right)\tag{4.28}$$

to $\mathcal{L}_0$, as well as *interaction terms*, for example

$$\mathcal{L}_g=g\left(\mathcal{F}_1\left(\phi_1^2-\phi_2^2\right)+2\mathcal{F}_2\phi_1\phi_2-\bar\psi\left(\phi_1-i\gamma_5\phi_2\right)\psi\right).\tag{4.29}$$

Each term of the resulting LAGRANGEan density $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_m+\mathcal{L}_g$ transforms into a divergence,

$$\begin{aligned}\delta_\varepsilon\mathcal{L}_m&=\partial_\mu(\,\bar\varepsilon V_m^\mu),\qquad V_m^\mu=-im\phi\gamma^\mu\psi\\ \delta_\varepsilon\mathcal{L}_g&=\partial_\mu(\,\bar\varepsilon V_g^\mu),\qquad V_g^\mu=-ig\phi^2\gamma^\mu\psi.\end{aligned}\tag{4.30}$$

This then gives rise to the following NOETHER current

$$\bar\varepsilon J^\mu=\,\bar\varepsilon\left(\gamma^\nu\gamma^\mu\partial_\nu\phi\psi+im\phi\gamma^\mu\psi+ig\phi\phi\gamma^\mu\right)\psi=-i\delta_\varepsilon\bar\psi\gamma^\mu\psi.\tag{4.31}$$

The corresponding spinorial *supercharge* has the form

$$\mathcal{Q}=\int_{x^0}d\boldsymbol{x}\,\left(\pi-\alpha^i\partial_i\phi+im\phi\gamma^0+ig\phi\phi\gamma^0\right)\psi,\tag{4.32}$$

where we have introduced $\pi=\pi_1+i\gamma_5\pi_2$. It generates the *on-shell* supersymmetry transformations, which are just the off-shell transformation (4.2,4.23) in which one eliminates the auxiliary fields through their equations of motion

$$\mathcal{F}=-m\phi-g\phi\phi.\tag{4.33}$$

This yields the following 'on-shell' LAGRANGEan density

$$\begin{aligned}\mathcal{L}\;&=\;\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi},\partial^\mu\boldsymbol{\phi})-\tfrac{1}{2}m^2(\boldsymbol{\phi},\boldsymbol{\phi})+\tfrac{i}{2}\bar\psi\partial\!\!\!/\psi-\tfrac{1}{2}m\bar\psi\psi\\ &\quad-mg\phi_1(\boldsymbol{\phi},\boldsymbol{\phi})-\tfrac{1}{2}g^2(\boldsymbol{\phi},\boldsymbol{\phi})^2-g\bar\psi(\phi_1-i\gamma_5\phi_2)\psi,\end{aligned}\tag{4.34}$$

where $\boldsymbol{\phi}$ denotes the doublet $(\phi_1,\phi_2)^T$, and the following on-shell transformations

$$\delta_\varepsilon\phi_1=\,\bar\varepsilon\psi,\quad\delta_\varepsilon\phi_2=i\,\bar\varepsilon\gamma_5\psi\quad\text{and}\quad\delta_\varepsilon\psi=-i\partial\!\!\!/\phi\varepsilon-(m\phi+g\phi\phi)\varepsilon.\tag{4.35}$$

They are generated by the supercharge (4.32).

Chapter 5

Representations of supersymmetric algebras

In a supersymmetric field theory the fields fall into supermultiplets which transform according to some representation of the superalgebra. In this chapter we consider representations which may occur in perturbatively renormalizable field theories. We consider the superalgebra in 4 dimensions in the WEYL basis and use the conventions introduced in section (3.3.7). Since the charge conjugation matrix in the chiral representation (3.81) is

$$\mathcal{C} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \quad \text{such that} \quad \mathcal{C}\gamma_0^T = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}, \tag{5.1}$$

the spinorial supercharge $\mathcal{Q}$ is MAJORANA if and only if

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_\alpha \\ \bar{\mathcal{Q}}^{\dot\alpha} \end{pmatrix} = \mathcal{Q}_\mathrm{c} \Longleftrightarrow \bar{\mathcal{Q}}_{\dot\alpha} = \mathcal{Q}_\alpha^\dagger, \quad \bar{\mathcal{Q}}^{\dot\alpha} = (\mathcal{Q}^\alpha)^\dagger. \tag{5.2}$$

The nontrivial anticommutator (4.17) takes the form

$$\{\mathcal{Q}_\alpha, \mathcal{Q}^\beta\} = 0 \quad \text{and} \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu,$$

and for convenience we recall the index structure of the relevant matrices,

$$(\sigma_\mu)_{\alpha\dot\alpha}, \quad (\tilde\sigma_\mu)^{\dot\alpha\alpha}, \quad (\sigma_{\mu\nu})_\alpha{}^\beta, \quad (\tilde\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}. \tag{5.3}$$

Under LORENTZ transformations the supercharges $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}^{\dot\alpha}$ transform with the chiral representations of the spin group. The hermitian adjoint of a left(right)-handed operator is a linear combination of right(left)-handed operators.

In a classic paper HAAG, LOPUSZANKI and SOHNIUS re-examined the result of COLEMAN and MANDULA by redefining the very notion of symmetry to encompass LIE superalgebras. They characterized the most general symmetry LIE superalgebra of an S-matrix [48]. The COLEMAN-MANDULA theorem applies to the bosonic sector of this algebra, which is a LIE algebra, so that the bosonic subalgebra is the direct product of the POINCARÉ algebra and an internal symmetry LIE algebra. The bosonic generators B_k, P_μ and $J_{\mu\nu}$ belong to the $(0,0), (\frac{1}{2},\frac{1}{2})$ and $(0,1)\oplus(0,1)$ representations of $SL(2,\mathbb{C})$. The novelty lies in the fermionic sector, which is generated by spinorial charges $\mathcal{Q}^i_\alpha$ in the $(\frac{1}{2},0)$ representation and their hermitian adjoints $\bar{\mathcal{Q}}^i_{\dot\alpha}=(\mathcal{Q}^i_\alpha)^\dagger$ in the $(0,\frac{1}{2})$ representation. Here i runs from 1 to some positive integer $\mathcal{N}$. The HAAG-LOPUSZANSKI-SOHNIUS theorem[1] states in part that the fermion symmetry generators can only belong to these representations.

5.1 Extended superalgebras

Let us now assume that there are $\mathcal{N}$ such spinorial $(\frac{1}{2},0)$ supercharges $\{\mathcal{Q}^1,\dots,\mathcal{Q}^\mathcal{N}\}$. They must commute with translations,

$$[\mathcal{Q}^i_\alpha, P_\mu]=0,\quad i=1,\dots,\mathcal{N}, \tag{5.4}$$

and have spin $\frac{1}{2}$ which fixes their commutators with the LORENTZ generators,

$$[J_{\mu\nu},\mathcal{Q}^i_\alpha]=\tfrac{1}{2i}(\sigma_{\mu\nu})_\alpha{}^\beta\mathcal{Q}^i_\beta \tag{5.5}$$

Thus the $\mathcal{N}$-extended superalgebra consists of the generators of the POINCARÉ algebra plus $\mathcal{N}$ spinorial supercharges with anticommutators

$$\{\mathcal{Q}^i_\alpha,\bar{\mathcal{Q}}^j_{\dot\beta}\}=2\delta^{ij}(\sigma^\mu)_{\alpha\dot\beta}P_\mu. \tag{5.6}$$

The missing supercommutators $\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}$ will be discussed soon. The sign in the anticommutator (5.6) is determined by the requirement that in a relativistic quantum field theory the energy should be a non-negative operator: we get for each value of the index i

$$\sum_{\alpha=1}^{2}\{\mathcal{Q}^i_\alpha,\mathcal{Q}^{i\dagger}_\alpha\}\overset{(5.2)}{=}\sum_{\alpha=1}^{2}\{\mathcal{Q}^i_\alpha,\bar{\mathcal{Q}}^i_{\dot\alpha}\}=2\,\mathrm{Tr}\,\sigma^\mu\,P_\mu=4P_0\quad\text{(no sum over i.)} \tag{5.7}$$

The left hand side is manifestly non-negative, since each term has this property,

$$\langle\psi|\{\mathcal{Q}^i_\alpha,\mathcal{Q}^{i\dagger}_\alpha\}\psi\rangle=||\mathcal{Q}^i_\alpha\psi||^2+||\mathcal{Q}^{i\dagger}_\alpha\psi||^2\ge 0,$$

and it follows that

- the *spectrum of $H=P_0$ in a theory with supersymmetry contains no negative eigenvalues.*

We denote the state (or family of states) with the lowest energy by $|0\rangle$ and call it *vacuum state*. The vacuum will have zero energy, $H|0\rangle=0$, if and only if

$$\mathcal{Q}^i_\alpha|0\rangle=0\quad\text{and}\quad\mathcal{Q}^{i\dagger}_\alpha|0\rangle=0\qquad\forall\,\alpha,i. \tag{5.8}$$

[1] A proof can be found in the textbook of Weinberg [3].

Any state with positive energy cannot be invariant under supersymmetry transformations. It follows in particular that every one-particle state $|1\rangle$ must have super partner states $\mathcal{Q}^i_\alpha|1\rangle$ or $\mathcal{Q}^{i\dagger}_\alpha|1\rangle$. The spin of these partners will differ by $\frac{1}{2}$ from that of $|1\rangle$. Thus

• *each supermultiplet must contain at least one boson and one fermion whose spins differ by* $\frac{1}{2}$.

The translation invariance of $\mathcal{Q}$ implies that $\mathcal{Q}$ does not change energy-momentum

$$P_\mu|p\rangle = p_\mu|p\rangle \Longrightarrow P_\mu\mathcal{Q}^i_\alpha|p\rangle = p_\mu\mathcal{Q}^i_\alpha|p\rangle \quad , \quad P_\mu\mathcal{Q}^{i\dagger}_\alpha|p\rangle = p_\mu\mathcal{Q}^{i\dagger}_\alpha|p\rangle, \tag{5.9}$$

and therefore

• *all states in a multiplet of unbroken supersymmetry have the same mass.*

Supersymmetry is spontaneously broken if the ground state will not be invariant under all supersymmetry transformations,

$$\mathcal{Q}^i_\alpha|0\rangle \neq 0 \quad \text{or} \quad \mathcal{Q}^{i\dagger}_\alpha|0\rangle \neq 0 \tag{5.10}$$

for same α and i. We conclude that

• *supersymmetry is spontaneously broken if and only if the energy of the lowest lying state is not exactly zero.*

In *extended supersymmetry* the $\mathcal{Q}^i$ may carry a representation of some internal symmetry,

$$[T_r, \mathcal{Q}^i_\alpha] = (t_r)^i{}_j\mathcal{Q}^j_\alpha. \tag{5.11}$$

Since we assume this so-called *R-symmetry* to be compact, the representation matrices t can be chosen Hermitian, $t_r = t_r^\dagger$.
Now let us consider the anticommutator $\{\mathcal{Q}, \mathcal{Q}\}$. It must be a linear combination of the bosonic operators in the representation $(0,0)$ and $(1,0)$ of the Lorentz group. The only three-dimensional $(1,0)$ representation in the bosonic sector is the (anti)selfdual part of $J_{\mu\nu}$. Such a term would not commute with the 4-momentum, whereas $\{\mathcal{Q}, \mathcal{Q}\}$ does. Thus we are left with

$$\{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} = 2\varepsilon_{\alpha\beta}Z^{ij}, \tag{5.12}$$

were Z^{ij} commutes with the space-time symmetries and hence is some linear combination of the internal symmetry generators,

$$Z^{ij} = \alpha^{rij}T_r. \tag{5.13}$$

Using the super-Jacobi identity one shows that the Z^{ij} commute with *all* generators of the superalgebra, including the T_r and thus with themselves,

$$[T_r, Z^{ij}] = 0 \Longrightarrow [Z^{ij}, Z^{kl}] = 0. \tag{5.14}$$

For this reason they are called *central charges.* The internal symmetries are generated by $\{T_r\} = \{T'_\ell, Z^{ij}\}$. In what follows we shall skip the prime at the non-central generators T'_ℓ.

Summary: It may be useful to collect the relevant (anti)commutation relations of the $\mathcal{N}$-extended superalgebra in 4 dimensions containing the POINCARÉ algebra (3.1-3.3) as a subalgebra. The supercharges commute with translations (5.4), transform as spinors under LORENTZ transformations (5.5) and transform under the R-symmetry as

$$[T_r, \mathcal{Q}^i_\alpha] = (t_r)^i{}_j \mathcal{Q}^j_\alpha \quad , \quad [T_r, \bar{\mathcal{Q}}^i_{\dot\alpha}] = -\bar{\mathcal{Q}}^j_{\dot\alpha}(t_r)_j{}^i. \tag{5.15}$$

The generators of the R-symmetry generate a LIE subalgebra,

$$[T_r, T_s] = iC_{rs}{}^t T_t \tag{5.16}$$

and commute with the bosonic generators of the POINCARÉ algebra,

$$[T_r, J_{\mu\nu}] = [T_r, P_\mu] = 0. \tag{5.17}$$

The supercharges fulfill the following anticommutation relations

$$\{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} = 2\varepsilon_{\alpha\beta} Z^{ij} \tag{5.18}$$

$$\{\bar{\mathcal{Q}}^i_{\dot\alpha}, \bar{\mathcal{Q}}^j_{\dot\beta}\} = 2\varepsilon_{\dot\alpha\dot\beta} \bar{Z}^{ij} \tag{5.19}$$

$$\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}^j_{\dot\alpha}\} = 2\delta^{ij}\sigma^\mu_{\alpha\dot\alpha} P_\mu \, , \tag{5.20}$$

where (Z^{ij}) is the antisymmetric *central charge matrix*. The central charges Z^{ij} commute with all generators of the super-POINCARÉ algebra. According to the HAAG-LOPUSZANSKI-SOHNIUS theorem [48] the operators $\{P_\mu, J_{\mu\nu}, T_r, Z^{ij}, \mathcal{Q}^i_\alpha\}$ generate the *most general* LIE superalgebra of an S-matrix. The bosonic ones generate the direct product of the POINCARÉ and internal symmetry group. The remaining (anti)commutators are given in (5.4-5.5), (5.15) and (5.18-5.20).

5.2 Representations

Let us discuss the representation theory of $\mathcal{N}$-extended supersymmetry in four dimensions. First we shall assume that the *central charges are zero*.

5.2.1 Massive representations without central charges

For vanishing central charges the supercharges anticommute,

$$\{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} = \{\bar{\mathcal{Q}}^i_{\dot\alpha}, \bar{\mathcal{Q}}^j_{\dot\beta}\} = 0. \tag{5.21}$$

For a massive particle we may choose the rest frame in which $P \sim (M, \boldsymbol{0})$. Then the relations (5.18) simplify as follows:

$$\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}^j_{\dot\alpha}\} = 2M\delta^{ij}(\sigma_0)_{\alpha\dot\alpha} = 2M\delta_{\alpha\dot\alpha}\delta^{ij}. \tag{5.22}$$

$\mathcal{Q}$ is a tensor operator of spin $\frac{1}{2}$, as follows from

$$[\mathcal{Q}^i, \boldsymbol{J}\,] = \tfrac{1}{2}(\boldsymbol{\sigma}\mathcal{Q}^i)$$

and in particular

$$[J_3, \mathcal{Q}^i_1] = -\tfrac{1}{2}\mathcal{Q}^i_1 \quad \text{and} \quad [J_3, \mathcal{Q}^i_2] = \tfrac{1}{2}\mathcal{Q}^i_2. \tag{5.23}$$

Therefore, the result of the action of $\mathcal{Q}$ on a state with spin s will be a linear combination of states with spin $s \pm \frac{1}{2}$. Now we define the $2\mathcal{N}$ fermionic creation and annihilation operators

$$A^i_\alpha = \frac{1}{\sqrt{2M}}\,\mathcal{Q}^i_\alpha \quad \text{and} \quad \bar{A}^i_{\dot\alpha} = \frac{1}{\sqrt{2M}}\,\bar{\mathcal{Q}}^i_{\dot\alpha} \tag{5.24}$$

which satisfy the simple algebra

$$\{A^i_\alpha, \bar{A}^j_{\dot\alpha}\} = \delta^{ij}\delta_{\alpha\dot\alpha}. \tag{5.25}$$

Building irreducible representations is now straightforward. We introduce a CLIFFORD *vacuum* $|\Omega\rangle$, which is annihilated by the A^i_α and generate the FOCK states by acting with the creation operators $\bar{A}^i_{\dot\alpha}$. A typical state would be

$$\bar{A}^{i_1}_{\dot\alpha_1}\cdots\bar{A}^{i_n}_{\dot\alpha_n}\,|\Omega\rangle. \tag{5.26}$$

It is antisymmetric under interchange of index-pairs $(\dot\alpha, i) \leftrightarrow (\dot\beta, j)$. Since there are $2\mathcal{N}$ creation operators there are $\binom{2\mathcal{N}}{n}$ states at the n'th oscillator level. The total number of states built on *one* vacuum state is

$$\sum_{n=0}^{2\mathcal{N}}\binom{2\mathcal{N}}{n} = (1+1)^{2\mathcal{N}} = 4^{\mathcal{N}}, \tag{5.27}$$

half of them being bosonic and half fermionic. If the vacuum sector is degenerate, which happens if the CLIFFORD vacuum $|\Omega\rangle$ is a member of a spin multiplet, then the number of states is

$$\text{NUMBER OF STATES} = 4^{\mathcal{N}}\cdot\ \text{DIMENSION OF VACUUM SECTOR}.$$

The operators $\bar{A}^i_{\dot 2}$ increase J_3 by 1/2, such that the *maximal spin* is equal to the spin s_0 of the ground-state plus $\mathcal{N}/2$,

$$s_{\max} = s_0 + \mathcal{N}/2.$$

The *minimal spin* is 0 if $\mathcal{N}/2 \geq s_0$ or $s_0 - \mathcal{N}/2$ otherwise.
Since renormalizability requires *massive matter* to have spin $\leq \frac{1}{2}$. We conclude from the above expression for $s_{\max}$ that we must have

- $\mathcal{N} = 1$ *for renormalizable coupling of massive matter.*

We see that in the absence of central charges, the only relevant massive multiplet is that of the massive WESS-ZUMINO model which has $\mathcal{N}=1$ and $s_0=0$. It contains a scalar, a pseudo-scalar and the two spin states of a massive MAJORANA spinor

$\mathcal{N}=1$	s^P :	0^+ $\frac{1}{2}$ 0^-
	states:	1 2 1

(5.28)

For example, the spin-zero states are $|\Omega\rangle$ and $\bar{A}_{\dot{1}}\bar{A}_{\dot{2}}|\Omega\rangle$.

