The Functional Renormalization Group Method – An Introduction

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La Plata

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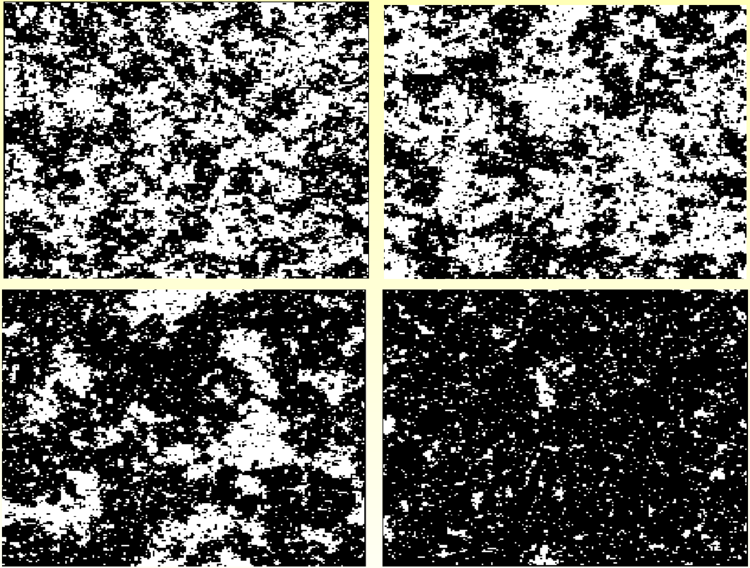

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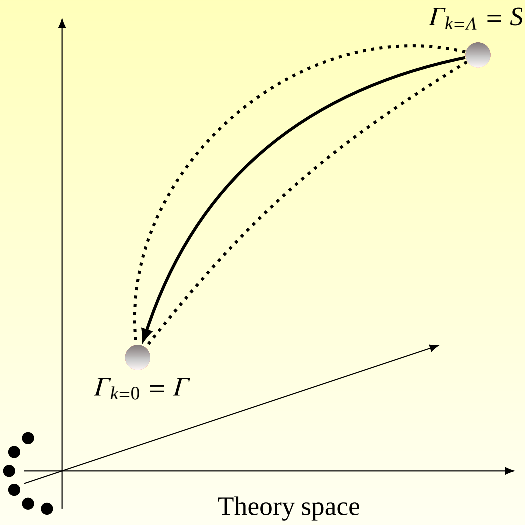
Introduction

- particular implementation of the renormalization group
- for continuum field theory, in momentum-space
- functional methods + renormalization group idea
- scale-dependent Schwinger functional or effective action
- conceptionally simple, technically demanding flow equations
- scale parameter k = adjustable screw of microscope
- large values of a momentum scale k: high resolution
- lowering k: decreasing resolution of the microscope
- known microscopic laws $\longrightarrow$ complex macroscopic phenomena
- non-perturbative

- flow of Schwinger functional $W_k[j]$: Polchinski equation
- flow of effective action $\Gamma_k[\varphi]$: Wetterich equation
- flow from classical action $S[\varphi]$ to effective action $\Gamma[\varphi]$
- applied to variety of physical systems
  - strong interaction
  - electroweak phase transition
  - asymptotic safety scenario
  - condensed matter systen
    e.g. Hubbard model, liquid He^4, frustrated magnets, superconductivity . . .
  - effective models in nuclear physics
  - ultra-cold atoms



$\Gamma_{k=\Lambda} = S$
$\Gamma_{k=0} = \Gamma$
Theory space

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Scale-dependent functionals

- generating functional of (Euclidean) correlation functions

$$Z[j] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]+(j,\phi)}, \quad (j,\phi) = \int \mathrm{d}^d x \, j(x)\phi(x)$$

- Schwinger functional $W[j] = \log Z[j] \rightarrow$ connected correlation functions
- effective action = Legendre transform of $W[j]$

$$\Gamma[\varphi] = (j,\varphi) - W[j] \quad \text{with} \quad \varphi(x) = \frac{\delta W[j]}{\delta j(x)} \tag{1}$$

$\rightarrow$ one-particle irreducible correlation functions

- last equation in (1) $\rightarrow j[\varphi]$, insert into first equation in (1)
- Γ: all properties of QFT in a most economic way

- add scale-dependent IR-cutoff term ΔS_k to classical action in functional integral $\to$ scale-dependent generating functional

$$Z_k[j] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]+(j,\phi)-\Delta S_k[\phi]}$$

- Scale-dependent Schwinger functional

$$W_k[j] = \log Z_k[j] \tag{2}$$

- regulator: quadratic functional with a momentum-dependent mass,

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \phi^*(p) R_k(p)\phi(p) \equiv \frac{1}{2}\int_p \phi^*(p) R_k(p)\phi(p)\ ,$$

$\to$ one-loop structure of flow equation

conditions on cutoff function $R_k(p)$

- should recover effective action for $k \to 0$:

$$R_k(p) \stackrel{k\to 0}{\longrightarrow} 0 \quad \text{for fixed} \quad p$$

- should recover classical action at UV-scale Λ:

$$R_k \stackrel{k\to\Lambda}{\longrightarrow} \infty$$

- regularization in the IR:

$$R_k(p) > 0 \quad \text{for} \quad p \to 0$$

possible cut-offs are

the exponential regulator: $R_k(p) = \dfrac{p^2}{\mathrm{e}^{p^2/k^2} - 1}$,

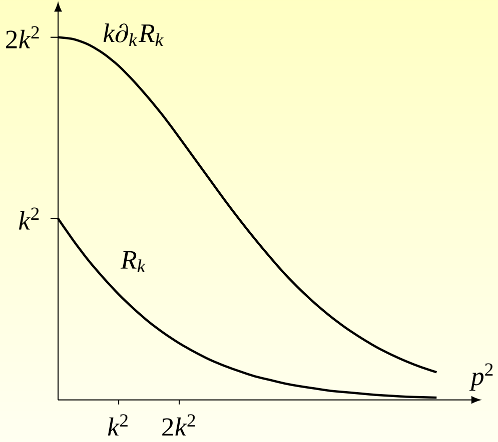
the optimized regulator: $R_k(p) = (k^2 - p^2)\,\theta\left(k^2 - p^2\right)$,

