Symmetries in Physics

Andreas Wipf

Theoretisch-Physikalisches Institut, FSU Jena

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Introduction

- *Wenn ein System eine gewisse Gruppe von Symmetrieoperationen besitzt, dann muss jede physikalische Beobachtungsgröße dieses Systems ebenfalls dieselbe Symmetrie besitzen.*
  Prinzip von NEUMANN
- *My work always tried to unite the true with the beautiful, but when I had to choose one over the other, I usually chose the beautiful*.
  HERMANN WEYL
- *Nowadays, group theoretical methods—especially those involving characters and representations, pervade all branches of quantum mechanics.*
  GEORGE MACKEY

- Emmy Noether:

  symmetries $\rightarrow$ conserved charges

  e.g. energy, momentum, angular momentum, electric charge, ...
- conserved charges generate symmetries

  e.g. momenta generate translations, electric charge generates phase rotations
- symmetries $\rightarrow$ groups, Lie-algebras
- groups: abstract, geometric, representations
- theory of representations $->$ ordering, classification, selection rules, simplify calculations, improve understanding, ...
- where: classical and quantum mechanics, atomic and molecular physics, solid state physics, crystals, particle physics, ...
- beyond Standard Model: string theory, AdS-CFT, ...

Symmetries form a group

- symmetries:

  transform object appearing in nature or formalism

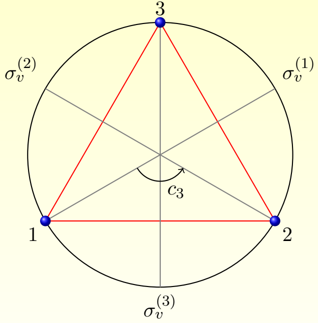
  e.g. atom, molecule, solid body, field, position, time, ...
- symmetry operations can be composed and inverted $\to$ group
- definition of group: $(G, \circ)$ set with composition $\circ$
  1. closed $g_1, g_2 \in G \mapsto g_1 \circ g_2 \in G$
  2. associative: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
  3. neutral element: $e \circ g = g \circ e = g$ for every $g \in G$
  4. exist inverse $g^{-1} \in G$ with $g \circ g^{-1} = g^{-1} \circ g = e$

- Präsentation: e.g. all words of characters $\{e, a, b\}$ with relations

$$\left\{a, b \,|\, a^3 = b^2 = e,\ b \circ a \circ b^{-1} = a^{-1} \Longleftrightarrow b \circ a = a^2 \circ b\right\}$$

which group is this?

- geometric definition: e.g. symmetries of equilateral triangle $\rightarrow \mathcal{D}_3$



- same groups: $a \cong$ rotation c_3, $b \cong$ reflection $\sigma_v^{(3)}$

finite, discrete, continuous groups

- atom and molecular physics: point groups
  finite subgroups of rotation group O(3) known
- crystals: space groups
  discrete subgroup of Euclidean group E_3 known
- space-time symmetries: Galilean, Lorentz or Poincaré groups
  $O(3)$, $O(1,3)$, translations, supersymmetry, ... classified
- particle physics: unitary Lie-Groups
  Standard model $SU_c(3) \times SU_L(2) \times U_Y(1)$ classified

finite groups more difficult than continuous compact groups

representations

- quantum mechanics, field theory:
  group elements are linear transformation in vector space $\mathcal{V}$
  (not necessarily in classical mechanics or general relativity)
- $g \mapsto D(g)$, $D(g)$ linear and invertible map $\mathcal{V} \longmapsto \mathcal{V}$

$$\text{representation:} \quad D(g_1 \circ g_2) = D(g_1)D(g_2)$$

- $D(g)$ form group $\mathrm{GL}(n)$, $n = \dim(\mathcal{V})$, n-dimensional reps
- reducible reps: in adapted base $D(g)$ has block-form

$$D(g) = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \quad \text{for all} \quad g \in G$$

- not reducible: irreducible reps, atoms of reps
- irreducible reps. has no proper invariant subspace $\subseteq \mathcal{V}$

- finite groups, most continuous groups:
  every reps is direct sum of irreducible reps
- reduction of reps: for all g

$$D = \begin{pmatrix} D_1 & 0 & 0 & \dots \\ 0 & D_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & D_r \end{pmatrix}, \quad D = D_1 \oplus D_2 \oplus \cdots \oplus D_r$$