5.2.2 Massless representations

For massless particles we can choose an inertial frame in which $P_\mu=(E,0,0,E)$. Then the nontrivial anti-commutation relations have the form

$$\{\mathcal{Q}^i_\alpha,\bar{\mathcal{Q}}^j_{\dot\alpha}\}=2E\delta^{ij}(\sigma_0+\tau_3)_{\alpha\dot\alpha}=\begin{pmatrix}4E&0\\0&0\end{pmatrix}\delta^{ij}\,, \tag{5.29}$$

the other anticommutators being zero. Since

$$\{\mathcal{Q}^i_2,\bar{\mathcal{Q}}^i_{\dot 2}\}=\mathcal{Q}^i_2\mathcal{Q}^{i\dagger}_2+\mathcal{Q}^{i\dagger}_2\mathcal{Q}^i_2=0, \tag{5.30}$$

the $\mathcal{Q}^i_2$ are represented by zero in a unitary theory. Thus we only have $\mathcal{N}$ non-trivial creation and annihilation operators

$$A^i=\frac{1}{2\sqrt{E}}\,\mathcal{Q}^i_1\quad\text{and}\quad\bar{A}^i=\frac{1}{2\sqrt{E}}\,\bar{\mathcal{Q}}^i_{\dot 1}, \tag{5.31}$$

and the FOCK-representation is $2^{\mathcal{N}}$-dimensional. It is much shorter than the massive one which contains $4^{\mathcal{N}}$ states.
The following LORENTZ generators commute with $P_\mu=(E,0,0,E)$:

$$J_1=J_{10}+J_{13},\quad J_2=J_{20}+J_{23}\quad\text{and}\quad J_3=-J_{12}. \tag{5.32}$$

Observe that J_3 is just the *helicity operator* λ for a massless particle moving in the 3-direction. The J_i generate the *non-compact* little group E_2 of translations and rotations in the 2-plane,

$$[J_1,J_3]=iJ_2,\quad[J_2,J_3]=-iJ_1\quad\text{and}\quad[J_1,J_2]=0. \tag{5.33}$$

In any *finite dimensional* unitary representation the generators J_1,J_2 are trivially represented. Note that A^i increases the helicity by $\frac{1}{2}$ and $\bar{A}^i$ decreases it by $\frac{1}{2}$,

$$[\lambda,A^i]=\tfrac{1}{2}A^i\quad\text{and}\quad[\lambda,\bar{A}^i]=-\tfrac{1}{2}\bar{A}^i. \tag{5.34}$$

We introduce the CLIFFORD vacuum $|\Omega\rangle$ with maximal helicity λ which is annihilated by all A^i. The states in an irreducible representation are gotten by acting with the creation operators on this state. For example,

$$\bar{A}^{i_1}\cdots\bar{A}^{i_n}|\Omega\rangle \tag{5.35}$$

has helicity $\lambda - n/2$. This way we get the following states

$$\begin{array}{|l|cccc|} \hline \text{helicity:} & \lambda & \lambda-\frac{1}{2} & \dots & \lambda-\frac{\mathcal{N}}{2} \\ \hline \text{multiplicity:} & 1 & \binom{\mathcal{N}}{1} & \dots & \binom{\mathcal{N}}{\mathcal{N}} \\ \hline \end{array} \tag{5.36}$$

The total number of states in an irreducible massless representation is

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}. \tag{5.37}$$

According to the CPT theorem a physical massless state contains both helicities λ and $-\lambda$. A *single* supermultiplet can contain all massless states for $\lambda = \mathcal{N}/4$. Otherwise the states must be doubled starting with a second Clifford vacuum with helicity $\lambda' = \mathcal{N}/2 - \lambda$. Here we will describe the important examples with helicities up to one.

1. $\mathcal{N} = 1$ **supersymmetry:** The number of Clifford-states in an irreducible multiplet is just $1 + 1$ and we need at least two Clifford vacua to built a CPT invariant model.

   - For the *chiral multiplet* $\lambda = \frac{1}{2}$ and $\lambda' = 0$ and we have the following states

helicity:	1/2	1 Majorana spinor
	0	2 real scalars
	−1/2	1 Majorana spinor

     These are the fields of the massless Wess-Zumino model.

   - The *vector multiplet* with $\lambda = 1$ and $\lambda' = -1/2$ consists of

helicity:	1	1 gauge field
	1/2	1 Majorana spinor
	−1/2	1 Majorana spinor
	−1	1 gauge field

2. $\mathcal{N} = 2$ **supersymmetry:** A irreducible representation of the $\mathcal{N} = 2$-extended superalgebra contains 3 different helicities and $1 + 2 + 1$ 'states'. The relevant multiplets are the hyper- and vector multiplet.

   - The simplest multiplet is the *hyper-multiplet* $\lambda = 1/2$ with:

helicity:	1/2	1 Majorana spinor
	0	2 real scalars
	−1/2	1 Majorana spinor

   - To built a *vector multiplet* we need two Clifford vacua with $\lambda = 1$ and $\lambda' = 0$ so that it contains

helicity:	1	1 gauge field
	1/2	2 MAJORANA spinors
	0	2 real scalars
	$-1/2$	2 MAJORANA spinors
	-1	1 gauge field

The two MAJORANA spinors combine to a DIRAC spinor. This multiplet has been considered in the SEIBERG-WITTEN solution to $\mathcal{N}=2$ supersymmetric gauge theories [49].

3. $\mathcal{N}=4$ **supersymmetry:** An irreducible supermultiplet with 4 supercharges consists of $1+4+6+4+1=16$ states.

   - The unique multiplet giving rise to a renormalizable field theory in flat spacetime is the *vector multiplet*. It has $\lambda=1$ and contains

helicity:	1	1 gauge field
	1/2	4 MAJORANA spinors
	0	6 real scalars
	$-1/2$	4 MAJORANA spinors
	-1	1 gauge field

This multiplet enters the celebrated AdS/CFT correspondence [50].

5.2.3 Non-zero central charges

In this case massive supermultiplets can be as short as massless ones. Under a $U(N)$ transformation of the super charges,

$$\mathcal{Q}^i_\alpha \to U^i{}_j \mathcal{Q}^j_\alpha \quad \text{and} \quad \bar{\mathcal{Q}}^i_{\dot\alpha} \to \bar{\mathcal{Q}}^j_{\dot\alpha} (U^\dagger)_j{}^i$$

the antisymmetric central charge matrix $Z=(Z^{ij})$ transforms as

$$Z \longrightarrow UZU^T, \quad \bar{Z} \longrightarrow \bar{U}\bar{Z}U^\dagger.$$

In can be shown that there exists a unitary U such that Z becomes block diagonal[2],

$$\begin{pmatrix} 0 & Z_1 & 0 & 0 & & \dots & \\ -Z_1 & 0 & 0 & 0 & & \dots & \\ 0 & 0 & 0 & Z_2 & & \dots & \\ 0 & 0 & -Z_2 & 0 & & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & & & & 0 & Z_{\mathcal{N}/2} \\ & \dots & & & & -Z_{\mathcal{N}/2} & 0 \end{pmatrix}. \tag{5.38}$$

[2] Consider even $\mathcal{N}$.

We have labeled the *real positive* eigenvalues by Z_m, $m = 1, 2, \ldots, \mathcal{N}/2$. We will split the index $i \to (a, m)$: $a = 1, 2$ labels positions inside the 2×2 blocks while m labels the blocks. Then

$$\{\mathcal{Q}^{am}_\alpha, \bar{\mathcal{Q}}^{bn}_{\dot\alpha}\} = 2M\delta_{\alpha\dot\alpha}\delta^{ab}\delta^{mn} \quad , \quad \{\mathcal{Q}^{am}_\alpha, \mathcal{Q}^{bn}_\beta\} = 2Z_n\varepsilon_{\alpha\beta}\varepsilon^{ab}\delta^{mn}\,. \tag{5.39}$$

Now we define the following fermionic oscillators

$$A^m_\alpha = \frac{1}{\sqrt{2}}\left(\mathcal{Q}^{1m}_\alpha + \varepsilon_{\alpha\beta}\mathcal{Q}^{\dagger 2m}_\beta\right) \quad , \quad B^m_\alpha = \frac{1}{\sqrt{2}}\left(\mathcal{Q}^{1m}_\alpha - \epsilon_{\alpha\beta}\mathcal{Q}^{\dagger 2m}_\beta\right)\,, \tag{5.40}$$

and similarly for the conjugate operators. Their anticommutators read

$$\begin{aligned} \{A^m_\alpha, A^n_\beta\} &= \{A^m_\alpha, B^n_\beta\} = \{B^m_\alpha, B^n_\beta\} = 0\,,\\ \{A^m_\alpha, A^{\dagger n}_\beta\} &= 2\delta_{\alpha\beta}\delta^{mn}(M + Z_m)\\ \{B^m_\alpha, B^{\dagger n}_\beta\} &= 2\delta_{\alpha\beta}\delta^{mn}(M - Z_m)\,. \end{aligned} \tag{5.41}$$

Unitarity requires that the right-hand sides in (5.41) be non-negative. This in turn implies the well-known BOGOMOL'NYI BOUND

$$M \geq \max|Z_m|\,. \tag{5.42}$$

It is required by the unitarity of the underlying supersymmetric theory.
Assume, for example, that $Z_m = M$ for all m. Then

$$\{B^m_\alpha, B^{\dagger m}_\alpha\} = 0, \tag{5.43}$$

implies that the B^m_α vanish identically and we are left with the following creation and annihilation operators

$$A^m_\alpha,\, A^{\dagger m}_\alpha, \quad i = m, \ldots, \tfrac{1}{2}\mathcal{N}, \quad \alpha = 1, 2.$$

They generate a multiplet with $2^{\mathcal{N}}$ states and not $4^{\mathcal{N}}$ states as one gets without central charges. More generally, consider $0 \leq r \leq \mathcal{N}/2$ of the Z_m's to be equal to M. Then $2r$ of the B-oscillators vanish identically and we are left with $2\mathcal{N} - 2r$ creation and annihilation operators. The representation has $2^{2\mathcal{N}-2r}$ states.
The maximal case $r = \mathcal{N}/2$ gives rise to the *short BPS multiplet* which has the same number of states as the massless multiplet. The other multiplets with $0 < r < \mathcal{N}/2$ are known as *intermediate BPS multiplets.*
BPS states are important probes of non-perturbative physics in theories with extended ($\mathcal{N} \geq 2$) supersymmetry. The BPS states are special for various reasons:

- Although they are massive, they form multiplets under extended SUSY which are shorter than the massive multiplets. Their mass is given in terms of their charges and HIGGS (moduli) expectation values.
- They are the only multiplets that can become massless when we vary coupling constants and Higgs expectation values without breaking supersymmetry.

- They describe solitonic excitations which (at rest) exert no force on each other.
- Their mass-formula is supposed to be exact if one uses renormalized values for the charges and moduli. If quantum corrections would spoil the relation of mass and charges without breaking supersymmetry, then the number of states in the supermultiplet would jump as function of the moduli parameter.

Chapter 6

Supersymmetric Yang-Mills Theories

The simplest $\mathcal{N}=1$ supersymmetric gauge theory is ABELian and contains a 'photon' and its massless superpartner, the 'photino'. The neutral photino does not couple to the photon and the resulting theory has no interactions. Although the ABELian theory has a free dynamics it shares many *algebraic properties* with interacting non-ABELian gauge theories. The transition from the free ABELian to the interacting non-ABELian theories is achieved by replacing ordinary by covariant derivatives. The non-ABELian models contain charged 'gluons' and their superpartners, the massless charged 'gluinos'. These supersymmetric YANG-MILLS theories are introduced in the second part of this chapter. Supersymmetry was applied first to ABELian gauge theories without using the superfield formalism by J. WESS and B. ZUMINO [51]. It was then extended to non-ABELian YANG-MILLS theories by S. FERRARA and B. ZUMINO [52] and A. SALAM and J. STRATHDEE [53]

6.1 $\mathcal{N}=1$ Abelian gauge theories

We have seen that a CPT-doubled $\mathcal{N}=1$ vector multiplet contains one gauge field and one massless MAJORANA spinor. In an off-shell version (and the WESS-ZUMINO gauge) we also need an uncharged pseudo-scalar field which later may be eliminated. Since a MAJORANA particle is its own antiparticle and thus uncharged, the LAGRANGEan density takes the simple form

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi+\tfrac{1}{2}\mathcal{G}^2,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{6.1}$$

The dimensions of the fields are

$$[A_\mu]=L^{-1},\quad [\psi]=L^{-3/2},\quad [\mathcal{G}]=L^{-2}. \tag{6.2}$$

In order to find the supersymmetry transformations of the fields it is useful to recall the hermiticity properties (3.80) for fermionic bilinears of MAJORANA spinors

and the results (3.75), were we calculated the sign in $\bar{\varepsilon}\gamma^{(n)}\psi = \pm\bar{\psi}\gamma^{(n)}\varepsilon$ for two MAJORANA spinors ε and ψ. We use the hermitian $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$.
Taking the dimensions and hermiticity properties into account we could guess the following supersymmetry transformations,

$$\begin{aligned} \delta_\varepsilon A_\mu &= i\,\bar{\varepsilon}\gamma_\mu\psi && (6.3)\\ \delta_\varepsilon\psi &= iF^{\mu\nu}\Sigma_{\mu\nu}\varepsilon + iq\mathcal{G}\gamma_5\varepsilon && (6.4)\\ \delta_\varepsilon\mathcal{G} &= q\,\bar{\varepsilon}\gamma_5\not{\partial}\psi, \quad q\in\mathbb{R}. && (6.5)\end{aligned}$$

The supersymmetry parameter ε is a constant MAJORANA spinor anti-commuting with itself and with ψ. Up to a surface term the LAGRANGEan (6.1) is invariant under these supersymmetry transformation,

$$\delta_\varepsilon\mathcal{L} = \bar{\varepsilon}\partial_\mu V^\mu, \quad V^\mu = \tfrac{1}{2}\left({}^*F^{\mu\nu}\gamma_5 - iF^{\mu\nu}\right)\gamma_\nu\psi + \tfrac{1}{2}q\mathcal{G}\gamma_5\gamma^\mu\psi. \tag{6.6}$$

In the proof, which I leave as an exercise, one needs the formulas (A.1) and (A.5) in the appendix and the BIANCHI *identity* ${}^*F^{\mu\nu}_{\ \ ,\nu} = 0$ for the *dual field strength tensor*

$${}^*F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{6.7}$$

At this point the real parameter q is not fixed. Only when we want to recover the superalgebra are we forced to choose $q=\pm 1$.