the quartic regulator: $R_k(p) = k^4/p^2$,

the sharp regulator: $R_k(p) = \dfrac{p^2}{\theta\left(k^2 - p^2\right)} - p^2$,

the Callan-Symanzik regulator: $R_k(p) = k^2$

exponential cutoff function and its derivative



Polchinski equation

- partial derivative of W_k in (2) is given by

$$\partial_k W_k[j] = -\frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \ \langle \phi(x)\partial_k R_k(x,y)\phi(y)\rangle_k$$

- relates to connected two-point function

$$G_k^{(2)}(x,y) \equiv \frac{\delta^2 W_k[j]}{\delta j(x)\delta j(y)} = \langle\phi(x)\phi(y)\rangle_k - \varphi(x)\varphi(y)$$

- Polchinski equation

$$\begin{aligned}\partial_k W_k[j] &= -\frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y\, \partial_k R_k(x,y) G_k^{(2)}(y,x) - \partial_k \Delta S_k[\varphi] \\ &= -\frac{1}{2}\mathrm{tr}\left(\partial_k R_k\, G_k^{(2)}\right) - \partial_k \Delta S_k[\varphi]\end{aligned}$$

Scale dependent effective action

- average field of the cutoff theory with j

$$\varphi(x) = \frac{\delta W_k[j]}{\delta j(x)} \tag{3}$$

- fixed source $\rightarrow$ average field depends on cutoff
- fixed average field $\rightarrow$ source depends on cutoff
- modified Legendre transformation:

$$\Gamma_k[\varphi] = (j, \varphi) - W_k[j] - \Delta S_k[\varphi] \tag{4}$$

- solve (3) for $j = j[\varphi] \rightarrow$ use solution in (4)
- Γ_k not Legendre transform of $W_k[j]$ for $k > 0$!
- Γ_k need not to be convex, but $\Gamma_{k\to 0}$ is convex

Derivation of Wetterich equation

- vary effective average action

$$\frac{\delta\Gamma_k}{\delta\varphi(x)} = \int \frac{\delta j(y)}{\delta\varphi(x)}\varphi(y) + j(x) - \int \frac{\delta W_k[j]}{\delta j(y)}\frac{\delta j(y)}{\delta\varphi(x)} - \frac{\delta \Delta S_k[\varphi]}{\delta\varphi(x)}$$

- terms chancel $\rightarrow$ effective equation of motion

$$\frac{\delta\Gamma_k}{\delta\varphi(x)} = j(x) - \frac{\delta}{\delta\varphi(x)}\Delta S_k[\varphi] = j(x) - (R_k\varphi)(x)$$

- flow equation: φ fixed, j depends on scale, differentiate Γ_k

$$\partial_k\Gamma_k = \int \mathrm{d}^d x\, \partial_k j(x)\varphi(x) - \partial_k W_k[j] - \int \frac{\partial W_k[j]}{\partial j(x)}\partial_k j(x) - \partial_k \Delta S_k[\varphi]$$

- two red contributions cancel
- $\partial_k W_k[j]$: only scale dependence of the parameters

$$\begin{aligned}\partial_k \Gamma_k &= -\partial_k W_k[j] - \partial_k \Delta S_k[\varphi] \\ &= -\partial_k W_k[j] - \frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \, \varphi(x) \partial_k R_k(x,y) \varphi(y)\end{aligned}$$

- use Polchinski equation $\rightarrow$

$$\partial_k \Gamma_k = \frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \, \partial_k R_k(x,y) \, G_k^{(2)}(y,x) \tag{5}$$

second derivative of W_k vs. second derivative of Γ_k:

$$\varphi(x) = \frac{\delta W_k[j]}{\delta j(x)} \quad \text{and} \quad j(x) = \frac{\delta \Gamma_k}{\delta \varphi(x)} + \int \mathrm{d}^d y \, R_k(x,y)\varphi(y)$$

- chain rule $\to$

$$\delta(x-y) = \int \mathrm{d}^d z \, \frac{\delta\varphi(x)}{\delta j(z)} \frac{\delta j(z)}{\delta\varphi(y)} = \int \mathrm{d}^d z \, G_k^{(2)}(x,z) \left\{ \Gamma_k^{(2)} + R_k \right\}(z,y)$$

- Hence

$$G_k^{(2)} = \frac{1}{\Gamma_k^{(2)} + R_k}, \qquad \Gamma_k^{(2)}(x,y) = \frac{\delta^2 \Gamma_k}{\delta\varphi(x)\delta\varphi(y)}$$

- insert into (5) $\to$ Wetterich equation

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{tr} \left(\frac{\partial_k R_k}{\Gamma_k^{(2)}[\varphi] + R_k} \right) \tag{6}$$

- non-linear functional integro-differential equation
- full propagator enters flow equation
- Polchinski and Wetterich equations = exact FRG equations
- Polchinski: simple polynomial structure
  favored in structural investigations
- Wetterich: second derivative in the denominator
  stabilizes flow in (numerical) solution
  mainly used in explicit calculations.
- in practice: truncation = projection onto finite-dim. space
- difficult: error estimate for flow
  $\rightarrow$ improve truncation, optimize regulator, check stability

Quadratic action

- at the cutoff

$$\Gamma_\Lambda[\varphi] = \frac{1}{2}\int \mathrm{d}^d x\, \varphi(-\Delta + m_\Lambda^2)\varphi\ ,$$