- reps decomposes into r irreducible reps
- main problems:
  find all irreducible reps of group
  decompose given reps. into these irreducible reps

Theorem (consequence of Schur's lemma)

Assume D decomposes into irreducible reps.,

$$D = D_1 \oplus D_2 \oplus \cdots \oplus D_r \,,$$

and that the linear map $H : \mathcal{V} \to \mathcal{V}$ commutes with all $D(g)$:

$$[H, D(g)] = 0, \quad \forall g \in G \tag{1}$$

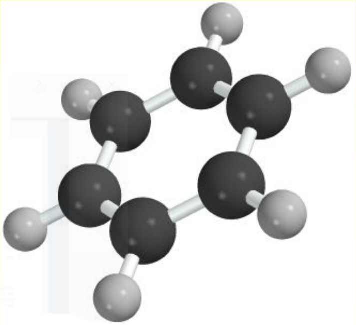
Then all invariant subspaces of D are eigenspaces of H.

allows for partial diagonalization of H,

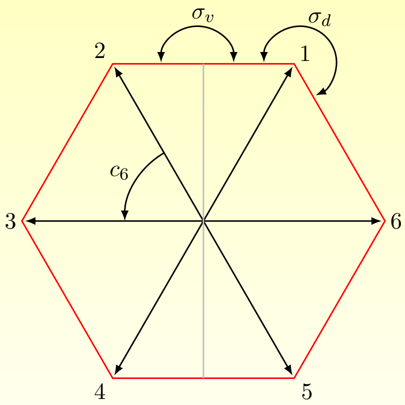
$$H = \begin{pmatrix} H_1 & 0 & 0 & \dots \\ 0 & H_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \\ 0 & \dots & \dots & H_r \end{pmatrix}$$

example: benzen molecule C_6H_6

- effective description with atomic orbitals
- 6 corners $\sim$ CH groups
- symmetries form dihedral group $\mathcal{D}_6$ with 12 elements



symmetry-adapted base in Hilbert space $\mathbb{C}^6$



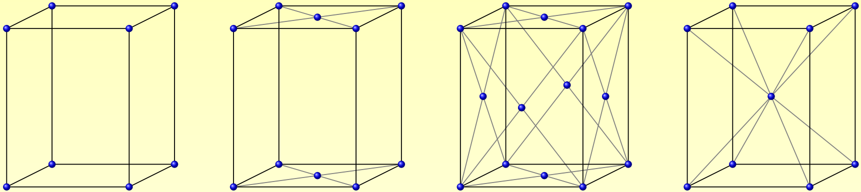
$$H = -\varepsilon \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

$$D = D_1^1 \oplus D_1^2 \oplus D_2^1 \oplus D_2^2$$

$$\Downarrow$$

$$H = \epsilon\,\text{diag}\left(-2, 2, 1, 1, -1, -1\right)$$

example: 7 crystal systems and 32 classes



orthorhombic Bravais lattices

crystal system	group	# Bravais	# classes	space group Nr.
triclinic	$\mathcal{S}_2$	1	2	$1, 2$
monoclinic	$\mathcal{C}_{2h}$	2	3	$3, 4, \dots, 15$
orthorhombic	$\mathcal{D}_{2h}$	4	3	$16, 17, \dots, 74$
tetragonal	$\mathcal{D}_{4h}$	2	7	$75, 76, \dots, 142$
trigonal	$\mathcal{D}_{3d}$	1	5	$143, 144, \dots, 167$
hexagonal	$\mathcal{D}_{6h}$	1	7	$168, 169, \dots, 194$
cubic	$\mathcal{O}_h$	3	5	$195, 196, \dots, 230$

continuous symmetries

- continuous symmetries $\to$ Lie group G
- infinitesimal deviation from identity: $G \ni g \approx \mathbb{1} + X + \dots$
- $\{X\}$ form Lie-Algebra $\mathfrak{g}$ of Lie group G
  vector space and $[X, Y] = -[Y, X]$, bilinear, Jacobi-identity
- $X_1, \dots, X_n$ basis of n-dimensional Lie-Algebra

$$[X_a, X_b] = f_{ab}{}^c X_c, \quad \text{structure constants} \;\; f_{ab}{}^c$$