6.1.1 The closing of the algebra

We repeat what we have done for the WESS-ZUMINO model and calculate the commutators of two supersymmetry transformations. The commutator acting on the bosonic fields is easily computed,

$$\begin{aligned} [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]A_\mu &= -F^{\rho\sigma}\left(\bar{\varepsilon}_2\gamma_\mu\Sigma_{\rho\sigma}\varepsilon_1 - \bar{\varepsilon}_1\gamma_\mu\Sigma_{\rho\sigma}\varepsilon_2\right) \overset{(A.5)}{=} 2i\,\bar{\varepsilon}_2\gamma^\nu\varepsilon_1 F_{\mu\nu}\\ &= -2i\left(\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\right)\partial_\rho A_\mu + 2i\partial_\mu\left(\bar{\varepsilon}_2\not{A}\varepsilon_1\right). \end{aligned} \tag{6.8}$$

The first term in the second line is the expected infinitesimal translation of the vector field. The last term we did not encounter in the WESS-ZUMINO model. It is just a field dependent *gauge transformation* with gauge parameter

$$\lambda = 2i\,\bar{\varepsilon}_2\not{A}\varepsilon_1. \tag{6.9}$$

Similarly, using (A.1) and the BIANCHI identity one finds

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\mathcal{G} = -2iq^2\,\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\partial_\rho\mathcal{G}.$$

For the choices $q=\pm 1$ we find the expected commutator and hence we shall assume that $q\in\{-1,1\}$ in what follows. To calculate the commutator of two transformations on the photino field is a bit more tricky. First one obtains

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi = \left(i\left(\bar{\varepsilon}_1\gamma_5\gamma^\rho\partial_\rho\psi\right)\gamma_5\varepsilon_2 - 2\left(\bar{\varepsilon}_1\gamma^\nu\partial^\mu\psi\right)\Sigma_{\mu\nu}\varepsilon_2\right) - (\varepsilon_1\leftrightarrow\varepsilon_2).$$

With the help of (A.6), the (anti)commutators of the Clifford and Lorentz algebras and the relation (A.1) in the appendix one ends up with the expected result

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi=-2i(\,\bar\varepsilon_2\gamma^\rho\varepsilon_1)\,\partial_\rho\psi. \tag{6.10}$$

As in chapter 4 we introduce the *supercharge* via $\delta_\varepsilon(..)=i[\,\bar\varepsilon\mathcal{Q},..]$ and arrive at

$$\{\mathcal{Q}_\alpha,\bar{\mathcal{Q}}^\beta\}=2i\,(\gamma^\mu)_\alpha^{\;\beta}\,\partial_\mu-G_\alpha^{\;\beta}(A), \tag{6.11}$$

where $G_\alpha^{\;\beta}$ is the gauge transformation with gauge parameter $\lambda_\alpha^{\;\beta}=2i(\not{A})_\alpha^{\;\beta}$. On gauge invariant fields the last term on the right hand side vanishes and we recover the familiar superalgebra.

6.1.2 Noether charge

To find the Noether charge we must first calculate

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta_\varepsilon A_\nu+\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta_\varepsilon\psi=-F^{\mu\nu}\delta_\varepsilon A_\nu-\tfrac{i}{2}\delta_\varepsilon\bar\psi\gamma^\mu\psi. \tag{6.12}$$

Using the above expressions for the supersymmetry transformations this can be written as (we take $q=1$)

$$-\tfrac{3i}{2}F^{\mu\nu}\,\bar\varepsilon\gamma_\nu\psi+\tfrac{1}{2}\,{}^*F^{\mu\nu}\,\bar\varepsilon\gamma_\nu\gamma_5\psi+\tfrac{1}{2}\mathcal{G}\,\bar\varepsilon\gamma_5\gamma^\mu\psi. \tag{6.13}$$

Subtracting V^μ in (6.6) we find the Noether current $J^\mu=-i\delta_\varepsilon\bar\psi\gamma^\mu\psi|_{\mathcal{G}=0}$. Inserting the $3+1$ decompositions of the field strength tensor and its dual,

$$(F_{0i},F_{ij})=(E_i,-\epsilon_{ijk}B_k)\quad\text{and}\quad({}^*F_{0i},{}^*F_{ij})=(-B_i,-\epsilon_{ijk}E_k), \tag{6.14}$$

the corresponding conserved and gauge invariant Noether charge takes the form

$$\mathcal{Q}=\int d\boldsymbol{x}\,(i\pi_i-\gamma_5B_i)\,\gamma_i\psi,\quad\text{where}\quad\boldsymbol{\pi}=\boldsymbol{E} \tag{6.15}$$

is the momentum field conjugate to $\boldsymbol{A}$.

6.2 $\mathcal{N}=1$ SYM theories

We consider $SU(N)$ gauge theories without matter, sometimes called SUSY *gluodynamics*, with 'gluons' and massless 'gluinos'. The former are described by a vector field A_μ and the latter by a Majorana spinor field ψ. Off-shell we also need an auxiliary field $\mathcal{G}$. All fields take their values in the Lie algebra of the gauge group,

$$A_\mu=A_\mu^aT_a,\quad\psi=\psi^aT_a\quad\text{and}\quad\mathcal{G}=\mathcal{G}^aT_a. \tag{6.16}$$

The real *structure constants* in

$$[T_a, T_b] = f_{ab}{}^c T_c \tag{6.17}$$

are totally *antisymmetric* and we normalize the hermitian generators by $\mathrm{Tr}\, T_a T_b = \delta_{ab}$.
The gauge and matter fields transform under gauge transformations as

$$\begin{array}{rcl} A & \to & UAU^{-1} + iUdU^{-1} \\ \psi & \to & U\psi U^{-1} \\ \mathcal{G} & \to & U\mathcal{G}U^{-1} \end{array} \tag{6.18}$$

with group elements $U(x)$. With $U \sim \mathbb{1} + i\lambda$ the infinitesimal transformations read

$$\delta_\lambda A_\mu = D_\mu \lambda, \quad \delta_\lambda \psi = i[\lambda, \psi] \quad \text{and} \quad \delta_\lambda \mathcal{G} = i[\lambda, \mathcal{G}]. \tag{6.19}$$

The gauge invariant LAGRANGEan density is the expected generalization of (6.1),

$$\mathcal{L} = -\tfrac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{2}\mathrm{Tr}\,(\bar\psi \not{D}\psi) + \tfrac{1}{2}\mathrm{Tr}\, \mathcal{G}^2. \tag{6.20}$$

It contains the *covariant derivative* of the spinor field $D_\mu\psi = \partial_\mu\psi - i[A_\mu, \psi]$ and the gauge covariant *field strength tensor* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Both transform according to the adjoint representation.
The susy transformations are gotten from those of the ABELian model if we only replace ordinary by covariant derivatives,

$$\begin{array}{rcl} \delta_\varepsilon A_\mu & = & i\,\bar\varepsilon\gamma_\mu\psi \\ \delta_\varepsilon\psi & = & iF^{\mu\nu}\Sigma_{\mu\nu}\varepsilon + i\mathcal{G}\gamma_5\varepsilon \\ \delta_\varepsilon\mathcal{G} & = & \bar\varepsilon\gamma_5\not{D}\psi. \end{array} \tag{6.21}$$

To calculate the variation of the LAGRANGEan density one needs the formula $\delta F_{\mu\nu} = D_\mu\delta A_\nu - D_\nu\delta A_\mu$ which yields

$$\delta_\varepsilon F_{\mu\nu} = i\,\bar\varepsilon\,(\gamma_\nu D_\mu - \gamma_\mu D_\nu)\,\psi,$$

together with the identities

$$\delta(D_\mu\psi) = D_\mu\delta\psi - i[\delta A_\mu, \psi] \quad \text{and} \quad \delta_\varepsilon\bar\psi = -i\,\bar\varepsilon F^{\mu\nu}\Sigma_{\mu\nu} + i\,\bar\varepsilon\mathcal{G}\gamma_5. \tag{6.22}$$

As for the ABELian model one uses the BIANCHI identity $D_{[\rho}F_{\mu\nu]} = 0$ and obtains

$$\begin{array}{rcll} \delta_\varepsilon\mathcal{L} & = & \bar\varepsilon\partial_\mu V^\mu + \tfrac{i}{2}\mathrm{Tr}\,\left(\bar\psi\gamma^\mu[\,\bar\varepsilon\gamma_\mu\psi, \psi]\right) & \text{with} \qquad (6.23) \\ V^\mu & = & \tfrac{1}{2}\mathrm{Tr}\,\{({}^*F^{\mu\nu}\gamma_5 - iF^{\mu\nu})\,\gamma_\nu\psi\} + \tfrac{1}{2}\mathrm{Tr}\,(\mathcal{G}\gamma_5\gamma^\mu\psi)\,. & \qquad (6.24) \end{array}$$

If we can show that the last term in (6.23) vanishes, then we have proved the invariance of the action. It is not straightforward to show this, so let me indicate the proof. First we expand $\psi = \psi^a T_a$, $\bar\psi = \bar\psi^a T_a$ and rewrite this term as

$$\tfrac{i}{2}\mathrm{Tr}\,\left(\bar\psi\gamma^\mu[\,\bar\varepsilon\gamma_\mu\psi, \psi]\right) = \tfrac{1}{2}f_{abc}(\bar\psi^a\gamma^\mu\psi^b)(\,\bar\varepsilon\gamma_\mu\psi^c). \tag{6.25}$$

Then we insert the general FIERZ identity (3.73) for $\psi^c\bar\psi^a$ in the right hand side of

$$(\bar\psi^a\gamma^\mu\psi^b)(\bar\varepsilon\gamma_\mu\psi^c) = \bar\varepsilon\gamma_\mu(\psi^c\bar\psi^a)\gamma^\mu\psi^b, \tag{6.26}$$

and use the relations (A.3) in the appendix to arrive at the useful identity

$$\begin{aligned}(\bar\psi^a\gamma^\mu\psi^b)(\bar\varepsilon\gamma_\mu\psi^c) &= -(\bar\varepsilon\psi^b)(\bar\psi^a\psi^c) + \tfrac{1}{2}(\bar\varepsilon\gamma_\rho\psi^b)(\bar\psi^a\gamma^\rho\psi^c)\\ &\quad -\tfrac{1}{2}(\bar\varepsilon\gamma_\rho\gamma_5\psi^b)(\bar\psi^a\gamma_5\gamma^\rho\psi^c) + (\bar\varepsilon\gamma_5\psi^b)(\bar\psi^a\gamma_5\psi^c).\end{aligned} \tag{6.27}$$

Note that all but the second term on the right hand side are symmetric in a and c such that

$$f_{abc}(\bar\psi^a\gamma^\mu\psi^b)(\bar\varepsilon\gamma_\mu\psi^c) = \tfrac{1}{2}f_{abc}(\bar\psi^a\gamma^\mu\psi^c)(\bar\varepsilon\gamma_\mu\psi^b) = -\tfrac{1}{2}f_{abc}(\bar\psi^a\gamma^\mu\psi^b)(\bar\varepsilon\gamma_\mu\psi^c) \tag{6.28}$$

which implies, that the expression on the left is zero. We conclude that the last term in (6.23) is absent and that the susy variation of the LAGRANGEan density is just a divergence.
The conserved NOETHER current has the familiar form $J^\mu = -i\mathrm{Tr}\,\delta\bar\psi\gamma^\mu\psi|_{\mathcal{G}=0}$ such that the conserved supercharge reads

$$\mathcal{Q} = \int d\boldsymbol{x}\,\mathrm{Tr}\,\{(i\pi_i - B_i\gamma_5)\gamma_i\psi\}\,. \tag{6.29}$$

The commutator of two supersymmetry transformations on the fields is a translation plus a field dependent infinitesimal *gauge transformation*. After some lengthy but straightforward manipulations in which one uses (A.4,A.8), the BIANCHI identity, (A.4,A.6) and $2i\Sigma_{\mu\nu}\gamma^\nu = 3\gamma_\mu$ one ends up with the expected commutators

$$\begin{aligned}[][\delta_{\varepsilon_1},\delta_{\varepsilon_2}]A_\mu &= -2i\,(\bar\varepsilon_2\gamma^\nu\varepsilon_1)\,\partial_\nu A_\mu + D_\mu\lambda,\\ [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\mathcal{G} &= -2i\,(\bar\varepsilon_2\gamma^\rho\varepsilon_1)\,\partial_\rho\mathcal{G} + i[\lambda,\mathcal{G}],\\ [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi &= -2i(\bar\varepsilon_2\gamma^\rho\varepsilon_1)\partial_\rho\psi + i[\lambda,\psi].\end{aligned} \tag{6.30}$$

The gauge parameter $\lambda = 2i(\bar\varepsilon_2 A\!\!\!/\varepsilon_1)$ depends on the gauge field and the susy parameters ε_i. As in the ABELian case the superalgebra closes only on *gauge invariant fields*.

6.3 $\mathcal{N}=2$ SYM theories

In this section we consider theories with two supersymmetries. Realistic (i.e. chiral) models of particle interactions have at most one supersymmetry. However, there are good reasons for discussing models with extended susy. Their dynamics is under much better control which may lead to precise statements about their spectra, in perturbation theory and beyond. Some models possess *central charges* which are saturated by BPS-configurations. Magnetic monopoles of BPS type[1] are particular

[1]See [54] for a recent review on topological objects in gauge theories.

BPS-configurations. In [55] these monopoles were first discussed in the context of SYM with two supercharges. The rather simple $\mathcal{N}=2$ model has played an important role in recent developments about confinement in asymptotically free gauge theories. SEIBERG and WITTEN derived an analytic expression for the low energy effective action (the leading two terms in a derivative expansion) of this theory [49].

On shell the model contains one vector field A_μ, two MAJORANA spinors λ_1 and λ_2, one scalar field ϕ_1 and one pseudo-scalar field ϕ_2. All fields transform according to the adjoint representation of the gauge group. The two MAJORANA spinors define a DIRAC spinor as $\kappa\psi=\lambda_1+i\lambda_2$, where $\kappa=\sqrt{2}$, with inverse relation $\kappa\lambda_1=\psi+\psi_c$. Also the two MAJORANA supersymmtry parameters ε_1 and ε_2 are combined to a DIRAC parameter $\kappa\alpha=\varepsilon_1+i\varepsilon_2$. It follows, that

$$\begin{aligned} i\bar{\psi}\not{D}\psi &= \tfrac{i}{2}\sum\left(\bar{\lambda}_i\not{D}\lambda_i\right)-\tfrac{1}{2}\partial_\rho\left(\bar{\lambda}_1\gamma^\rho\lambda_2\right) \\ \bar{\alpha}M\psi &+ \bar{\alpha}_cM\psi_c=\sum\bar{\varepsilon}_iM\lambda_i \end{aligned} \tag{6.31}$$

hold true for any matrix M. Now we could return to the $\mathcal{N}=1$ vector multiplet and obtain the following transformation rule for the gauge potential,

$$\delta_\alpha A_\mu=i\sum\bar{\varepsilon}_i\gamma_\mu\lambda_i=i\bar{\alpha}\gamma_\mu\psi+i\bar{\alpha}_c\gamma_\mu\psi_c. \tag{6.32}$$

By dimensional arguments there are no other terms which are linear combinations of the ψ and ψ_c contracted with the supersymmetry parameters.

6.3.1 Action of Seiberg-Witten model

The SYM theory with two supercharges has the same particles as the SYM theory with one supercharge combined with the massless WESS-ZUMINO model. But since all fields are in one supermultiplet they all must transform according to the adjoint representation of the gauge group. To construct the supersymmetric action of the $\mathcal{N}=2$ model we could try to add the WESS-ZUMINO LAGRANGEan (4.34) with *massless* and LIE algebra valued fields and ∂_μ replaced by D_μ to the gauge model LAGRANGEan (6.20). If ϕ_1 and ϕ_2 are in the adjoint representation, then the self interaction term in (4.34), namely

$$-g\bar{\psi}\phi^\dagger\psi-\tfrac{1}{2}g^2\,\left(\phi_1^2+\phi_2^2\right)^2,\quad \phi=\phi_1+i\gamma_5\phi_2, \tag{6.33}$$

should be replaced by something like

$$-g\mathrm{Tr}\,\bar{\psi}[\phi^\dagger,\psi]+\tfrac{1}{2}g^2\,\mathrm{Tr}\,\left([\phi_1,\phi_2]^2\right),\quad \phi=\phi_1+i\gamma_5\phi_2. \tag{6.34}$$

Hence we would guess that the on-shell LAGRANGEan density of the $\mathcal{N}=2$ vector multiplet has the form

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{4}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\sum\mathrm{Tr}\,(D_\mu\phi_i)^2+\tfrac{1}{2}g^2\,\mathrm{Tr}\,([\phi_1,\phi_2]^2) \\ &\quad +i\mathrm{Tr}\,\bar{\psi}\not{D}\psi-g\mathrm{Tr}\,\bar{\psi}[\phi^\dagger,\psi] \end{aligned} \tag{6.35}$$

where ψ is a DIRAC spinor. Expanding the LIE algebra valued fields A_μ, ϕ_i, ψ in a trace-orthonormal basis $\{T_a\}$, for example $\psi = \psi^a T_a$, such that

$$
\begin{aligned}
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu \\
(D_\mu \phi)^a &= \partial_\mu \phi^a + g f^a{}_{bc} A^b_\mu \phi^c,
\end{aligned}
\tag{6.36}
$$

we obtain for the component fields

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \tfrac{1}{2} (D_\mu \phi_i)^a (D^\mu \phi_i)_a - \tfrac{1}{2} g^2 f^a{}_{bc} f_{apq} (\phi^b_1 \phi^p_1 \phi^c_2 \phi^q_2) \\
&+ i \bar{\psi}^a \gamma^\mu \Big(\partial_\mu \psi_a + g f_{abc} A^b_\mu \psi^c \Big) + i g f_{abc} \phi^a_1 \bar{\psi}^b \psi^c + g f_{abc} \phi^a_2 \bar{\psi}^b \gamma_5 \psi^c,
\end{aligned}
\tag{6.37}
$$

where one sums over all indexes. The model contains only *one coupling constant* g and the self-coupling of the scalar fields and YUKAWA couplings are determined by g. There is a potential term in the LAGRANGEan but it has *flat directions* whenever $[\phi_1, \phi_2] = 0$. Classically the model is scale invariant but this invariance is spontaneously broken by a non-zero expectation value of the scalar fields.
For the LEGENDRE transformation from the LAGRANGEan to the HAMILTONian formulation we need the momenta conjugate to the vector, scalar and DIRAC fields,

$$
\pi_i = F_{0i} = E_i, \quad \pi_{\phi_i} = D_0 \phi_i \quad \text{and} \quad \pi_\psi = i \psi^\dagger. \tag{6.38}
$$

In the convenient WEYL or temporal gauge $A_0 = 0$ these relations simplify to $\pi_i = \dot{A}_i$ and $\pi_{\phi_i} = \dot{\phi}_i$. The HAMILTONian density $\mathcal{H}$ takes the form

$$
\begin{aligned}
\mathcal{H} &= \mathrm{Tr} \Big\{ \tfrac{1}{2} \left(\boldsymbol{\pi}^2 + \boldsymbol{B}^2 \right) + \tfrac{1}{2} \sum \left(\pi^2_{\phi_i} + (\boldsymbol{D} \phi_i)^2 \right) - i \psi^\dagger \gamma^0 \gamma^i D_i \psi \\
&\qquad + g \bar{\psi} [\phi_1, \psi] - i g \bar{\psi} \gamma_5 [\phi_2, \psi] - \tfrac{1}{2} g^2 \left[\phi_1, \phi_2 \right]^2 \Big\}.
\end{aligned}
\tag{6.39}
$$

This density is (formally) hermitian and the bosonic part is non-negative. The latter follows from the fact that the square of the antihermitian matrix $[\phi_1, \phi_2]$ is non-positive.