- solution of the FRG-equation

$$\Gamma_k[\varphi] = \Gamma_\Lambda[\varphi] + \frac{1}{2}\log\det\left(\frac{-\Delta + m_\Lambda^2 + R_k}{-\Delta + m_\Lambda^2 + R_\Lambda}\right) \tag{7}$$

- last term for optimized cutoff

3d: $$\frac{1}{6\pi^2}\left(m_\Lambda^3 \arctan\frac{m_\Lambda(k-\Lambda)}{m_\Lambda^2 + k\Lambda} + m_\Lambda^2(\Lambda - k) + \frac{k^3}{3} - \frac{\Lambda^3}{3}\right)\ ,$$

4d: $$\frac{1}{64\pi^2}\left(m_\Lambda^4 \log\frac{m_\Lambda^2 + k^2}{m_\Lambda^2 + \Lambda^2} + m_\Lambda^2\left(\Lambda^2 - k^2\right) + \frac{k^4}{2} - \frac{\Lambda^4}{2}\right)$$

Functional renormalization in QM

- anharmonic oscillator

$$S[q] = \int \mathrm{d}\tau \left(\frac{1}{2}\dot{q}^2 + V(q)\right) ,$$

- here LPA (local potential approximation)

$$\Gamma_k[q] = \int \mathrm{d}\tau \left(\frac{1}{2}\dot{q}^2 + u_k(q)\right) \tag{8}$$

- low-energy approximation
  leading order in gradient expansion
- scale-dependent effective potential u_k
- neglected: higher derivative terms, mixed terms $q^n \dot{q}^m$

- flow equation contains $\Gamma_k^{(2)} = -\partial_\tau^2 + u_k''(q)$
- LPA: sufficient to consider a constant $q \to$ momentum space

$$\int \mathrm{d}\tau \, \frac{\partial u_k(q)}{\partial k} = \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}\tau' \frac{\partial R_k}{\partial k}(\tau - \tau') \frac{1}{-\partial_\tau^2 + u_k''(q) + R_k}(\tau' - \tau)$$
$$= \frac{1}{2} \int \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\partial_k R_k(p)}{p^2 + u_k''(q) + R_k(p)}$$

- choose optimal regulator function

$$R_k(p) = (k^2 - p^2)\,\theta\left(k^2 - p^2\right) \Longrightarrow \partial_k R_k(p) = 2k\theta(k^2 - p^2)$$

- non-linear partial differential equation for u_k:

$$\partial_k u_k(q) = \frac{1}{\pi} \frac{k^2}{k^2 + u_k''(q)}$$

- minimum of $u_k(q)$ not ground state energy
  differs by q-independent contribution
- free particle limit fixes subtraction in flow equation

$$\partial_k u_k(q) = \frac{1}{\pi}\left(\frac{k^2}{k^2+u_k''(q)} - 1\right) = -\frac{1}{\pi}\frac{u_k''(q)}{k^2+u_k''(q)} \tag{9}$$

- assume $u_\Lambda(q)$ even $\to$ $u_k(q)$ even
- polynomial ansatz

$$u_k(q) = \sum_{n=0,1,2\dots} \frac{1}{(2n)!}\, a_{2n}(k)\, q^{2n}\ ,$$

- scale-dependent couplings a_{2n}
- Insert into (9), compare coefficients of powers of q^2
- $\to$ infinite set of coupled ode's

$$\frac{\mathrm{d}a_0}{\mathrm{d}k} = -\frac{1}{\pi} a_2 \Delta_0, \qquad \Delta_0 = \frac{1}{k^2 + a_2} ,$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}k} = -\frac{k^2}{\pi} a_4 \Delta_0^2 ,$$

$$\frac{\mathrm{d}a_4}{\mathrm{d}k} = -\frac{k^2 \Delta_0^2}{\pi} \left(a_6 - 6 a_4^2 \Delta_0 \right) ,$$

$$\frac{\mathrm{d}a_6}{\mathrm{d}k} = -\frac{k^2 \Delta_0^2}{\pi} \left(a_8 - 30 a_4 a_6 \Delta_0 + 90 a_4^3 \Delta_0^2 \right) ,$$

$$\vdots$$

- initial condition: a_{2n} at cutoff = parameters in classical potential
- projection onto space of polynomials up to given degree n

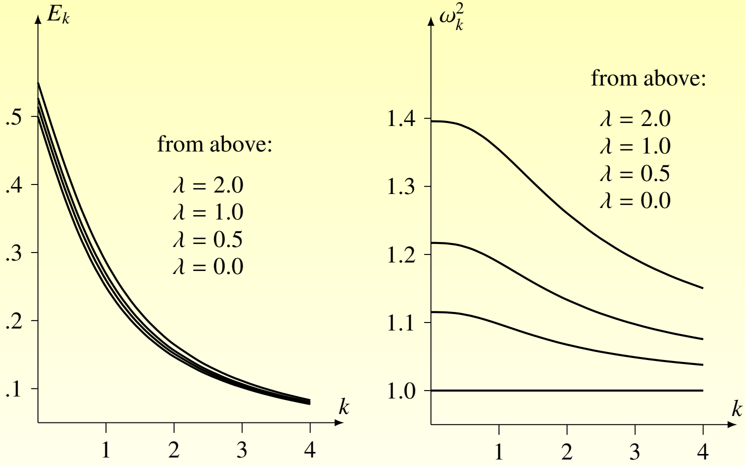
- e.g. crude truncation $a_6 = a_8 = \cdots = 0$: finite set of ode's
- use standard notation

$$a_0 = E, \quad a_2 = \omega^2 \quad \text{and} \quad a_4 = \lambda\,,$$

- $\Rightarrow$ truncated system of flow equations

$$\frac{\mathrm{d}E_k}{\mathrm{d}k} = -\frac{\omega_k^2}{\pi}\Delta_0, \quad \frac{\mathrm{d}\omega_k^2}{\mathrm{d}k} = -\frac{k^2\lambda_k}{\pi}\Delta_0^2, \quad \frac{\mathrm{d}\lambda_k}{\mathrm{d}k} = \frac{6k^2\lambda_k^2}{\pi}\Delta_0^3$$

- solve numerically (eg. with `octave`)
- initial conditions $E_\Lambda = 0,\ \omega_\Lambda = 1$, varying λ at the cutoff scale
- $\to$ scale-dependent couplings E_k and ω_k^2
- hardly change for $k \gg \omega$
- variation near typical scale $k \approx \omega$



The flow of the couplings E_k and ω_k^2 ($E_\Lambda = 0$, $\omega_\Lambda^2 = 1$).