- every reps of Lie groups $\to$ reps of Lie algebra
- every reps of Lie algebra $\to$ reps of univ. covering of Lie Group
- every reps of Lie algebra $\to$ projective reps. of Lie Group

coverings: $\mathrm{SU}(2) \mapsto \mathrm{SO}(3), \mathrm{SL}(2, \mathbb{C}) \mapsto \mathrm{SO}_+^\uparrow, \dots$

space-time symmetries

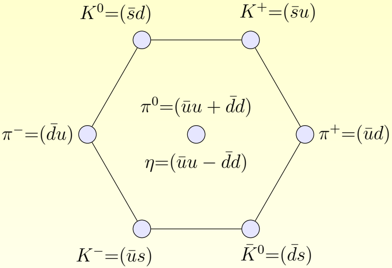
- rotations in space $x \to Rx \to$ rotation group SO(3)
  infinitesimal rotations form $\mathfrak{so}(3)$, basis L_1, L_2, L_3
  angular momentum CR: $[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ikj}L_k$
  irreducible reps. characterized by $\boldsymbol{L}^2$ and L_3
- change inertial system $x' = \Lambda x + a \to$ Poincaré group
  infinitesimal translations P_μ
  infinitesimal Lorentz-transformation $M_{\mu\nu} = -M_{\nu\mu}$
- Poincaé algebra

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \mathrm{i}\big(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\big) \\
[M_{\mu\nu}, P_\rho] &= \mathrm{i}\big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\big) \\
[P_\mu, P_\nu] &= 0
\end{aligned}
$$

- irred. reps characterized by mass and spin (mass and helicity)

global symmetries in particle physics

- strong interaction: approximate SU(3) flavor-symmetry
- eigenstates of H fall into irreducible mutipletts of SU(3)



octet of pseudo-scalar mesons with $J^P = 0^-$

$\Omega, J/\psi$

- more SU(3) multiplets
- order in zoo of mesons and baryons
- spectacular success: discovery of Ω, as predicted by SU(3)

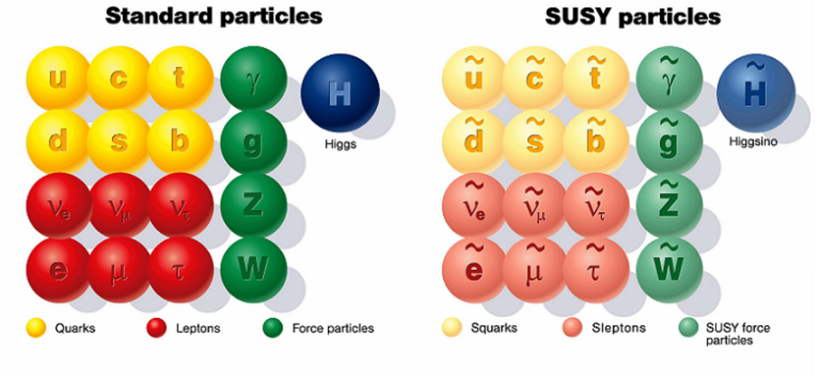
The importance of group theory was emphasized very recently when some physicists using group theory predicted the existence of a particle that had never been observed before, and described the properties it should have. Later experiments proved that this particle really exists and has those properties.
IRVING ADLER

from global to local

- gauge principle
  global symmetry can be made local
  introduces gauge potentials
- quantization: potentials describe exchange particles
  photon
  $Z, W^{\pm}$ bosons of electroweak interaction
  gluons of strong interaction
  gravitons in gravity
- all fundamental relativistic theories are gauge theories
  elektrodynamik, weak interaction, strong interaction, (gravity)

$$\mathrm{U}(1), \quad \mathrm{SU}_L(2) \times \mathrm{U}_Y(1), \quad \mathrm{SU}(3)$$

- gauge-principle fixes most part of interactions
  Symmetry principles have moved to a new level of importance in this century and especially in the last few decades; there are symmetry principles that dictate the very existence of all the known forces of nature.
  S. WEINBERG
- powerful symmetry principles → classification and calculations!
- are symmetries helpful for „physics beyond the Standard Model“?
- warning:
  grand unified theories GUTs: $\mathrm{SU}(5), \mathrm{SO}(10)$ → proton decay?
  supersymmetry → new particles?
  string theory: $\mathrm{SO}(32), E_6, E_7, E_8$ → not even wrong
- rich mathematical structures, e.g.
  conformal symmetry → infinite-dimensional Virasoro,
  Kac-Moody and W algebras


Standard particles
SUSY particles
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Quarks
Leptons
Force particles
Squarks
Sleptons
SUSY force particles