6.3.2 Susy transformations and invariance of S

Now we shall fix the variations of the remaining fields ϕ_i and ψ such that $\mathcal{L}$ is invariant up to surface terms. The calculations parallel those of the $\mathcal{N}=1$ models and I need not give many details here. With

$$
\delta D_\mu \phi_i = D_\mu \delta \phi_i - i g [\delta A_\mu, \phi_i] \quad , \quad \delta \not{D} \psi = \not{D} \delta \psi - i g [\delta A_\mu, \gamma^\mu \psi] \tag{6.40}
$$

and $\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\mu \delta A_\mu$ one can express the variations of the terms in the LAGRANGEan density (6.35) as functions of $\delta A_\mu, \delta \phi_i$ and $\delta \psi$. To fix the transformation rules for the scalar fields one considers the terms in $\delta \mathcal{L}$ which are *cubic* in the DIRAC field. After using the transformation rules (6.32) for the vector fields and a suitable

FIERZ identity one obtains the transformation rules for the scalar fields such that the terms trilinear in ψ cancel in $\delta\mathcal{L}$,

$$\delta_\alpha\phi_1 = i(\bar\alpha\psi - \bar\alpha_c\psi_c) \quad \text{and} \quad \delta_\alpha\phi_2 = \bar\alpha\gamma_5\psi - \bar\alpha_c\gamma_5\psi_c. \tag{6.41}$$

Again, the variations of ϕ_1 and ϕ_2 can only be linear combinations of ψ and ψ_c contracted with the supersymmetry parameters. Next one fixes the transformation rules for ψ such that the terms in $\delta\mathcal{L}$ which are cubic in the bosonic fields ϕ_i cancel. This happens for

$$\delta_\alpha\psi = iF^{\mu\nu}\Sigma_{\mu\nu}\alpha - D_\mu\phi\,\gamma^\mu\alpha + g\,[\phi_1,\phi_2]\gamma_5\alpha \tag{6.42}$$

Let us summarize our findings. The action (6.35) can only be invariant under susy transformations if they have the forms (6.32,6.41) and (6.42). Note that a constant background field ϕ_1 is left invariant by an arbitrary symmetry transformation. It does not 'break supersymmetry'.
Actually one can show that the action is invariant under the above transformation. The corresponding NOETHER currents reads

$$\begin{aligned} J^\mu &= \bar\alpha\,\{{}^*F^{\mu\nu}\gamma_\nu\gamma_5 - iF^{\mu\nu}\gamma_\nu + i\gamma^\nu D_\nu\phi\gamma^\mu + ig[\phi_1,\phi_2]\gamma^\mu\gamma_5\}\,\psi \\ &+ \bar\psi\,\{{}^*F^{\mu\nu}\gamma_\nu\gamma_5 + iF^{\mu\nu}\gamma_\nu - i\gamma^\mu D_\nu\phi\gamma^\nu + ig[\phi_1,\phi_2]\gamma^\mu\gamma_5\}\,\alpha \end{aligned} \tag{6.43}$$

and the NOETHER charge takes the simple form

$$\mathcal{Q} = \bar\alpha\int d\boldsymbol{x}\,(R+S)\,\psi + \int d\boldsymbol{x}\,\bar\psi\,(R-S)\,\alpha, \tag{6.44}$$

where we have defined the fields

$$\begin{aligned} R &= \gamma_5\gamma^i\pi_i - i\gamma^0 D_i\phi\gamma^i + i\gamma^0\gamma_5[\phi_1,\phi_2] \\ S &= -i\gamma^i\pi_i + i\pi_{\phi_1} + \pi_{\phi_2}\gamma_5. \end{aligned} \tag{6.45}$$

The conserved supercharge $\mathcal{Q}$ can be decomposed into two MAJORANA charges. These MAJORANA charges satisfy the $\mathcal{N}=2$ superalgebra.
Exercise: Check, that the commutator of two susy transformations on any field is

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\Phi = a^\rho\partial_\rho\Phi + \delta_\lambda\Phi \tag{6.46}$$

with parameter $a^\rho = -2i\,(\bar\alpha_2\gamma^\rho\alpha_1 - \bar\alpha_1\gamma^\rho\alpha_2)$ for infinitesimal translation. δ_λ is a small gauge transformation with field dependent gauge parameter

$$\lambda = -a^\rho A_\rho - 2i\bar\alpha_2\phi^\dagger\alpha_1 + 2i\bar\alpha_1\phi^\dagger\alpha_2, \quad \phi = \phi_1 + i\gamma_5\phi_2. \tag{6.47}$$

6.4 Chiral basis

We use our conventions in section 3.3 to rewrite the action (6.35) and supersymmetry transformations of the $\mathcal{N}=2$ model in terms of WEYL fermions φ and $\bar\chi$ in

$$\psi=\begin{pmatrix}\varphi_\alpha\\ \bar\chi^{\dot\alpha}\end{pmatrix}\quad\text{and}\quad \bar\psi=(\,\chi^\alpha,\bar\varphi_{\dot\alpha}\,) \tag{6.48}$$

Also we combine the two scalar fields ϕ_1 and ϕ_2 to a scalar field with values in the complexified LIE algebra,

$$\kappa\phi=\phi_1+i\phi_2\quad\text{with}\quad\kappa=\sqrt{2}. \tag{6.49}$$

For example, the YUKAWA term in (6.35) takes the form

$$\bar\psi[\phi_1-i\gamma_5\phi_2,\psi]=\kappa\chi[\phi,\varphi]+\kappa\bar\varphi[\phi^\dagger,\bar\chi]. \tag{6.50}$$

In terms of the WEYL spinors and complex scalar the action reads

$$\begin{aligned}\mathcal{L}\;&=\;-\tfrac14\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}+\mathrm{Tr}\,(D_\mu\phi D^\mu\phi^\dagger)-\tfrac12 g^2\,\mathrm{Tr}\,([\phi,\phi^\dagger]^2)\\ &\quad+\mathrm{Tr}\,\Big\{i\chi(\sigma D)\bar\chi+i\bar\varphi(\tilde\sigma D)\varphi-\kappa g\chi[\phi,\varphi]-\kappa\bar\varphi[\phi^\dagger,\bar\chi]\Big\}\end{aligned} \tag{6.51}$$

To rewrite the susy-transformations we also use a chiral basis for the susy parameter,

$$\alpha=\begin{pmatrix}\theta_\alpha\\ \bar\zeta^{\dot\alpha}\end{pmatrix}\quad\text{and}\quad\bar\alpha=(\,\zeta^\alpha\,,\bar\theta_{\dot\alpha}\,)\,. \tag{6.52}$$

Next we insert the relations (A.9) in the appendix into the susy transformations (6.32,6.41,6.42) and obtain

$$\delta_\alpha A_\mu \;=\; i\left(\theta\sigma_\mu\bar\varphi+\bar\theta\tilde\sigma_\mu\varphi+\zeta\sigma_\mu\bar\chi+\bar\zeta\tilde\sigma_\mu\chi\right), \tag{6.53}$$

$$\kappa\delta_\alpha\phi \;=\; i\theta\chi+i\bar\theta\bar\chi \tag{6.54}$$

$$\kappa\delta_\alpha\phi^\dagger \;=\; i\zeta\varphi+i\bar\zeta\bar\varphi \tag{6.55}$$

$$\delta_\alpha\varphi \;=\; \Big(\tfrac12 F^{\mu\nu}\sigma_{\mu\nu}-ig[\phi,\phi^\dagger]\Big)\,\theta-\kappa(\sigma_\mu D^\mu\phi^\dagger)\,\bar\zeta, \tag{6.56}$$

$$\delta_\alpha\bar\chi \;=\; \Big(\tfrac12 F^{\mu\nu}\tilde\sigma_{\mu\nu}+ig[\phi,\phi^\dagger]\Big)\,\bar\zeta-\kappa\,(\tilde\sigma_\mu D^\mu\phi)\,\theta \tag{6.57}$$

This finishes our introduction into the $\mathcal{N}=2$ supersymmetric YANG-MILLS theory.

Chapter 7

Supersymmetry, Solitons and Fluctuations

In this chapter we construct solitonic type solutions of the $\mathcal{N}=2$ supersymmetric gauge theory considered in chapter 6. In MINKOWSKI spacetime these are static solutions of the field equations with quantized magnetic charge [56]. At a distance these magnetically charged solutions look like DIRAC monopoles [57]. For the EUCLIDean field theory we consider solutions which are localized in space and 'time', socalled instantons. Both types of solutions are left invariant by half of the supersymmetries. The other supersymmetries transform the solitonic solutions into fermionic zero-modes, the JACKIW-REBBI modes [58].

7.1 Field equations

The EULER-LAGRANGE equations for the action (6.35) are the following hyperbolic field equations:

• The variation with respect to the gauge potential yields the YANG-MILLS equations

$$D_\mu F^{\mu\nu} = ig[\phi_1, D^\nu\phi_1] + ig[\phi_2, D^\nu\phi_2] - g[\bar{\psi}, \gamma^\nu\psi], \tag{7.1}$$

where $[\bar{\psi}, M\psi] \equiv if^a{}_{bc}\bar{\psi}^b M\psi^c T_a$.

• The variations with respect to the scalar and pseudo-scalar fields yield the KLEIN-GORDON-type equations

$$\begin{aligned} D_\mu D^\mu \phi_1 &= -g[\bar{\psi}, \psi] \;+\; g^2[\phi_2, [\phi_2, \phi_1]] \\ D_\mu D^\mu \phi_2 &= -ig[\bar{\psi}, \gamma_5\psi] + g^2[\phi_1, [\phi_1, \phi_2]], \end{aligned} \tag{7.2}$$

• The variation with respect to the spinor field yields the DIRAC equation

$$\not{D}\psi = g[\phi_1 - i\gamma_5\phi_2, \psi]. \tag{7.3}$$

A particular class of solutions is obtained if we set

$$\psi = 0 \quad \text{and} \quad \phi_2 = 0, \tag{7.4}$$

in which case the field equations for the remaining fields reduce to

$$D_\mu F^{\mu\nu} = ig[\phi_1, D^\nu\phi_1] \quad \text{and} \quad D_\mu D^\mu \phi_1 = 0. \tag{7.5}$$

If we further set $\phi_1 = 0$ then we obtain the pure YANG-MILLS equations $D_\mu F^{\mu\nu} = 0$, which in the EUCLIDean sector possess (anti)selfdual *instanton solutions*.
Now we have prepared the ground for discussing the intriguing relation between supersymmetry and solutions of the field equations. Consider a classical background configuration with vanishing ϕ_2 and ψ. Since the variation $\delta_\alpha\phi_2$ vanishes identically for $\psi = 0$ such a configuration is left invariant by the susy transformations if

$$\delta_\alpha\psi = 0 \overset{(6.42)}{\Longleftrightarrow} iF^{\mu\nu}\Sigma_{\mu\nu}\alpha = \not{D}\phi_1\alpha. \tag{7.6}$$

To see what second order equation is implied by this first order equation we act with $\not{D}$ on it. Using (A.1) and the BIANCHI identity the left hand side yields

$$i\not{D}\, F^{\mu\nu}\Sigma_{\mu\nu}\alpha = D_\mu F^{\mu\nu}\gamma_\nu\alpha$$

and the right hand side becomes

$$\not{D}\not{D}\phi_1\alpha = D_\mu D^\mu\phi_1\alpha + g[F_{\mu\nu}, \phi_1]\Sigma^{\mu\nu}\alpha.$$

Now we may use the first order equations (7.6) once more to rewrite the last term as $ig[\phi_1, D_\mu\phi_1]\gamma^\mu\alpha$. Thus we have shown that the first order equation $\delta_\alpha\psi = 0$ implies the *second order equation*

$$\left(D_\mu F^{\mu\nu} - ig[\phi_1, D^\nu\phi_1]\right)\gamma_\nu\alpha = D_\mu D^\mu\phi_1\,\alpha. \tag{7.7}$$

If α is an arbitrary DIRAC, WEYL or MAJORANA spinor, then both sides must vanish separately. To see this we note that $\bar\alpha\gamma_\nu\alpha$ vanishes for MAJORANA spinors and $\bar\alpha\alpha$ for WEYL spinors. The resulting two equations are just the *field equations* (7.5) for the scalar and gauge field. This then proves that for backgrounds with vanishing ϕ_2 and ψ the first order equation $\delta_\alpha\psi = 0$ implies the second order field equations (7.5) for ϕ_1 and A_μ.
The opposite is of course not true. If (7.6) would hold for all susy parameters α then $F_{\mu\nu}$ would vanish and ϕ_1 would be constant, up to a gauge transformation. Thus only *trivial background fields* respect all supersymmetries. The idea is to impose the first order condition $\delta_\alpha\psi = 0$ only for part of supersymmetry, that is for a restricted set of supersymmetry parameters α.