- $\omega = \omega_{k=0} > 0 \ \Rightarrow$ effective potential minimal at origin
- ground state energy: $E_0 = \min(u_{k=0})$
- energy of *first excited state*

$$E_1 = E_0 + \sqrt{u''_{k=0}(0)} = E_0 + \omega_{k=0}$$

- already good results with simple truncation

energies for different λ
different truncations und regulators
units of $\hbar\omega$

	ground state energy				energy of first excited state			
cutoff	optimal order 4	optimal order 12	Callan order 4	exact result	optimal order 4	optimal order 12	Callan order 4	exact result
$\lambda = 0$	0.5000	0.5000	0.5000	0.5000	1.5000	1.5000	1.5000	1.5000
$\lambda = 1$	0.5277	0.5277	0.5276	0.5277	1.6311	1.6315	1.6307	1.6313
$\lambda = 2$	0.5506	0.5507	0.5504	0.5508	1.7324	1.7341	1.7314	1.7335
$\lambda = 3$	0.5706	0.5708	0.5703	0.5710	1.8177	1.8207	1.8159	1.8197
$\lambda = 4$	0.5885	0.5889	0.5882	0.5891	1.8923	1.8968	1.8898	1.8955
$\lambda = 5$	0.6049	0.6054	0.6045	0.6056	1.9593	1.9652	1.9562	1.9637
$\lambda = 6$	0.6201	0.6207	0.6196	0.6209	2.0205	2.0278	2.0168	2.0260
$\lambda = 7$	0.6343	0.6350	0.6336	0.6352	2.0771	2.0857	2.0728	2.0836
$\lambda = 8$	0.6476	0.6484	0.6469	0.6487	2.1299	2.1397	2.1250	2.1374
$\lambda = 9$	0.6602	0.6611	0.6594	0.6614	2.1794	2.1905	2.1741	2.1879
$\lambda = 10$	0.6721	0.6732	0.6713	0.6735	2.2263	2.2385	2.2205	2.2357
$\lambda = 20$	0.7694	0.7714	0.7679	0.7719	2.5994	2.6209	2.5898	2.6166

Recall flow equation in LPA:

$$\partial_k u_k(q) = -\frac{1}{\pi}\frac{u_k''(q)}{k^2 + u_k''(q)}$$

- negative ω^2 in V: local maximum at 0 and two minima
- denominator minimal where u_k'' minimal (maximum of u_k)
- denominator positive for large scales

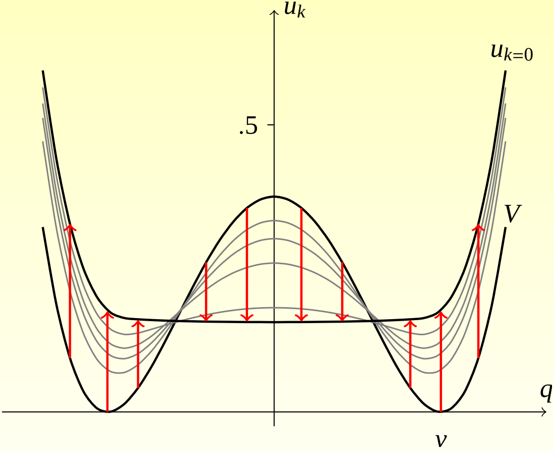
  $\Rightarrow$ denominator remains positive during the flow
- flow equation $\Rightarrow$

  $u_k(q)$ increases toward infrared if $u_k''(q)$ is positive

  $u_k(q)$ decreases toward infrared if $u_k''(q)$ is negative

  $\Rightarrow$ double-well potential flattens during flow, becomes convex
- convexity expected on general grounds

solution of partial differential equation, $\omega^2 = -1$, $\lambda = 1$



- energies of ground state and first excited state:
  less good, less stable
- fourth-order polynomials $\rightarrow$ inaccurate results for weak couplings
- numerical solution of the flow equation does better
- decreasing λ (increasing barrier) $\rightarrow$ increasingly difficult
- to detect splitting induced by instanton effects:
  must go beyond leading order LPA

energies for $\omega^2 = -1$ and varying λ
optimized regulator, units of $\hbar\omega$

	ground state energy				energy of first excited state			
	optimal order 4	optimal order 12	pde	exact	optimal order 4	optimal order 12	pde	exact
$\lambda = 1$			-0.8732	-0.8556			-0.7887	-0.8299
$\lambda = 2$		-0.2474	-0.2479	-0.2422		0.0049	0.0063	-0.0216
$\lambda = 3$	0.2473	-0.0681	-0.0679	-0.0652	-0.2241	0.3514	0.3500	0.3307
$\lambda = 4$	-0.0186	0.0286	0.0290	0.0308	0.3511	0.5753	0.5755	0.5598
$\lambda = 5$	0.0654	0.0949	0.0953	0.0967	0.5835	0.7455	0.7462	0.7324
$\lambda = 6$	0.1234	0.1457	0.1461	0.1472	0.7509	0.8842	0.8851	0.8723
$\lambda = 7$	0.1688	0.1871	0.1876	0.1885	0.8851	1.0021	1.0030	0.9909
$\lambda = 8$	0.2063	0.2223	0.2228	0.2236	0.9987	1.1052	1.1061	1.0944
$\lambda = 9$	0.2671	0.2530	0.2535	0.2543	1.1863	1.1972	1.1981	1.1866
$\lambda = 10$	0.2386	0.2803	0.2808	0.2816	1.0978	1.2805	1.2814	1.2701
$\lambda = 20$	0.4536	0.4632	0.4639	0.4643	1.7866	1.8638	1.8648	1.8538