7.2 Bogomol'nyi bound and monopoles

Let us consider magnetic monopole solutions in more detail. Hence we assume that the only non-vanishing fields (A_μ, ϕ_1) are static and choose the WEYL or temporal gauge $A_0 = 0$. With $2\Sigma_{ij} = -\epsilon_{ijk}\sigma_0 \otimes \tau_k$ the first order equation (7.6) reads

$$0 = \delta_\alpha\psi = i\begin{pmatrix} \boldsymbol{\tau B} & 0 \\ 0 & \boldsymbol{\tau B} \end{pmatrix}\alpha - \begin{pmatrix} 0 & \boldsymbol{\tau D}\phi_1 \\ -\boldsymbol{\tau D}\phi_1 & 0 \end{pmatrix}\alpha. \tag{7.8}$$

This system of equations is satisfied if

$$\alpha = \begin{pmatrix} \theta \\ \bar{\zeta} \end{pmatrix} \quad \text{with} \quad \theta \pm i\bar{\zeta} = 0 \quad \text{and} \quad \boldsymbol{B} = \pm \boldsymbol{D}\phi_1 \tag{7.9}$$

hold true. The first condition means that only an $\mathcal{N} = 1$ susy is left intact and the second condition is the first order BOGOMOL'NYI *monopole equation.* Solutions of this equation describe magnetic monopoles. To see this we rewrite the energy or mass (6.46) for a static configuration (A_i, ϕ_1) as follows,

$$\begin{aligned} M &= \tfrac{1}{2} \int d\boldsymbol{x}\, \mathrm{Tr}\, \left(\boldsymbol{B}^2 + (\boldsymbol{D}\phi_1)^2\right) \\ &= \tfrac{1}{2} \int d\boldsymbol{x}\, \mathrm{Tr}\, (\boldsymbol{B} \pm \boldsymbol{D}\phi_1)^2 \mp \int d\boldsymbol{x} \mathrm{Tr}\, (\boldsymbol{B} \cdot \boldsymbol{D}\phi_1)\,, \end{aligned} \tag{7.10}$$

from which the celebrated BOGOMOL'NYI *bound* follows at once,

$$M \geq \Big| \int \mathrm{Tr}\,(\boldsymbol{B} \cdot \boldsymbol{D}\phi_1) \Big|. \tag{7.11}$$

Using the BIANCHI identity for the 'chromo-magnetic' field, $\boldsymbol{D} \cdot \boldsymbol{B} = 0$, the integrand can be written as a divergence,

$$\mathrm{Tr}\,(\boldsymbol{B} \cdot \boldsymbol{D}\phi_1) = \nabla\, \mathrm{Tr}\,(\boldsymbol{B}\, \phi_1).$$

Inserting this into the BOGOMOL'NYI bound and using the GAUSS theorem yields

$$M \geq \Big| \oint \mathrm{Tr}\,(\boldsymbol{B}\, \phi_1) d\boldsymbol{S}\, \Big|. \tag{7.12}$$

So far we did not specify the gauge group. In what follows we assume that it is the group $SU(2)$ with generators

$$T_a = \frac{1}{\sqrt{2}} \tau_a, \quad \text{and} \quad f_{abc} = \epsilon_{abc}.$$

A smooth configuration has finite energy (7.10) if both $\boldsymbol{B}$ and $\boldsymbol{D}\phi_1$ vanish sufficiently fast at large distances $r = |\boldsymbol{x}|$ from the monopole core. If we further assume that the length of the scalar field tends to a constant value v,

$$v^2 = \lim_{|\boldsymbol{x}| \to \infty} \mathrm{Tr}\, \phi_1^2(\boldsymbol{x}), \tag{7.13}$$

then the last surface integral in (7.12) is to be interpreted as *magnetic flux* of the magnetic field $\boldsymbol{b}$ belonging to unbroken $U(1)$ gauge group. More precisely, far away from the monopole the 'chromo-magnetic' field becomes ABELian,

$$\boldsymbol{B} \stackrel{|\boldsymbol{x}| \to \infty}{\longrightarrow} \frac{1}{v^2} \phi_1 \mathrm{Tr}\,(\phi_1 \boldsymbol{B}) = \hat{\phi}_1 \boldsymbol{b}, \tag{7.14}$$

and the non-ABELian monopole becomes an ABELian DIRAC monopole [57]. As a consequence the BOGOMOL'NYI bound simplifies to

$$M \geq v|\Phi|, \quad \text{where} \quad \Phi = \oint \boldsymbol{b} d\boldsymbol{S} \tag{7.15}$$

is the magnetic charge of the monopole. The last surface integral is over a large 2-sphere surrounding the pole. Since $|\boldsymbol{D}\phi_1|$ tends to zero at spatial infinity it follows that

$$v^2 A \stackrel{r\to\infty}{\longrightarrow} \frac{i}{g}[\phi_1, d\phi_1] + \phi_1 a, \quad a = \mathrm{Tr}\,(\phi_1 A). \tag{7.16}$$

If we now compute the leading order behavior of the non-ABELian field strength we find the result (7.14), that is $F = \hat{\phi}_1 f$ with ABELian field strength

$$f = \frac{1}{2} f_{ij} dx^i \wedge dx^j = \frac{i}{gv^3} \mathrm{Tr}\,(\phi_1 d\phi_1 \wedge d\phi_1) + da. \tag{7.17}$$

One can show that far away from the monopole core the YANG-MILLS equations for F simplify to the MAXWELL equations for f [59].
With the help of (7.17) the surface integral in (7.16) defining the magnetic charge can be rewritten as

$$\Phi = \oint f = \frac{1}{g} \oint \mathrm{Tr}\,(i\hat{\phi}_1 d\hat{\phi}_1 \wedge d\hat{\phi}_1). \tag{7.18}$$

The field $\hat{\phi}_1$ has a constant length 1 in the asymptotic region and the map

$$\hat{x} \longrightarrow \phi_1(\hat{x}) = \lim_{r\to\infty} \phi_1(r\hat{x}) \tag{7.19}$$

is from the asymptotic sphere S^2 surrounding the monopole to the unit sphere S^2 in the internal space. The last surface integral in (7.18) is just $4\pi N/g$, where N denotes the *winding number* of this map [59]. Thus we obtain the celebrated DIRAC *quantization condition* for the magnetic charge of a monopole,

$$\Phi = \frac{4\pi}{g} N. \tag{7.20}$$

It relates the gauge coupling to the charges of the monopoles.
Now the BOGOMOL'NYI bound (7.15) for the mass can be written as

$$M \geq \frac{4\pi N}{g} v. \tag{7.21}$$

This inequality becomes an equality for

$$\boldsymbol{B} = \pm \boldsymbol{D}\phi_1, \tag{7.22}$$

that is for BPS-monopoles which leave half of the supersymmetries intact. The monopole solutions of the Bogomol'nyi equations (7.22) have been constructed in [60]. Two such monopoles neither repel nor attract each other. This must be so since the Bogomol'nyi bound is attained independent of the distance between the constituent monopoles. When one changes the collective coordinates, and in particular the distance between BPS monopoles, then the energy does not change and this implies that there is no interaction between the constituents. This behavior is typical for BPS states which saturate a Bogomol'nyi type bound.

7.3 Jackiw-Rebbi modes

We have characterized BPS configurations as those fields which are left invariant by part of the original supersymmetry. Now we shall see that the other supersymmetry transformations, which do not leave the BPS configurations invariant, generate the so-called fermionic Jackiw-Rebbi zero-modes [58]. These are time-independent solutions of the Dirac equation in the BPS background fields. Note that the Dirac equation in the Weyl gauge $A_0 = 0$ reads

$$i\dot{\psi} = -i\alpha^i D_i\psi + g\gamma^0[\phi_1, \psi] \quad \text{with} \quad \alpha^i = \gamma^0\gamma^i. \tag{7.23}$$

For the static monopole background (A_i, ϕ_1) we may factorize the time dependence and arrive at the time-independent Dirac equation

$$E\psi = -i\boldsymbol{\alpha}\boldsymbol{D}\psi + g\gamma^0[\phi_1, \psi] = H\psi. \tag{7.24}$$

Now we shall prove that $\delta_\alpha\psi$ is a zero-mode of H if (A_i, ϕ_1) satisfy the Bogomol'nyi equation. For $\boldsymbol{B} = \boldsymbol{D}\phi_1$ the susy variation of the Dirac spinor simplifies to

$$\delta_\alpha\psi = \delta_\kappa\psi = \boldsymbol{D}\phi_1 \begin{pmatrix} i\boldsymbol{\tau}\,\kappa \\ \boldsymbol{\tau}\,\kappa \end{pmatrix}, \quad \text{where} \quad \alpha = \begin{pmatrix} \theta \\ \bar{\zeta} \end{pmatrix}, \quad \kappa = \theta + i\bar{\zeta}. \tag{7.25}$$

Since $D_i D^i\phi_1 = 0$ we conclude that

$$\begin{aligned} \gamma^j D_j\delta\psi &= D_j D_i\phi_1 \begin{pmatrix} \tau_j\tau_i\kappa \\ -i\tau_j\tau_i\kappa \end{pmatrix} = \tfrac{1}{2}\epsilon_{jik}[D_j, D_i]\phi_1 \begin{pmatrix} i\tau_k\kappa \\ \tau_k\kappa \end{pmatrix} \\ &= ig[B_k, \phi_1] \begin{pmatrix} i\tau_k\,\kappa \\ \tau_k\,\kappa \end{pmatrix} = -ig[\phi_1, \delta\psi], \end{aligned} \tag{7.26}$$

which is equivalent to the Dirac equation (7.24) with zero energy,

$$H\delta_\kappa\psi = 0. \tag{7.27}$$

Thus we have found an explicit expression for the Jackiw-Rebbi zero-modes in the vicinity of magnetic BPS-monopoles. Supersymmetry not only opens up an elegant way to characterize and construct BPS-monopoles, it also gives the associated Jackiw-Rebbi modes. For the physical consequences of these modes I refer to [54].

7.4 $\mathcal{N}=2$-SYM in Euclidean spacetime

Any supersymmetric theory contains both bosonic and fermionic fields and the transition from LORENTZian to EUCLIDean signature is not as straightforward as it is for purely bosonic models. In spacetimes with EUCLIDean signature the gamma matrices must be hermitian and thus we choose

$$(\gamma_0, \gamma_i)_M = (\gamma_0, -i\gamma_i)_E, \tag{7.28}$$

where the index M refers to MINKOWSKIan and E to EUCLIDean spacetime. From

$$\mathcal{L}_M \sim \tfrac{1}{2}\mathrm{Tr}\,(F_{0i})^2 - \tfrac{1}{4}\mathrm{Tr}\,(F_{ij})^2 + \dots$$

we infer that the LAGRANGEan becomes negative definite, irrespective whether we multiply the time coordinate or the space coordinates with i. Because

$$\mathcal{L}_M = \tfrac{1}{2}\mathrm{Tr}\,(D_0\phi_1)^2 - \tfrac{1}{2}\mathrm{Tr}\,(D_i\phi_1)^2 + \dots$$

we prefer to continue the time coordinate for these terms to become negative definite as well. Hence we choose

$$(\partial_0, \partial_i)_M = (i\partial_0, \partial_i)_E, \tag{7.29}$$

such that

$$(A_0, A_i, F_{0i}, F_{ij})_M = (iA_0, A_i, iF_{0i}, F_{ij})_E \quad \text{and} \quad \not{D}_M = i\not{D}_E.$$

In EUCLIDean spacetime the DIRAC term must have the form

$$\pm i\psi^\dagger \not{D}\psi$$

in order to be hermitian and $SO(4)$ invariant. This tells us, that

$$(\psi, \bar\psi)_M = (\psi, i\psi^\dagger)_E\,, \quad \text{such that} \quad (i\bar\psi\not{D}\psi)_M = -i(\psi^\dagger\not{D}\psi)_E \tag{7.30}$$

holds true. For the choices

$$(\phi_1)_M = (\phi_1)_E \quad \text{and} \quad (\phi_2)_M = (\phi_2)_E \tag{7.31}$$

the YUKAWA interaction term

$$-g\mathrm{Tr}\,(\bar\psi[\phi_1, \psi]) = -ig\mathrm{Tr}\,(\psi^\dagger[\phi_1, \psi])$$

becomes antihermitean in EUCLIDean spacetime. More generally, one can show that there are no consistent transformation rules of the coordinates, γ-matrices and fields such that the resulting EUCLIDean action is hermitean and bounded from below [61]. This is not as bad as it seems, since in the path integral formulation of finite temperature field theories the mass term in the EUCLIDean LAGRANGEan $\mathcal{L}_E =$

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$\psi^\dagger(i\not{D}+im)\psi+\dots$ is not hermitean either. Although the eigenvalues of $i\not{D}+im$ are not real, they come in pairs $im\pm\lambda$ such that the partition function stays real, since $(im+\lambda)(im-\lambda)$ is real. Actually, in a spontaneous broken phase the YUKAWA term may induce a mass term for the fermions and we even expect an *antihermitean* YUKAWA interaction in EUCLIDean spacetime. Hence we may accept the transformation rules (7.28-7.31) in which case

$$\begin{aligned}\mathcal{L}_E &= \tfrac{1}{4}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\mathrm{Tr}\,(D_\mu\phi_1)^2+\tfrac{1}{2}\mathrm{Tr}\,(D_\mu\phi_2)^2-\tfrac{1}{2}g^2\,\mathrm{Tr}\,\left([\phi_1,\phi_2]^2\right)\\ &\quad+i\mathrm{Tr}\,\psi^\dagger\not{D}\psi+ig\mathrm{Tr}\,\left(\psi^\dagger[\phi_1,\psi]\right)+g\mathrm{Tr}\,\left(\psi^\dagger\gamma_5[\phi_2,\psi]\right).\end{aligned}\tag{7.32}$$

The indices are raised and lowered with the EUCLIDean metric $\delta_{\mu\nu}$. A problem with this LAGRANGEan will be discussed below.

In the chiral representation the hermitean γ-matrices have the form

$$\gamma_\mu=\begin{pmatrix}0&\sigma_\mu\\ \sigma_\mu^\dagger&0\end{pmatrix}\quad\text{with}\quad(\sigma_\mu)=(\sigma_0,i\tau_i)=(\sigma^\mu),\tag{7.33}$$

and the hermitean γ_5 is block diagonal $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3=\tau_3\otimes\sigma_0$. Note that the σ_μ in the EUCLIDean space differ from the one in MINKOWSKIan spacetime. Since

$$\gamma_{i0}=i\tau_3\otimes\tau_i\quad\text{and}\quad\gamma_{ij}=i\epsilon_{ijk}\sigma_0\otimes\tau_k\tag{7.34}$$

the anti-hermitean generators of EUCLIDean spin rotations

$$\gamma_{\mu\nu}=\begin{pmatrix}\sigma_{\mu\nu}&0\\0&\tilde\sigma_{\mu\nu}\end{pmatrix}\tag{7.35}$$

contain selfdual and antiselfdual objects,

$$\tilde\sigma_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\tilde\sigma_{\alpha\beta}\quad\text{and}\quad\sigma_{\mu\nu}=-\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}.\tag{7.36}$$

In the EUCLIDean space there exists a YANG-MILLS theory with $\mathcal{N}=2$ extended supersymmetry. The field transformations follow from the transformations in the LORENTZian model with the appropriate replacements (7.28-7.31),

$$\begin{aligned}\delta_\alpha A_\mu &= i(\alpha^\dagger\gamma_\mu\psi-\psi^\dagger\gamma_\mu\alpha)\\ \delta_\alpha\phi_1 &= -\alpha^\dagger\psi+\psi^\dagger\alpha\\ \delta_\alpha\phi_2 &= i(\alpha^\dagger\gamma_5\psi-\psi^\dagger\gamma_5\alpha).\end{aligned}\tag{7.37}$$

Here we encounter a first problem: The second formula shows that ϕ_1 should be *antihermitean*. But with an antihermitean ϕ_1 the term $\mathrm{Tr}\,(D\phi_1)^2$ and the potential term quartic in the scalar fields become *unbounded from below*. The corresponding model is unstable, as has been pointed out in [61]. For a recent work on this annoying stability and hermiticity problem you may consult Belitsky et al. [62].

The susy transformation of the DIRAC spinor reads

$$\delta_\alpha\psi=-iF^{\mu\nu}\Sigma_{\mu\nu}\alpha-iD_\mu\phi\gamma^\mu\alpha+g\,[\phi_1,\phi_2]\gamma_5\alpha,\tag{7.38}$$

where we have set $\phi_1 + i\gamma_5\phi_2 = \phi$. To prove the invariance of S one uses the general EUCLIDean FIERZ identity

$$\begin{aligned} 4\psi\chi^\dagger &= -(\chi^\dagger\psi) - \gamma_\mu(\chi^\dagger\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\chi^\dagger\gamma^{\mu\nu}\psi) \\ &\quad + \gamma_5\gamma_\mu(\chi^\dagger\gamma_5\gamma^\mu\psi) - \gamma_5(\chi^\dagger\gamma_5\psi). \end{aligned} \tag{7.39}$$

The corresponding NOETHER current takes the form

$$\begin{aligned} J^\mu &= \alpha^\dagger\left\{i\gamma_\nu({}^*F^{\mu\nu}\gamma_5 + F^{\mu\nu}) + ig[\phi_1,\phi_2]\gamma^\mu\gamma_5 - \not{D}\phi\gamma^\mu\right\}\psi \\ &+ \psi^\dagger\left\{i\gamma_\nu({}^*F^{\mu\nu}\gamma_5 - F^{\mu\nu}) + ig[\phi_1,\phi_2]\gamma^\mu\gamma_5 + \gamma^\mu D_\nu\phi\gamma^\nu\right\}\alpha, \end{aligned} \tag{7.40}$$

and is just the continuation of the current (6.43) in the SEIBERG-WITTEN model.

7.4.1 Instantons as BPS configurations

Here we are interested in background configurations which preserve half of the EUCLIDean supersymmetries. We assume that all fields with the exception of the gauge potential vanish. This condition is preserved by supersymmetry transformations if

$$0 = \delta_\alpha\psi = -iF^{\mu\nu}\Sigma_{\mu\nu}\alpha, \qquad \alpha = \begin{pmatrix}\theta\\ \bar\zeta\end{pmatrix} \tag{7.41}$$

Decomposing this condition into its chiral parts yields

$$F^{\mu\nu}\sigma_{\mu\nu}\theta = F^{\mu\nu}\tilde\sigma_{\mu\nu}\bar\zeta = 0. \tag{7.42}$$

There are two ways to fulfill these conditions:

$$i)\quad \theta = 0 \quad , \quad F^{\mu\nu}\tilde\sigma_{\mu\nu} = 0 \tag{7.43}$$

$$ii)\quad \bar\zeta = 0 \quad , \quad F^{\mu\nu}\sigma_{\mu\nu} = 0. \tag{7.44}$$

In the first case only the $\bar\zeta$-supersymmetry survives and $F^{\mu\nu}$ is *anti-selfdual* whereas in the second case the θ-supersymmetry survives and $F^{\mu\nu}$ is *selfdual*. As expected on general ground these selfdual and anti-selfdual gauge fields are *solutions* of the classical field equations which in the present case reduce to the EUCLIDean YANG-MILLS equation $D_\nu F^{\mu\nu} = 0$. As for the monopoles solutions we can interprete the instantons as BPS-states which preserve half of the EUCLIDean supersymmetry.
Now we pick a selfdual instanton configuration $\bar A_\mu$ and consider its supersymmetry variation. For a pure gauge field background the only nontrivial variation is

$$\delta_\alpha\psi = -iF^{\mu\nu}\Sigma_{\mu\nu}\alpha, \tag{7.45}$$

and since $\bar A_\mu$ is a classical solution we have

$$S[\bar A_\mu] = S[\bar A_\mu, \delta_\alpha\psi] \sim S[\bar A_\mu] + (\delta_\alpha\psi^\dagger, i\not{D}\,\delta_\alpha\psi)$$

which indicates (only indicates since $i\not{D}$ is indefinite) that $\delta_\alpha\psi$ is a zero-mode of the Dirac operator. This is easily proven by acting with $i\not{D}$ on $\delta_\alpha\psi$,

$$i\not{D}\,\delta_\alpha\psi = i\gamma_\nu D_\mu\left({}^*F^{\mu\nu}\gamma_5 - F^{\mu\nu}\right)\alpha = 0, \tag{7.46}$$

where we used the Bianchi identity and the Yang-Mills equation. This shows that the susy variation $\delta_\alpha\psi$ is a zeromode of the Dirac operator. For a selfdual gauge field the zero-mode is chiral

$$\delta_\alpha\psi = -\tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \tilde{\sigma}^{\mu\nu}F_{\mu\nu}\bar{\zeta}\end{pmatrix}, \tag{7.47}$$

and for a anti-selfdual gauge field the mode has opposite chirality.
Actually, from the *index theorem* we infer that in an $SU(2)$-instanton background with instanton number

$$q = \frac{1}{8\pi^2}\int d^4x\,\mathrm{Tr}\,(F\wedge F)$$

the operator $i\not{D}$ possesses at least

$$\tfrac{2}{3}(2j+1)(j+1)j\cdot q \tag{7.48}$$

righthanded zero-modes in the spin-j representation. For $\mathcal{N}=2$ SYM the fermions transform according to the adjoint representation such that there are at least $4q$ zero modes.

7.4.2 Small fluctuations about instantons

Now we wish to relate the various fluctuation fields about selfdual instantons [63]. We start with perturbing an arbitrary background field by small fluctuations

$$A_\mu = \bar{A}_\mu + a_\mu, \quad \phi_1 = \bar{\phi}_1 + \delta\phi_1, \quad \phi_2 = \bar{\phi}_2 + \delta\phi_2. \tag{7.49}$$

The Dirac field is considered as a fluctuation itself. We perform the Taylor expansion of the (unstable) Euclidean action about the bosonic background,

$$S = S_0 + S_1 + S_2 + \dots, \tag{7.50}$$

where S_i is of order i in the fluctuation fields. The first term S_1 yields the Euler-Lagrange equations and vanishes for backgrounds which solve the field equations. For general backgrounds the expression for S_2 is rather complicated. But for vanishing $\bar{\phi}_i$ and without gauge fixing (see below) it becomes rather simple

$$S_2 = \int d^4x\left(a_\mu M_{\mu\nu}a_\nu + \delta\phi_1 M\delta\phi_1 + \delta\phi_2 M\delta\phi_2 + \psi^\dagger M_\psi\psi\right), \tag{7.51}$$

with fluctuation operators

$$\begin{aligned} M_{\mu\nu} &= -D^2\delta_{\mu\nu} + D_\mu D_\nu + 2ig\,\mathrm{ad}\,(F_{\mu\nu}), \\ M &= -D^2 \quad\text{and}\quad M_\psi = i\not{D}. \end{aligned} \tag{7.52}$$

All covariant derivatives act on fields in the adjoint representation.
In a one-loop approximation to the partition function one needs the eigenvalues of these fluctuation operators,

$$\begin{aligned} M_{\mu\nu}a_\nu &= \lambda^2 a_\nu \qquad (7.53)\\ M_i\delta\phi_i &= \lambda^2\delta\phi_i \qquad (7.54)\\ M_\psi\psi &= \lambda\psi. \qquad (7.55)\end{aligned}$$

Let us assume that ψ is an eigenmode of the DIRAC operator, $M_\psi\psi = i\not{D}\psi = \lambda\psi$. For selfdual gauge fields the squared DIRAC operator simplifies as follows,

$$\not{D}^2 = D^2 + \Sigma^{\mu\nu}\,\mathrm{ad}(F_{\mu\nu}) = D^2 - \tfrac{i}{2}\begin{pmatrix}0 & 0\\ 0 & \tilde\sigma^{\mu\nu}\mathrm{ad}(F_{\mu\nu})\end{pmatrix}. \qquad (7.56)$$

We see that the chiral spinor φ in the decomposition

$$\psi = \begin{pmatrix}\varphi\\ \bar\chi\end{pmatrix} \qquad (7.57)$$

fulfills a WEYL type equation without PAULI term,

$$-D^2\varphi = \lambda^2\varphi.$$

This equation is identical to the eigenvalue equation for the fluctuations $\delta\phi_i$ of the scalar and pseudoscalar fields. Hence every left-handed eigenmode φ of the squared DIRAC operator transforms into an scalar and pseudoscalar eigenmode of D^2 with the same eigenvalue,

$$\delta_\alpha\phi_1 = -\theta^\dagger\varphi + \varphi^\dagger\theta \quad\text{and}\quad \delta_\alpha\varphi_2 = i\theta^\dagger\varphi - i\varphi^\dagger\theta. \qquad (7.58)$$

As in (7.41) we denoted the chiral parts of α by θ and $\bar\zeta$.
The same procedure applies to the eigenvalue equation of the vector fluctuations in an selfdual instanton background. We start with a righthanded fermionic eigenmode of $-\not{D}^2$ which fulfills

$$\left(-D^2 + \tfrac{i}{2}\tilde\sigma^{\mu\nu}\mathrm{ad}(F_{\mu\nu})\right)\bar\chi = \lambda^2\bar\chi.$$

We multiply this equation from the left with σ_ν and use the identity

$$\sigma_\rho\tilde\sigma_{\mu\nu} = \delta_{\rho\mu}\sigma_\nu - \delta_{\rho\nu}\sigma_\mu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma, \qquad (7.59)$$

together with the selfduality condition ${}^*F_{\nu\mu} = F_{\mu\nu}$. This leads to

$$\left(-D^2\delta_{\mu\nu} + 2i\,\mathrm{ad}(F_{\mu\nu})\right)\sigma_\nu\bar\chi = \lambda^2\sigma_\mu\bar\chi. \qquad (7.60)$$

Acting with D_μ on this equation and using $[D_\mu, D_\nu] = -i\mathrm{ad}(F_{\mu\nu})$ we find

$$-D^2(\sigma^\mu D_\mu\bar\chi) = \lambda^2(\sigma^\mu D_\mu\bar\chi). \qquad (7.61)$$

Now we take the vector field which is gotten as supersymmetry transformation of a righthanded eigenmode,

$$a_\mu = i\theta^\dagger \sigma_\mu \bar{\chi} + \text{h.c.} \tag{7.62}$$

For eigenvalue $\lambda \neq 0$ this vector field has a non-vanishing divergence,

$$D_\mu a_\mu = \lambda \theta^\dagger \varphi + \text{h.c.}. \tag{7.63}$$

It can be used to construct the following 'source free' vector field

$$b_\mu = a_\mu + \frac{1}{\lambda^2} D_\mu (D_\nu a_\nu) \tag{7.64}$$

which satisfies the so-called *background gauge condition*

$$D_\mu b_\mu = 0. \tag{7.65}$$

The difference between b_μ and a_μ is an infinitesimal gauge transformation. Actually the proof of (7.65) is simple when one recalls (7.61),

$$\begin{aligned} D_\mu b_\mu &= D_\mu a_\mu + \frac{1}{\lambda^2} D^2 \Big(i\theta^\dagger (\sigma D \bar{\chi}) + \text{h.c.} \Big) \\ &= D_\mu a_\mu - i \Big(\theta^\dagger (\sigma D \bar{\chi}) + \text{h.c} \Big) = D_\mu a_\mu - D_\mu a_\mu = 0. \end{aligned}$$

We return to the fluctuation operator $M_{\mu\nu}$ for the gauge bosons in (7.52). It is just the sum of the operator on the left in (7.60), plus $D_\mu D_\nu$ which annihilates b_ν. We conclude that b_ν is an eigenmode of the fluctuation operator with eigenvalues λ^2,

$$M_{\mu\nu} b_\nu = \lambda^2 b_\mu, \qquad D_\mu b_\mu = 0. \tag{7.66}$$

Thus we have diagonalized $M_{\mu\nu}$ on fluctuations fulfilling the background gauge condition (7.65).

We summarize our findings: the fluctuation operators $M, M_{\mu\nu}$ for the scalars and gauge field fluctuations (in the background gauge) have the same spectra as the fluctuation operator $-\not{D}^2$ for the Dirac field. The eigenmodes are related by supersymmetry. This important result can be used to calculate the one-loop partition function of the Euclidean Yang-Mills theory with $\mathcal{N} = 2$ supersymmetry.

7.5 One-loop β-function

In this section we shall calculate the one-loop β-function and anomalous dimension of A_μ for the $\mathcal{N} = 2$ SYM. We shall use the results in [64] which relate the finite size scaling of the partition function to the β-functions and anomalous dimensions. After our previous fluctuation analysis it is natural to use the background gauge fixing. Hence we add a gauge fixing term $\frac{1}{2}(D_\mu a_\mu)^2$ to the LAGRANGEan such that the eigenvalue equation for the fluctuations a_μ in (7.53) is modified to

$$K_{\mu\nu}a_\nu = (M_{\mu\nu} - D_\mu D_\nu)a_\nu = \lambda^2 a_\mu. \tag{7.67}$$

The general formula in [64, 65] for the *one-loop* generating functional in the *q-instanton sector* and spacetime volume V simplifies for the $\mathcal{N} = 2$ SYM as follows,

$$\begin{aligned} Z_q(V,g,\eta) &= e^{W_q(V,g)} = \frac{1}{\mathcal{N}}\int \mathcal{D}(a_\mu,\psi^\dagger,\psi,\phi_1,\phi_2)\, e^{-S+\int(\psi^\dagger\eta+\eta^\dagger\psi)} \\ &= e^{-S(\bar{A}_\mu)}\left(\frac{2g^2\pi}{V}\right)^{d_H/2}\frac{1}{V_H}\int\prod_1^p d\gamma_r(\det J)^{1/2}\frac{\det' M_\psi \det' M_{gh}}{\det' M \det'^{1/2}(-K_{\mu\nu})} \\ &\qquad \times \prod_n (\eta^\dagger,\psi_n)(\psi_n^\dagger,\eta)\exp\left(-\int \eta^\dagger G'\eta\right). \end{aligned}$$

We expanded the action about a classical gauge field background such that the determinants depend on this background field. In the non-perturbative sectors with non-vanishing instanton number the fluctuation operators, and in particular M_ψ, may admit zero-modes. These zeromodes must be omitted when one computes the determinants and the product of the non-zero eigenvalues is denoted by $\det'$ in the above formula. Since the fermions are of DIRAC type we get the determinant of the DIRAC operator. For MAJORANA fermions we would get the square root of this determinant. For the background gauge fixing the fluctuation operator for the ghosts is $M_{gh} = -D^2$ and coincides with the operator M for the scalar fields. Each of the two scalar fields yields a square root of $\det' M$ in the denominator. Also we have used the dimension d_H and volume V_H of the stability group H which commutes with the $su(2)$-algebra defined by the instanton [65]. For the gauge group $SU(2)$ we have $d_H = 0$. The fluctuation operator $K_{\mu\nu}$ for the gauge bosons may possess p additional zero-modes arising from the variation of the collective parameters $\{\gamma_r\}$. J denotes the Jacobian when one converts the p expansion parameters (in the expansion of the gauge field) into collective parameters.

In [64] it was shown that the β-functions and anomalous dimensions are the same in *all instanton sectors* and hence it suffices to consider the perturbative $q = 0$ sector and $\eta = 0$. Hence we skip the index q in what follows. To extract the β-function for the gauge coupling we keep an arbitrary background gauge field. We use the *zeta-function* regularization to 'calculate' the determinant of an selfadjoint and non-negative operator A,

$$\log\det A = -\frac{d}{ds}\,\zeta_A(s)|_{s=0}, \quad \zeta_A(s) = \sum_{\lambda_n>0}\lambda_n^{-s}. \tag{7.68}$$

The so defined determinant has a simple scaling property,

$$\log\det\left(\frac{1}{\lambda}A\right) = \log\det A - \log\lambda\cdot\zeta_A(0). \tag{7.69}$$

Now we rescale the quantization volume and background field as

$$V \longrightarrow \tilde{V} = \lambda V \quad \text{and} \quad A_\mu(x) \longrightarrow \tilde{A}_\mu(\tilde{x}) = \lambda^{-1} A_\mu(x), \tag{7.70}$$

where $\tilde{x} = \lambda x$, and compute how the SCHWINGER *functional* changes. We take the general results in [64], applied to the present situation, and find

$$W[\lambda V, \tilde{A}, g] = W(V, A, g) + \frac{\log\lambda}{16\pi^2} \int \mathrm{Tr}\left(a_2^{A_\mu}(x) - a_2^{\psi}(x)\right), \tag{7.71}$$

where the second SEELEY-DEWITT coefficients of the various fluctuation operators appeared. We used that the scalar and ghost operators are equal such that their contributions cancel. Setting

$$X = g^2\, \mathrm{Tr_A} F^{\mu\nu} F_{\mu\nu},$$

where $\mathrm{Tr_A}$ is the trace in the adjoint representation, the needed coefficients are [66]

$$a_2^{A_\mu}(x) = -\tfrac{5}{3}X, \quad a_2^{gh} = \frac{1}{12}X \quad \text{and} \quad a_2^{\psi}(x) = -\tfrac{2}{3}X. \tag{7.72}$$

Thus we obtain

$$W[\lambda V, \tilde{A}, g] = W[V, A, g] - \frac{\log\lambda}{16\pi^2} C_\mathrm{A}\, g^2 \int F^a_{\mu\nu} F^{\mu\nu}_a, \tag{7.73}$$

where C_A is the second-order CASIMIR of the adjoint representation. This then implies the following scaling law for the *effective action*, that is the generating functional of the one-particle irreducible FEYNMAN graphs [64],

$$\Gamma[\lambda V, \tilde{A}, g] = \frac{1}{4g^2} Z_3 \int F^{\mu\nu}_a F^a_{\mu\nu} \quad \text{with} \quad Z_3 = 1 - \frac{\log\lambda}{4\pi^2} C_\mathrm{A}\, g^2. \tag{7.74}$$

The effective action stays invariant if the background field and gauge coupling scale in a non-canonical way,

$$\Gamma\left[\lambda V, \tilde{A}, g\right] = \Gamma\left[V, Z_3^{1/2} A, Z_3^{-1/2} g\right] \tag{7.75}$$

For the gauge group $SU(2)$ the second order CASIMIR in the adjoint representation is 2 and the coupling runs with the inverse size μ of the quantization volume as

$$g^2(\mu) = \frac{2\pi^2 g^2}{2\pi^2 + g^2 \log\mu}, \quad \text{where} \quad \mu = \frac{1}{\lambda}. \tag{7.76}$$

Hence the *β-function* and *anomalous dimension* of the gauge fields are

$$\beta(g) = \mu\frac{\partial}{\partial\mu} g(\mu) = -\frac{1}{4\pi^2} g^3(\mu) \tag{7.77}$$

$$\gamma_A(g) = \mu\frac{\partial}{\partial\mu} \log Z_3 = \frac{1}{2\pi^2} g^2(\mu). \tag{7.78}$$

The theory is asymptotically free, similarly as QCD. Actually, it has been shown in [67] that there is only the one-loop contributions (7.77) to the perturbative β-function of $\mathcal{N} = 2$ SYM. The exact non-perturbative β-function has been calculated in [68].

I leave it as an exercise to calculate the β-function and anomalous dimension for the Yang-Mills theory with $\mathcal{N} = 1$ supersymmetry. Again you should find that the model is asymptotically free.

Chapter 8

$N = 4$ Super-Yang-Mills Theory

The largest supersymmetry that can be represented on a multiplet with spins ≤ 1 is the one with four Majorana supercharges and for this reason the $\mathcal{N} = 4$ model [69] is called *maximally extended*. The renewed interest in this model is twofold. On one hand, it is expected to be S-dual and the *complete effective action* (including all instanton and anti-instanton effects) should organize into an $SL(2, \mathbb{Z})$ invariant expression. On the other hand, not unrelated to the previous, it appears in the celebrated AdS/CFT correspondence [50, 70].
On-shell the theory contains one gauge field A_μ, four Majorana spinors[1] and six scalar fields. All fields transform according to the adjoint representation of the gauge group.