Scalar Field Theory

- QM = 1-dimensional field theory
- Now: Euclidean scalar field theory in d dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)$$

- first local potential approximation

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\partial_\mu \varphi)^2 + u_k(\varphi) \right)$$

- second functional derivative: $\Gamma_k^{(2)} = -\Delta + u_k''(\varphi)$
- flow of effective potential: may assume constant average field

$$\partial_k u_k(q) = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\partial_k R_k(p)}{p^2 + u_k''(q) + R_k(p)} \tag{10}$$

- optimized regulator:
  $\Rightarrow$ volume of the d-dimensional ball divided by $(2\pi)^d$,

$$\mu_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$$

- p-integration can be done $\rightarrow$ flow equation

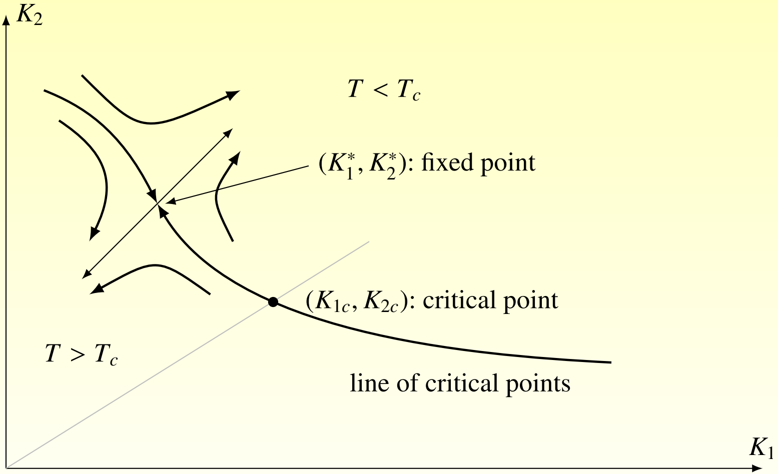
$$\partial_k u_k(\varphi) = \mu_d \frac{k^{d+1}}{k^2 + u_k''(\varphi)} , \qquad (11)$$

- dimensions enters via k^{d+1} and μ_d
- nonlinear partial differential equation
- polynomial ansatz for even potential

- flow equations for infinite set of couplings

$$
\begin{aligned}
k\frac{\mathrm{d}a_0}{\mathrm{d}k} &= +\mu_d k^{d+2}\Delta_0, \qquad \Delta_0 = \frac{1}{k^2+a_2}\ , \\
k\frac{\mathrm{d}a_2}{\mathrm{d}k} &= -\mu_d k^{d+2}\Delta_0^2 a_4\ , \\
k\frac{\mathrm{d}a_4}{\mathrm{d}k} &= -\mu_d k^{d+2}\Delta_0^2\left(a_6-6a_4^2\Delta_0\right)\ , \\
k\frac{\mathrm{d}a_6}{\mathrm{d}k} &= -\mu_d k^{d+2}\Delta_0^2\left(a_8-30a_4a_6\Delta_0+90a_4^3\Delta_0^2\right)\ , \\
\vdots \qquad & \quad \vdots
\end{aligned}
$$

Fixed points



- critical hyper-surface on which $\xi = \infty$
- RG trajectory moves away from critical surface
- If flow begins on critical surface $\rightarrow$ stays on surface
- most critical points are not fixed point
- $d \geq 3$: expect a finite set of isolated fixed points
- fixed point $K^* = (K_1^*, K_2^*, \dots)$
- RG flow in the vicinity of fixed point $K = K^* + \delta K$
- linearize flow around fixed point

$$K_i' = K_i^* + \delta K_i' = R_i\big(K_j^* + \delta K_j\big) = K_i^* + \frac{\partial R_i}{\delta K_j}\Big|_{K^*} \delta K_j + O(\delta K^2)$$

- linearized RG transformation,

$$\delta K'_i = \sum_j M_i^{\ j} \delta K_j, \qquad M_i^{\ j} = \frac{\partial R_i}{\partial K_j}\Big|_{K^*}$$

- eigenvalues and left-eigenvectors Φ_α of matrix M

$$\sum_j \Phi_\alpha^j M_j^{\ i} = \lambda_\alpha \Phi_\alpha^i = b^{y_\alpha} \Phi_\alpha^i$$

- subset of $\{\Phi_\alpha\}$ span space tangential to critical surface at K^*
- every λ_α defines a critical exponent y_α
- consider the new variables

$$g_\alpha = \sum_i \Phi_\alpha^i \delta K_i \ .$$

- We have

$$g'_\alpha = \sum_i \Phi^i_\alpha \delta K'_i = \sum_{ij} \Phi^i_\alpha M_i^j \delta K_j = \sum_j b^{y_\alpha} \Phi^j_\alpha \delta K_j = b^{y_\alpha} g_\alpha \ . \quad (12)$$

- $y_\alpha > 0$: deviation g_α increases, flow moves point $K^* + g_\alpha$ away from the fixed point $K^* \to$ relevant perturbation
- $y_\alpha < 0$: deviation g_α decreases, flow carries point $K^* + g_\alpha$ towards the fixed point $K^* \to$ irrelevant perturbation
- $y_\alpha = 0$: marginal coupling
- relevant couplings determine important scaling laws
- all TD critical exponent functions of relevant exponents
- relevant couplings and exponents determine IR-physics