8.1 Scale invariance in one-loop

Without knowing the action we can already calculate the one-loop β-function of this theory. Under scale transformations the Schwinger functional changes as

$$\begin{aligned} W[\lambda V, \tilde{A}, g] &= W(V, A, g) + \frac{\log \lambda}{16\pi^2} \int \mathrm{Tr} \left(6a_2^{\phi}(x) - 2a_2^{gh}(x) - 2a_2^{\psi}(x) + a_2^{A_\mu}(x) \right) \\ &= W(V, A, g) + \frac{\log \lambda}{16\pi^2} \int \mathrm{Tr} \left(4a_2^{\phi}(x) - 2a_2^{\psi}(x) + a_2^{A_\mu}(x) \right), \end{aligned}$$

where the Seeley-deWitt coefficients have been given in (7.72). We have used the field content of the theory and in particular that all fields transform under the adjoint representation of the gauge group. We have also used that in the background gauge the scalars and ghosts have the same fluctuation operators. The sum of the Seeley-deWitt coefficients is zero,

$$\left(\frac{4}{12} + \frac{4}{3} - \frac{5}{3} \right) X = 0, \quad X = g^2 \mathrm{Tr}_{\mathrm{A}} F^{\mu\nu} F_{\mu\nu},$$

[1] or equivalently 4 Weyl spinors which may be grouped into two Dirac spinors.

and the 1-loop action is scale invariant. Hence the β-function and wave function renormalization are trivial,

$$\beta(g) = 0 \quad \text{and} \quad Z_3(g) = 1. \tag{8.1}$$

In [71] it was found that the β-function remains zero up to three loops and that therefore there were no divergent graphs at all to that order. This led everyone to suspect that the theory may be a finite field theoretical model in four space-time dimensions and arguments were put forward to proof the finiteness to all orders in perturbation theory. Some arguments are based on the relation between the trace anomaly in the energy-momentum tensor, the β-function and conformal invariance, other arguments used the explicit matching of bosonic and fermionic counting in the light-cone gauge and yet other arguments were based on the non-renormalisation theorem and the background gauge. The vanishing of the β-function to all orders has been shown in [72].

8.2 Kaluza-Klein reduction

Models with extended supersymmetry are intimately related to LAGRANGEan field theories in higher dimensions. The idea of employing higher dimensions to construct 4-dimensional models with extended supersymmetry has been pioneered by J. SCHERK and co-workers [69]. We shall see how the maximally extended gauge model fits into a world in higher dimensions. We derive this theory by a sort of KALUZA-KLEIN reduction of a $\mathcal{N} = 1$ super-YANG-MILLS theory in higher dimensions. After compactification all but four components of the vector potential in higher dimensions become scalar fields. Since the $\mathcal{N} = 4$ SYM theory has 6 scalar fields and a gauge field with 4 components we must reduce a gauge theory with a $4 + 6 = 10$-component gauge potential, that is a gauge theory in 10 dimensions.
$\mathcal{N} = 1$ SYM theory in 10 dimensions has the gauge invariant action

$$S = \int d^{10}x \left(-\tfrac{1}{4}\mathrm{Tr}\, F_{mn}F^{mn} + \tfrac{i}{2}\mathrm{Tr}\, \bar{\Psi}\not{D}\Psi\right) \tag{8.2}$$

and is invariant under the *on-shell* susy transformations

$$\delta_\alpha A_m = i\bar{\alpha}\Gamma_m\Psi \quad \text{and} \quad \delta_\alpha\Psi = iF^{mn}\Sigma_{mn}\alpha. \tag{8.3}$$

The only difference to the 4-*dimensional* transformations (6.3, 6.4) (with $\mathcal{G} = 0$) is that ψ and α are MAJORANA-WEYL spinors.
The proof that the 10-dimensional action is invariant is very similar to the corresponding proof for the $\mathcal{N} = 1$ theory in 4 dimensions. On the way one needs to show that the quartic term

$$\tfrac{i}{2}\mathrm{Tr}\,(\bar{\Psi}\Gamma^m[(\bar{\alpha}\Gamma_m\Psi), \Psi]) = \tfrac{1}{2}f_{abc}(\bar{\Psi}^a\Gamma^m\Psi^b)(\bar{\alpha}\Gamma_m\Psi^c) \tag{8.4}$$

vanishes. Again one applies a suitable FIERZ identity. In 10 dimensions the general identity is more complicated than in 4 dimensions and is given in (A.10) in the appendix. After a lengthy calculation on finds the following NOETHER current

$$J^m = -i\text{Tr}\,(F^{mn}\Gamma_n\Psi) + \tfrac{i}{2}\text{Tr}\,(F_{pq}\Gamma^{pqm}\Psi) \tag{8.5}$$

8.2.1 Reduction of Yang-Mills term

In 10 spacetime dimensions a gauge potential A_m and gauge coupling constant $\tilde{g}$ have length-dimensions

$$[A_m] = L^{-4}, \quad \tilde{g} = L^3 \Longrightarrow [\tilde{g}A_m] = L^{-1}. \tag{8.6}$$

First we perform the KALUZA-KLEIN reduction of the YANG-MILLS action

$$S_{YM} = -\tfrac{1}{4}\int d^{10}x\,\text{Tr}\,F_{mn}F^{mn} \tag{8.7}$$

on $\mathbb{R}^4 \times T^6$. As internal space we choose the torus T^6 with volume $\tilde{V}$ and set

$$\tilde{V}^{\frac{1}{2}}A_m = (A_\mu, \Phi_a), \qquad m = 0,\dots,9, \quad \mu = 0,\dots,4, \quad a = 1,\dots,6. \tag{8.8}$$

We assume that *all fields* are independent of the *internal coordinates* $x^4,\dots,x^9$. This assumption may be justified on dynamical grounds for internal spaces with tiny volumes. On length scales much larger then the typical size of the internal space the modes which depend on the internal coordinates are not excited. With this crucial assumption the 10-dimensional YANG-MILLS action reduces to the action of a 4-dimensional YANG-MILLS-HIGGS model

$$\begin{aligned} S_{YM} \quad &\longrightarrow \quad S_{YMH} = \int d^4x\,\mathcal{L}_{YMH} \\ \mathcal{L}_{YMH} \;&=\; -\tfrac{1}{4}\text{Tr}\,F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\sum_a \text{Tr}\,D^\mu\Phi_a D_\mu\Phi_a + \tfrac{1}{4}g^2\sum_{ab}\text{Tr}\,[\Phi_a,\Phi_b]^2. \end{aligned} \tag{8.9}$$

We used $\Phi_a = -\Phi^a$ and that the dimensionful coupling constant $\tilde{g}$ in 10 dimensions and the *dimensionless coupling constant* g in 4 dimensions are related as

$$\tilde{g}^2 = \tilde{V}g^2. \tag{8.10}$$

Not unexpected we got the action of a four-dimensional YANG-MILLS-HIGGS theory with 6 HIGGS fields in the adjoint representation. Typical is the quartic potential $\sim \sum \text{Tr}\,[\Phi_a,\Phi_b]^2$ with many flat directions.

8.2.2 Spinors in 10 dimensions

In 10 dimensions a DIRAC spinor has 32 complex components. But the $\mathcal{N} = 4$ SYM theory in 4 dimensions has 4 MAJORANA spinors which together have only 16 real

components. Hence we must assume that the spinors in 10 dimensions are both Weyl and Majorana. Such spinors do exist, see chapter 3.
We start with Dirac matrices γ^μ in 4-dimensional Minkowski space-time and give an explicit realization for the matrices $\Gamma^m, m = 0, \dots, 9$ in 10 dimensions. We make the ansatz

$$\Gamma_\mu = \Delta \otimes \gamma_\mu, \quad \Gamma_{3+a} = \Delta_a \otimes \gamma_5, \qquad \mu = 0, 1, 2, 3, \quad a = 1, \dots, 6, \tag{8.11}$$

with 8×8 matrices Δ and Δ_a. They must satisfy

$$\Delta^2 = \mathbb{1}_6, \quad [\Delta, \Delta_a] = 0 \quad \text{and} \quad \{\Delta_a, \Delta_b\} = -2\delta_{ab}\mathbb{1}_8 \tag{8.12}$$

in order for

$$\{\Gamma_m, \Gamma_n\} = 2\eta_{mn}, \qquad \eta = \text{diag}(1, -1, \dots, -1) \tag{8.13}$$

to hold true. Because Γ^0 is hermitian and the $\Gamma^{m>0}$ antihermitian we conclude that Δ is hermitian and Δ_a antihermitian.
Since Δ commutes with all matrices we may choose it to be the identity,

$$\Delta = \mathbb{1}_8. \tag{8.14}$$

Note that the hermitian $i\Delta_a$ generate the Euclidean Clifford algebra in 6 dimensions and that the $[\Delta_a, \Delta_b]$ generate the group $Spin(6)$. In 6 Euclidean dimensions there is a Majorana representation with *real and antisymmetric* Δ_a. An explicit representation is given in the appendix.
We choose a Majorana representation in 4-dimensions such that the γ_μ and $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ are imaginary. As charge conjugation matrix in 4 dimensions we take $\mathcal{C}_4 = -\gamma_0$. The real Δ_a lead to imaginary Γ_m such that the charge conjugation matrix in 10 dimensions is simply

$$\mathcal{C}_{10} = -\Gamma_0 = \mathbb{1}_8 \otimes \mathcal{C}_4 \quad \text{with} \quad \mathcal{C}_4 = -\gamma_0. \tag{8.15}$$

For the chiral projections we need $\Gamma_{11} = -\Gamma_0 \cdots \Gamma_9 = \Gamma_{11}^\dagger$ which takes the form

$$\Gamma_{11} = \Gamma_* \otimes \gamma_5, \qquad \Gamma_* = -i\Delta_1 \cdots \Delta_6, \qquad \Gamma_*^\dagger = \Gamma_* = -\Gamma_*^T. \tag{8.16}$$

A spinor $\Psi = \xi \otimes \psi$ is Majorana in 10 dimensions if ξ is real and ψ a Majorana spinor in 4 dimensions. Hence an arbitrary Majorana *spinor* has the expansion

$$\Psi = \sum_{r=1}^{8} \boldsymbol{e}_r \otimes \psi_r, \tag{8.17}$$

where the $\boldsymbol{e}_r$ form a basis of $\mathbb{R}^8$ and the ψ_r are Majorana spinors in 4 dimensions. A spinor has positive chirality if

$$\Psi = \Gamma_{11}\Psi = (\Gamma_* \otimes \gamma_5)\Psi. \tag{8.18}$$

To characterize WEYL spinors we expand the first factor of $\xi\otimes\psi$ in eigenvectors of the imaginary and hermitian Γ_*. Let $\boldsymbol{g}_1,\dots,\boldsymbol{g}_4\in\mathbb{C}^8$ be the orthonormal (and necessarily complex) eigenvectors of Γ_* with eigenvalue 1. Together with the complex conjugate eigenvectors $\boldsymbol{g}_1^*,\dots,\boldsymbol{g}_4^*$ with eigenvalue -1 they form a basis of $\mathbb{C}^8$. A spinor with *positive chirality* has the expansion

$$\Psi=\sum_{p=1}^{4}\left(\boldsymbol{g}_p\otimes\psi_p^+ + \boldsymbol{g}_p^*\otimes\psi_p^-\right),\quad\text{where}\quad \gamma_5\psi^\pm=\pm\psi^\pm, \tag{8.19}$$

that is $\psi^\pm$ are the chiral parts of ψ. A MAJORANA-WEYL *spinor* in 10 dimensions has at the same time the expansion (8.16) and thus has the form

$$\Psi=\sum_{p=1}^{4}\left(\boldsymbol{g}_p\otimes\psi_p^+ + \boldsymbol{g}_p^*\otimes\psi_p^-\right) \tag{8.20}$$

with four MAJORANA spinors $\psi_p=\psi_p^+ + \psi_p^-$ in 4 dimensions. The important formula (8.20) assigns to each MAJORANA-WEYL spinor in 10 dimensions four MAJORANA spinors in 4 dimensions and vice versa. It is the analog of (8.8) for spinor fields.

8.2.3 Reduction of the Dirac term

In ten and four dimensions a spinor field has the dimension

$$[\Psi]=L^{-9/2}\quad\text{and}\quad[\psi]=L^{-3/2}, \tag{8.21}$$

respectively. In the general expansion (8.20) for a MAJORANA-WEYL spinor in 10 dimensions we rescale the spinor such that the ψ_p in

$$\tilde V^{\frac12}\Psi=\sum_p\left(\boldsymbol{g}_p\otimes\psi_p^+ + \boldsymbol{g}_p^*\otimes\psi_p^-\right) \tag{8.22}$$

have correct length-dimension. In the reduction from ten to four dimensons we assume that the spinor field does not depend on the internal coordinates and with this assumption and (8.8,8.10) we obtain

$$\begin{aligned}\tilde V^{\frac12}D_\mu\Psi &= \sum_p\left(\boldsymbol{g}_p\otimes D_\mu\psi_p^+ + \boldsymbol{g}_p^*\otimes D_\mu\psi_p^-\right),\\ \tilde V^{\frac12}D_{3+a}\Psi &= -ig\sum_p\left(\boldsymbol{g}_p\otimes[\Phi_a,\psi_p^+] + \boldsymbol{g}_p^*\otimes[\Phi_a,\psi_p^-]\right),\end{aligned} \tag{8.23}$$

where the covariant derivative $D_\mu=\partial_\mu-ig[A_\mu,.]$ contains the gauge potential and coupling constant in 4 dimensions. Now we are ready to dimensionally reduce the DIRAC term from ten to four dimensions. One obtains the DIRAC terms for the four 'gluino' fields in 4 dimensions plus a particular YUKAWA interaction between the scalar fields Φ_a and the 'gluino' fields ψ_p. This interaction contains the non-zero matrix elements of $\Delta^a \equiv \Delta_a$ in the eigenbasis of Γ_*,

$$\Delta^a_{pq} = (\boldsymbol{g}^*_p, \Delta^a \boldsymbol{g}_q) \quad \text{and} \quad \bar{\Delta}^a_{pq} = (\boldsymbol{g}_p, \Delta^a \boldsymbol{g}^*_q). \tag{8.24}$$

The other matrix elements vanish since the hermitian Γ_* anticommutes with Δ^a. The matrix elements Δ^a_{pq} define a 4-dimensional matrix and completely determine the 8-dimensional matrix Δ^a. Collecting the various terms we end up with the LAGRANGEan density for the $\mathcal{N} = 4$ supersymmetric gauge theory in 4 dimensions,

$$\begin{aligned} \mathcal{L} \;=\; & -\tfrac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mathrm{Tr}\, D^\mu\Phi_a D_\mu\Phi_a + \tfrac{1}{4}g^2\mathrm{Tr}\,[\Phi_a, \Phi_b]^2 \\ & +\tfrac{1}{2}\mathrm{Tr}\,\left(i\bar{\psi}_p \not{D}\psi_p + g\Delta^a_{pq}\bar{\psi}_p[\Phi_a, \psi^+_q] - g\bar{\Delta}^a_{pq}\bar{\psi}_p[\Phi_a, \psi^-_q]\right), \end{aligned} \tag{8.25}$$

where $a = 1, \dots, 6$ and $p = 1, \dots, 4$. The CLEBSCH-GORDAN type coefficients Δ^a_{pq} are needed for the density (8.25) to be invariant under R-transformations.

8.2.4 R-symmetry

Vector and spinor fields in 10 dimensions transform under LORENTZ transformations in 10 dimensions as

$$A(x) \longrightarrow \tilde{\Lambda}A(\tilde{\Lambda}^{-1}x) \quad \text{and} \quad \Psi(x) \longrightarrow \tilde{S}\Psi(\tilde{\Lambda}^{-1}x), \tag{8.26}$$

where the spin and LORENTZ transformation are related via

$$\tilde{S}^{-1}\Gamma^m\tilde{S} = \tilde{\Lambda}^m_{\ n}\Gamma^n. \tag{8.27}$$

The spin rotations are generated by the matrices $\Gamma_{mn} = \frac{1}{2}[\Gamma_m, \Gamma_n]$ and the explicit relation in (8.27) reads

$$\tilde{S} = \exp\left(\tfrac{1}{2}\omega_{mn}\Gamma^{mn}\right) \Longrightarrow (\tilde{\Lambda})^m_{\ n} = (e^\omega)^m_{\ n}. \tag{8.28}$$

The LAGRANGEan density is a scalar field such that the action is LORENTZ invariant. In the dimensional reduction to $\mathbb{R}^4$ we demanded the fields not to depend on the internal coordinates. This condition is not compatible with all LORENTZ transformations in ten dimensions. Only LORENTZ transformations which do not mix coordinates on $\mathbb{R}^4$ with internal coordinates are admitted. They have the form

$$\tilde{\Lambda} \longrightarrow \begin{pmatrix} \Lambda & 0 \\ 0 & R \end{pmatrix} \in SO(1,3) \times SO(6) \subset SO(1,9), \tag{8.29}$$

where Λ is a LORENTZ transformation in 4 dimensions and $R \in SO(6)$[2]. With our choice for the Γ_m in (8.11) the generators of the corresponding spin transformations in 10 dimensions read

$$\Gamma_{\mu\nu} = \mathbb{1}_8 \otimes \gamma_{\mu\nu} \quad , \quad \Gamma_{3+a,3+b} = \Delta_{ab} \otimes \mathbb{1}_4 \tag{8.30}$$

[2]actually, only $\det\Lambda \cdot \det R = 1$ is required.