Fixed point analysis for scalar models

- introduce the dimensionless field and potential,

$$\varphi = k^{(d-2)/2}\sqrt{\mu_d}\,\chi \quad \text{and} \quad u_k(\varphi) = k^d \mu_d v_k(\chi)$$

- flow equation in terms of dimensionless quantities

$$k\partial_k v_k + d v_k - \frac{d-2}{2}\chi v_k' = \frac{1}{1+v_k''}, \quad v_k' = \frac{\partial v_k}{\partial \chi}\dots$$

- at a fixed point: $\partial_k v_k = 0 \Rightarrow$
- fixed point equation for effective potential (ode)

$$d v_* - \frac{d-2}{2}\chi v_*' = \frac{1}{1+v_*''}$$

- constant solution $dv_* = 1 \to$ trivial Gaussian fixed point
- are there non-Gaussian fixed points?
- answer depends on the dimension d of spacetime
- even classical potential $\to v_k$ even as well:

$$v_k(\chi) = w_k(\varrho), \quad \text{with} \quad \varrho = \frac{\chi^2}{2}$$

- flow equation for $w_k(\varrho)$

$$k\partial_k w_k(\varrho) + dw_k(\varrho) - (d-2)\,\varrho w_k'(\varrho) = \frac{1}{1 + w_k'(\varrho) + 2\varrho w_k''(\varrho)}$$

- fixed point equation

$$dw_*(\varrho) - (d-2)\,\varrho w_*'(\varrho) = \frac{1}{1 + w_*'(\varrho) + 2\varrho w_*''(\varrho)}$$

- 2d theories: ∞ many fixed-point solutions [Morris 1994]
- also true for 2d Yukawa theories [Synatschke et al.]
- polynomial truncation to order m:

$$w^{(m)} = \sum_{n=0}^{m} c_n \, \varrho^n$$

- flow equation for couplings

$$\begin{aligned}
k\partial_k c_0 &= -d c_0 + \Delta_0, \qquad \Delta_0 = (1+c_1)^{-1}\,, \\
k\partial_k c_1 &= -2c_1 - 6c_2\Delta_0^2\,, \\
k\partial_k c_2 &= (d-4)c_2 - 15c_3\Delta_0^2 + 36c_2^2\Delta_0^3 \\
k\partial_k c_3 &= (2d-6)c_3 - 28c_4\Delta_0^2 + 180c_2c_3\Delta_0^3 - 216c_2^3\Delta_0^4\,, \\
k\partial_k c_4 &= (3d-8)c_4 - 45c_5\Delta_0^2 + (336c_2c_4 + 225c_3^2)\Delta_0^3 \\
&\quad - 1620c_2^2c_3\Delta_0^4 + 1296c_2^4\Delta_0^5 \\
&\;\;\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

Scalar fields in three dimensions

- expect nontrivial fixed point in $d = 3$
- first: polynomial truncation $\Rightarrow$ set $c_k = 0$ for $k > m$
- insert into above system of equations with lhs = 0

  $\Rightarrow$ m algebraic equations for the $m+1$ fixed-point couplings

$$0 = f_0(c_0^*, c_1^*) = f_1(c_1^*, c_2^*) = \cdots = f_{m-1}(c_1^*, \dots, c_m^*)$$

- polynomials in $c_0^*, c_2^*, \dots, c_m^*$ and $\Delta_0 = 1/(1 + c_1^*)$
- prescribe c_1^* (= slope at origin) and thus Δ_0
- solve the system for $c_0^*, c_2^*, c_3^*, \dots, c_m^*$ in terms of c_1^*
- algebraic computer program $\rightarrow$ solution for m up to 42

- explicit expression for the lowest fixed-point couplings

$$
\begin{aligned}
c_0^* &= \frac{1}{3}\frac{1}{1+c_1^*}\\
c_2^* &= -\frac{c_1^*(1+c_1^*)^2}{3}\\
c_3^* &= \frac{c_1^*(1+c_1^*)^3(1+13c_1^*)}{45}\\
c_4^* &= -\frac{c_1^{*2}(1+c_1^*)^4(1+7c_1^*)}{21}\,,\\
&\vdots\\
c_m^* &= c_1^{*2}(1+c_1^*)^m P_{m-3}(c_1^*)\,,
\end{aligned}
$$

- P_k polynomial of order k

- trivial solution (Gaussian fixed point $w'_* = 1$)

$$c_0^* = \frac{1}{3}, \quad 0 = c_2^* = c_3^* = c_4^* = \dots$$

- search for other fixed points:
- set $c_m^* = 0 \to P_{m-3}(c_1^*) = 0$
- polynomials P_k has many real roots c_1^*
- for each m choose c_1^* such that convergence for large m
- the approximating polynomials converge to a power series with maximal radius of convergence
- example:
  $m = 20 \Rightarrow c_1^* = -.186066$
  $m = 42 \Rightarrow c_1^* = -.186041$
- calculate other $c_k^* \Rightarrow$ fixed point solution

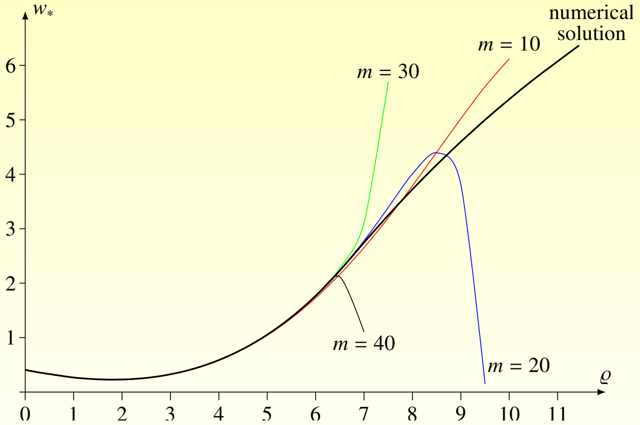
- With n! multiplied fixed-point coefficients c_n^*

	c_0^*	c_1^*	c_2^*	c_3^*	c_4^*	c_5^*	c_6^*
$m = 20$	0.409534	-0.186066	0.082178	0.018981	0.005253	0.001104	-0.000255
$m = 42$	0.409533	-0.186064	0.082177	0.018980	0.005252	0.001104	-0.000256
	c_7^*	c_8^*	c_9^*	c_{10}^*	c_{11}^*	c_{12}^*	c_{13}^*
$m = 20$	-0.000526	-0.000263	0.000237	0.000632	0.000438	-0.000779	-0.002583
$m = 42$	-0.000526	-0.000263	0.000236	0.000629	0.000431	-0.000799	-0.002643
	c_{14}^*	c_{15}^*	c_{16}^*	c_{17}^*	c_{18}^*	c_{19}^*	c_{20}^*
$m = 20$	-0.002029	0.007305	0.028778	0.034696	-0.077525	-0.381385	0.000000
$m = 42$	-0.002216	0.006677	0.026544	0.026320	-0.110498	-0.517445	-0.587152

- c_k^* stable when one increases polynomial order m ($m \gtrsim 2k$)

Polynomial approximations vs. numerical solution

numerics: shooting method with seventh-order Runge-Kutta



- fine-tune slope at origin $\rightarrow w_*'(0) \approx -0.186064249376$
- Polynomial of degree 42 $\rightarrow w_*'(0) \approx -0.186064279993$

Critical exponents

- flow equation in the vicinity of fixed-point solution w_*
- set $w_k = w_* + \delta_k$, linearize the flow in small δ_k
- $\rightarrow$ linear differential equation for the small fluctuations

$$k\partial_k\delta_k = -\,d\delta_k + (d-2)\,\varrho\delta_k' - \left(dw_* - (d-2)\,\varrho w_*'\right)^2 \left(\delta_k' + 2\varrho\delta_k''\right)$$

- insert the polynomial approximation for fixed-point solution
- polynomial ansatz for the perturbation $\rightarrow$

$$\delta_k(\varrho) = \sum_{n=0}^{m-1} d_n \varrho^n \quad \varrho = \frac{\chi^2}{2}$$

- linear system for the coefficients d_m

$$k\partial_k \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{m-1} \end{pmatrix} = M\,(c_0^*) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{m-1} \end{pmatrix}$$

- critical exponents = eigenvalues of m-dimensional matrix M
- $\rightarrow$ up to order $m = 46$ with algebraic computer program

m	$\nu = -1/\omega_1$	ω_2	ω_3	ω_4	ω_5
10	0.648617	0.658053	2.985880	7.502130	17.913494
14	0.649655	0.652391	3.232549	5.733445	9.324858
18	0.649572	0.656475	3.186784	5.853987	9.141093
22	0.649554	0.655804	3.170538	5.977066	8.522811
26	0.649564	0.655629	3.182910	5.897290	8.844632
30	0.649562	0.655791	3.180847	5.903039	8.907607
34	0.649561	0.655749	3.178636	5.922910	8.702583
38	0.649562	0.655731	3.180577	5.908885	8.814225
42	0.649562	0.655755	3.180216	5.909910	8.847386
46	0.649562	0.655746	3.179541	5.915754	8.738608

- convergence
- two negative exponents $\omega_0 = -3$ and $\omega_1 = -1/\nu$
- ω_0 ground state energy, unrelated to critical behavior
- $\omega_2, \omega_3, \omega_4, \dots$ all positive (irrelevant)
- LPA-prediction: $\nu = 0.649562$ (high-T expansion: $\nu = 0.630$)

Wave function renormalization

- next-to-leading in derivative expansion $\to$ wave function renormalization $Z_k(p, \varphi)$
- difficult non-linear parabolic partial differential RG-equations
- first step: neglect field and momentum dependence $\to$

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2} Z_k \, (\partial_\mu \varphi)^2 + u_k(\varphi) \right) .$$

- second functional derivative $\Gamma_k^{(2)} = -Z_k \Delta + u_k''(\varphi)$
- flow equation (simplification for $R_k \to Z_k R_k$):

$$\int \mathrm{d}^d x \left(\frac{1}{2} (\partial_k Z_k) \, (\partial_\mu \varphi)^2 + \partial_k u_k(\varphi) \right) = \frac{1}{2} \mathrm{tr} \left(\frac{\partial_k (Z_k R_k)}{Z_k (p^2 + R_k) + u_k''(\varphi)} \right)$$

- simple: flow of effective potential:

$$\partial_k u_k = \frac{\mathcal{Z}_k}{Z_k k^2 + u_k''}, \quad \mathcal{Z}_k = \frac{\mu_d}{d+2}\,\partial_k\left(k^{d+2} Z_k\right) .$$

- more difficult: flow of Z_k
- project flow on operator $(\partial\phi)^2$
- must admit non-homogeneous fields $\to [p^2, u_k''(\varphi)] \neq 0$
- final answer

$$k\partial_k Z_k = -\mu_d\, k^{d+2}\left(Z_k a_3 \Delta_0^2\right)^2, \quad \Delta_0 = \frac{1}{Z_k k^2 + a_2}$$

see A. Wipf, Lecture Notes in Physics 864

- anomalous dimension

$$\eta = -k\partial_k \log Z_t$$

Linear O(N) models

- scalar field $\phi \in \mathbb{R}^N$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)$$

- $O(N)$ invariant potential
- fixed-point analysis: dimensionless quantities χ and ν_k
- invariant dimensionless composite field

$$\varrho = \frac{1}{2}\sum_{i=1}^{N} \chi_i^2$$

- set $\nu_k(\chi) = w_k(\varrho)$

- flow equation in LPA (optimized regulator)

$$k\partial_k w_k + dw_k - (d-2)\,\varrho w_k' = \frac{N-1}{1+w_k'} + \frac{1}{1+w_k'+2\varrho w_k''}$$