The Δ_{ab} generate the $Spin(6) \sim SU(4)$ subgroup of $Spin(1,9)$. Since the $\Gamma_{\mu\nu}$ act trivially on the first factor of $\xi \otimes \psi$ and the other generators in (8.30) act trivially on the second factor, the admitted spin rotations act as

$$\tilde{S}(\xi \otimes \psi) = (S_6\, \xi) \otimes (S\psi), \tag{8.31}$$

where S_6 and S define $SO(6)$ and Lorentz transformations according to

$$S_6^{-1}\Delta^a S_6 = R^a_{\ b}\Delta^b \quad \text{and} \quad S^{-1}\gamma^\mu S = \Lambda^\mu_{\ \nu}\gamma^\nu. \tag{8.32}$$

It follows that the $SO(6)$-factor of the admitted Lorentz transformations act as global and compact *internal* $SU(4)$ R-symmetry,

$$\Phi_a(x) \longrightarrow R_a^{\ b}\Phi_b(x), \quad \Psi(x) \longrightarrow S_6\xi \otimes \psi, \quad A_\mu(x) \longrightarrow A_\mu(x) \tag{8.33}$$

and the $SO(1,3)$-factor as Lorentz transformations in 4 dimensions,

$$\Phi_a(x) \longrightarrow \Phi_a(\Lambda^{-1}x), \;\; \psi_p(x) \longrightarrow S\psi_p(\Lambda^{-1}x), \;\; A_\mu(x) \longrightarrow \Lambda_\mu^{\ \nu} A_\nu(\Lambda^{-1}x). \tag{8.34}$$

In the reduced theory the first factor in $\xi\otimes\psi$ has disappeared and we must reinterpret the R-symmetry $\Psi \to S_6\xi \otimes \psi$ as a transformation of the 4-dimensional spinor ψ. To find this transformation we note that the real S_6 commutes with Γ_* such that

$$S_6\boldsymbol{g}_p = U_{qp}\boldsymbol{g}_q \quad \text{and} \quad S_6\boldsymbol{g}^*_p = U^*_{qp}\boldsymbol{g}^*_q, \qquad U, U^* \in SU(4). \tag{8.35}$$

These relations define group-isomorphisms between $Spin(6)$ and $SU(4)$. Therefore, under the R-symmetry the spinors transform as

$$\psi_p^+ \longrightarrow \sum_q U_{pq}\psi_q^+ \quad \text{and} \quad \psi_p^- \longrightarrow \sum_q U^*_{pq}\psi_q^-. \tag{8.36}$$

Note that for Majorana spinors ψ_p the R-transformed objects $\sum_q \left(U_{pq}\psi_q^+ + U^*_{pq}\psi_p^-\right)$ are Majorana spinors as well.

By construction the action (8.25) of the $\mathcal{N} = 4$ SYM-theory must be invariant under R-transformations. The invariance is easily seen for the terms without fermions and the Dirac term. To prove it directly for the Yukawa terms one uses the first relation in (8.32) and the definitions in (8.24) giving rise to the following group homomorphism between $SO(6)$ and $SU(4)$:

$$U_{rp}\Delta^a_{rs}\, U_{sq} = R^a_{\ b}\Delta^b_{pq} = \left(S_6^{-1}\Delta^a S_6\right)_{pq}. \tag{8.37}$$

These relations implies that the Yukawa terms are invariant under R-transformations.

8.3 Susy transformation of reduced theory

The supersymmetries of the $\mathcal{N} = 1$ SYM-theory in ten dimensions reduce to supersymmetries of the $\mathcal{N} = 4$ SYM-theory in four dimensions. To derive these transformations we insert the expansion (8.20) for a Majorana-Weyl spinor into the supersymmetry transformations (8.2) (and set $\tilde V=1$)

$$\begin{aligned}
\delta_\alpha A_\mu &= i\bar\alpha\,(\mathbb{1}_8\otimes\gamma_\mu)\,\Psi && (8.38)\\
\delta_\alpha \Phi_a &= i\bar\alpha\,(\Delta_a\otimes\gamma_5)\,\Psi && (8.39)\\
\delta_\alpha \Psi &= \Big(iF_{\mu\nu}\mathbb{1}_8\otimes\Sigma^{\mu\nu}+D_\mu\Phi_a\,(\Delta^a\otimes\gamma^\mu\gamma_5)+\frac{g}{2i}[\Phi_a,\Phi_b]\Delta^{ab}\otimes\mathbb{1}_4\Big)\alpha && (8.40)
\end{aligned}$$

as well as the corresponding expansion for the supersymmetry parameter α:

$$\alpha=\sum\left(\boldsymbol{g}_p\otimes\varepsilon_p^+ +\boldsymbol{g}_p^*\otimes\varepsilon_p^-\right). \tag{8.41}$$

Using (8.21) we obtain the variations of the vector potential and scalar fields

$$\begin{aligned}
\delta_\alpha A_\mu &= i\sum_p \bar\varepsilon_p\gamma_\mu\psi_p && (8.42)\\
\delta_\alpha \Phi_a &= i\sum_{pq}\Delta^a_{pq}\;\bar\varepsilon_p\gamma_5\psi_q+i\sum_{pq}(\Delta^a\Gamma_*)_{pq}\;\bar\varepsilon_p\psi_q. && (8.43)
\end{aligned}$$

To find the transformations of the Majorana spinors is less simple. We insert into (8.40) and

$$\delta_\alpha\Psi=\sum\left(\boldsymbol{g}_p\otimes(\delta_\alpha\psi_p)^+ +\boldsymbol{g}_p^*\otimes(\delta_\alpha\psi_p)^+\right)$$

the expansion (8.41) for α and compare coefficients. With the definition

$$\Delta^{ab}_{pq}\equiv(\boldsymbol{g}_p,\Delta^{ab}\boldsymbol{g}_q) \tag{8.44}$$

the variations of the 4-dimensional Majorana spinors can be written as

$$\begin{aligned}
\delta\psi_p &= iF_{\mu\nu}\Sigma^{\mu\nu}\varepsilon_p+\not{D}\Phi_a\sum_q\left(\Delta^a_{pq}\varepsilon_q^+-\bar\Delta^a_{pq}\varepsilon_q^-\right)\\
&\qquad +\frac{g}{2i}\,[\Phi_a,\Phi_b]\sum_q\Big(\Delta^{ab}_{pq}\varepsilon_q^+ +\bar\Delta^{ab}_{pq}\varepsilon_q^-\Big). && (8.45)
\end{aligned}$$

Also the Noether current is derived from the current (8.5) by dimensional reduction. The four currents depend linearly on the ψ_p, the field strength and its dual and the covariant derivatives of the scalar fields. Their explicit forms read

$$\begin{aligned}
J^\mu_p &= -\mathrm{Tr}\,({}^*F^{\mu\nu}\gamma_5+iF^{\mu\nu})\,\gamma_\nu\psi_p\\
&\quad -i\sum_q\mathrm{Tr}\left\{D_\alpha\Phi_a\left(\Delta^a_{pq}\,\gamma^\alpha\gamma^\mu\psi_q^+-\bar\Delta^a_{pq}\,\gamma^\alpha\gamma^\mu\psi_q^-\right)\right\} && (8.46)\\
&\quad -\tfrac{g}{2}\sum_q\mathrm{Tr}\,\Big\{[\Phi_a,\Phi_b]\,\Big(\Delta^{ab}_{pq}\,\gamma^\mu\psi_q^+ +\bar\Delta^{ab}_{pq}\,\gamma^\mu\psi_q^-\Big)\Big\}.
\end{aligned}$$

From these 4 Noether currents one derives the 4 conserved supercharges of the $\mathcal{N}=4$ supersymmetric YM theory

$$\mathcal{Q}^p=\int d\boldsymbol{x}\;J^0_p,\qquad p=1,\dots,4. \tag{8.47}$$

They fulfill the anticommutation relations

$$\{\mathcal{Q}^p_\alpha, \mathcal{Q}^q_\beta\} = (\mathcal{C}_4\gamma^\mu)_{\alpha\beta}\,\delta^{pq}P_\mu. \tag{8.48}$$

These charges commute with the 4-momentum P_μ and transform as MAJORANA spinors under LORENTZ transformations.
Actually the $\mathcal{N} = 4$ SYM is classically scale invariant since all couplings in the LAGRANGEan (8.25) are dimensionless. Since the β-function vanishes to *all orders* in perturbation theory the quantized theory is scale invariant as well. Thus the supersymmetry algebra can be extended to a $\mathcal{N} = 4$ *superconformal algebra*. This enlarged symmetry leads to stringent conditions on the particle spectrum of the theory. The particles fall into representations of the superconformal algebra. When one tries to argue in favor of the AdS-CFT correspondence one needs these representations. Unfortunately, at this point I must refer to the literature, see for example the excellent and exhaustive review [70].

8.4 Omissions

In these lectures I had to omit many interesting aspects of supersymmetric theories. Probably I should have introduced *superfields* [73] from the very beginning to shorten some of the more lengthy calculations in the component formalism. In particular when it comes to gauge theories the superspace formulation is superior. I did not discuss FEYNMAN-diagram in supersymmetric theories and cancellations of divergences, although these chancellation lead to a vanishing β-function in $\mathcal{N} = 4$ susy Yang-Mills theory. Of course, for constructing realistic models for particle physics the breaking of supersymmetry is of paramount interest. I did not talk about this aspect of supersymmetric field theories either. It just would take another series of lectures at the troisieme cycle. But this issue is discussed in most of the texts [1]-[7].
One of the most interesting aspects of $\mathcal{N} = 1$ SUSY gluodynamics is the nonvanishing *gluino condensate*, which vanishes perturbatively, but does not vanish in the nonperturbative treatment and can be computed exactly. I refer to the review article by M. Shifman and A. Vainshtein [74] for the history of dynamical supersymmetry breaking and the role of instantons in susy gauge theories.
Also, the more recent results of SEIBERG and WITTEN on the low-energy effective action of $\mathcal{N} = 2$ SYM and the intriguing AdS-CFT correspondence have not been dealt with. You may consult the quoted literature, for example the recommended book of WEINBERG [3] or the review of Sachs [10] for the SEIBERG-WITTEN solution and the review [70] for the correspondence. I very much regretted to find no time for supersymmetric field theories on the lattice [75, 76, 77], a topic which attracted my attention during the past years [31].

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Appendix A

Useful formula

In this appendix we collected some useful formula which are used in the main body of the paper.

A.1 Gamma matrices and Fierz identities

For example, the product of generators of the spingroup and the gamma-matrices in 4 dimensions are reduced according to

$$\begin{aligned} \Sigma_{\mu\nu}\gamma_\rho &= \tfrac{i}{2}\eta_{\mu\rho}\gamma_\nu - \tfrac{i}{2}\eta_{\nu\rho}\gamma_\mu - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5, && \text{(A.1)} \\ \gamma_\rho\Sigma_{\mu\nu} &= -\tfrac{i}{2}\eta_{\mu\rho}\gamma_\nu + \tfrac{i}{2}\eta_{\nu\rho}\gamma_\mu - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 && \text{(A.2)} \end{aligned}$$

We also use the conjugation formula

$$\begin{aligned} \gamma_\mu\gamma_\rho\gamma^\mu &= (2-d)\gamma_\rho \\ \gamma_\mu\gamma_{\rho\sigma}\gamma^\mu &= (d-4)\gamma_{\rho\sigma} \\ \gamma_\mu\gamma_5\gamma_\rho\gamma^\mu &= (2-d)\gamma_\rho\gamma_5, \end{aligned} \qquad \text{(A.3)}$$

and the simple relation

$$i\gamma_5[\gamma_\mu,\gamma_\nu] + \epsilon_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma = 0, \qquad \text{(A.4)}$$

The following particular Fierz identities follow from the general identity (3.73)

$$\begin{aligned} \bar\psi\gamma^\rho\Sigma_{\mu\nu}\varepsilon &= \bar\varepsilon\Sigma_{\mu\nu}\gamma^\rho\psi, && \text{(A.5)} \\ \alpha_2\bar\alpha_1 - \alpha_1\bar\alpha_2 &= -\tfrac{1}{2}\gamma_\rho(\bar\alpha_1\gamma^\rho\alpha_2) + \gamma_{\rho\sigma}(\bar\alpha_1\gamma^{\rho\sigma}\alpha_2) && \text{(A.6)} \end{aligned}$$

For Majorana spinors the in (a,c) antisymmetric part of (6.27) can be written as

$$(\bar\psi^a\gamma^\mu\psi^b)(\bar\varepsilon\gamma_\mu\psi^c) + (\bar\psi^b\gamma^\mu\psi^c)(\bar\varepsilon\gamma_\mu\psi^a) + (\bar\psi^c\gamma^\mu\psi^a)(\bar\varepsilon\gamma_\mu\psi^b) = 0. \qquad \text{(A.7)}$$

Another useful identity is

$$\bar\varepsilon\gamma_5\gamma^\rho\Sigma_{\mu\nu}\psi = \bar\psi\Sigma_{\mu\nu}\gamma^\rho\gamma_5\alpha \qquad \text{(A.8)}$$

Sometimes one needs to switch from the chiral to the DIRAC basis. Then the following identities are useful:

$$\begin{aligned}\bar\alpha\psi-\bar\alpha_c\psi_c &= \theta\chi+\bar\theta\bar\chi+\zeta\lambda+\bar\zeta\bar\lambda\\ \bar\alpha\gamma_5\psi-\bar\alpha_c\gamma_5\psi_c &= \theta\chi+\bar\theta\bar\chi-\zeta\lambda-\bar\zeta\bar\lambda\\ \bar\alpha\gamma_\mu\psi+\bar\alpha_c\gamma_\mu\psi_c &= \theta\sigma_\mu\bar\lambda+\bar\theta\tilde\sigma_\mu\lambda+\zeta\sigma_\mu\bar\chi+\bar\zeta\tilde\sigma_\mu\chi.\end{aligned}\tag{A.9}$$

In constructing the LAGRANGEan of the $\mathcal{N}=4$ SYM theory we used the general FIERZ-Identity in 10 dimensions

$$\begin{aligned}32\Psi\bar\Phi \;=\;& -\bar\chi\psi-\Gamma_m X^m+\tfrac{1}{2!}\Gamma_{mn}X^{mn}+\tfrac{1}{3!}\Gamma_{mnp}X^{mnp}-\tfrac{1}{4!}\Gamma_{mnpq}X^{mnpq}\\ & -\tfrac{1}{5!}\Gamma_{mnpqr}X^{mnpqr}-\tfrac{1}{4!}\Gamma_{11}\Gamma_{mnpq}X_{11}^{mnpq}-\tfrac{1}{3!}\Gamma_{11}\Gamma_{mnp}X_{11}^{mnp}\\ & +\tfrac{1}{2!}\Gamma_{11}\Gamma_{mn}X_{11}^{mn}+\Gamma_{11}\Gamma_m X_{11}^m-\Gamma_{11}X_{11}\end{aligned}\tag{A.10}$$

where we have abbreviated

$$X^{m\cdots}=\bar\Phi\Gamma^{m\cdots}\Psi \quad\text{and}\quad X_{11}^{m\cdots}=\bar\Phi\Gamma_{11}\Gamma^{m\cdots}\Psi.$$

A.2 Majorana representation in 6 Euclidean dimensions

In our dimensional reduction from 10 to 4 dimensions we used a particular representation for the EUCLIDean γ-matrices belonging to the internal 6 dimensional space. To construct this MAJORANA representation we make for the Δ_a in (8.11) the ansatz

$$\Delta_i=i\tau_1\otimes\alpha_i \quad\text{and}\quad \Delta_{3+i}=i\tau_3\otimes\tilde\alpha_i,\qquad i=1,2,3,$$

so that

$$\{\alpha_i,\alpha_j\}=\{\tilde\alpha_i,\tilde\alpha_j\}=2\delta_{ij}\mathbb{1}_4,\quad [\alpha_i,\tilde\alpha_j]=0,\quad \alpha_i^\dagger=\alpha_i,\quad \tilde\alpha_i^\dagger=\tilde\alpha_i$$

must hold. A possible solution for the α-matrices is

$$\begin{aligned}\alpha_1=\tau_2\otimes\tau_1,\quad \alpha_2=\tau_0\otimes\tau_2,\quad \alpha_3=\tau_2\otimes\tau_3\\ \tilde\alpha_1=\tau_1\otimes\tau_2,\quad \tilde\alpha_2=-\tau_3\otimes\tau_2,\quad \tilde\alpha_3=\tau_2\otimes\tau_0.\end{aligned}$$

These hermitean, imaginary and hence antisymmetric matrices obey

$$\alpha_i\alpha_j=\delta_{ij}+i\epsilon_{ijk}\alpha_k \quad\text{and}\quad \tilde\alpha_i\tilde\alpha_j=\delta_{ij}+i\epsilon_{ijk}\tilde\alpha_k,\quad [\alpha_i,\tilde\alpha_j]=0.$$

They lead to *real and antisymmetric* matrices Δ_a which fulfill the EUCLIDean CLIFFORD algebra in 6 dimensions.

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