- contribution of the $N-1$ Goldstone modes
- contribution of massive radial mode
- large N: Goldstone modes give dominant contribution
- linearize about fixed-point solution: $w_k = w_* + \delta_k$
- fluctuation δ_k obeys the linear differential equation

$$k\partial_k\delta_k = -d\delta_k + (d-2)\,\varrho\delta_k' - \frac{(N-1)\,\delta_k'}{(1+w_*')^2} - \frac{\delta_k' + 2\varrho\delta_k''}{(1+w_*'+2\varrho w_*'')^2}$$

- proceed as before: polynomial truncation to high order (40)
  $\rightarrow$ slope at origin of fixed-point solution
- find always Wilson-Fisher fixed point
- eigenvalue $\omega_0 = -3$ of the scaling operator 1 not listed

N	1	2	3	100	1000
$-w'_*(0)$	0.186064	0.230186	0.263517	0.384172	0.387935
$\nu = -1/\omega_1$	0.64956	0.70821	0.76113	0.99187	0.99923
ω_2	0.6556	0.6713	0.6990	0.97218	0.99844
ω_3	3.1798	3.0710	3.0039	2.98292	2.99554

- extract asymptotic formulas

$$w'_*(0) \approx -0.3881 + \frac{0.4096}{N}, \quad \nu \approx 0.9998 - \frac{0.9616}{N}$$

Large N Limit

- rather simple flow equation ($t = \log(k/\Lambda) \Rightarrow \partial_t = k\partial_k$)

$$\partial_t w_k = (d-2)\,\varrho w_k' - d w_k + \frac{N}{1+w_k'}$$

$$\Rightarrow \partial_t w' = (d-2)\,\varrho w'' - 2w' - \frac{N}{(1+w')^2} w''$$

- can be solved exactly with methods of characteristics
- analytic relation between fixed point solution and perturbation in

$$s(t,\rho) \approx w_*(\rho) + \mathrm{e}^{\omega t}\delta(\rho)$$

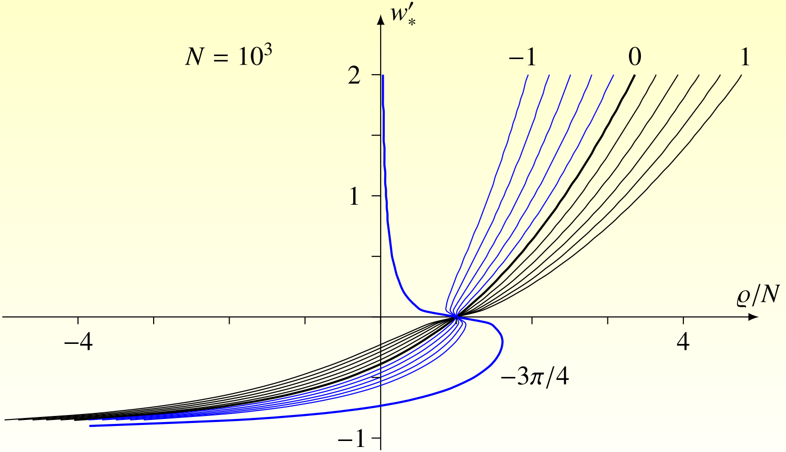
- result:

$$w(t,\varrho) \approx w_*(\varrho) + \text{const} \times \mathrm{e}^{\omega t}\, w_*'(\varrho)^{(\omega+d)/2}\ .$$

- if perturbation regular → all critical exponents

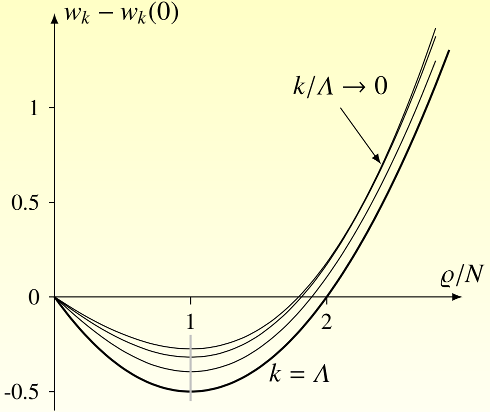
$$\omega \in \{2n - d \mid n = 0, 1, 2, \dots\}$$

- one-parameter family of fixed point solutions



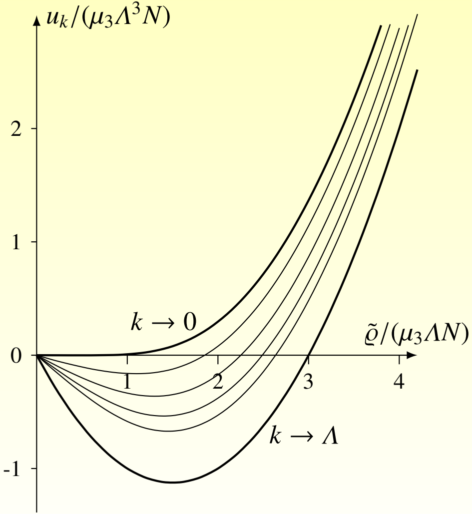
flow of dimensionless eff. potential for critical κ

$\lambda_\Lambda = 1,\ \kappa_\Lambda = \kappa_{\rm crit}$ (κ_Λ: dimensionless minimum of V)



flow of dimensionful eff. potential above critical κ

$\lambda_\Lambda = 1,\ \kappa_\Lambda = 1.3\kappa_{\rm crit} \ \Rightarrow$ broken phase



flow of dimensionful eff. potential below critical κ

$\lambda_\Lambda = 1,\ \kappa_\Lambda = 0.5\kappa_{\text{crit}}\ \Rightarrow$ symmetric phase