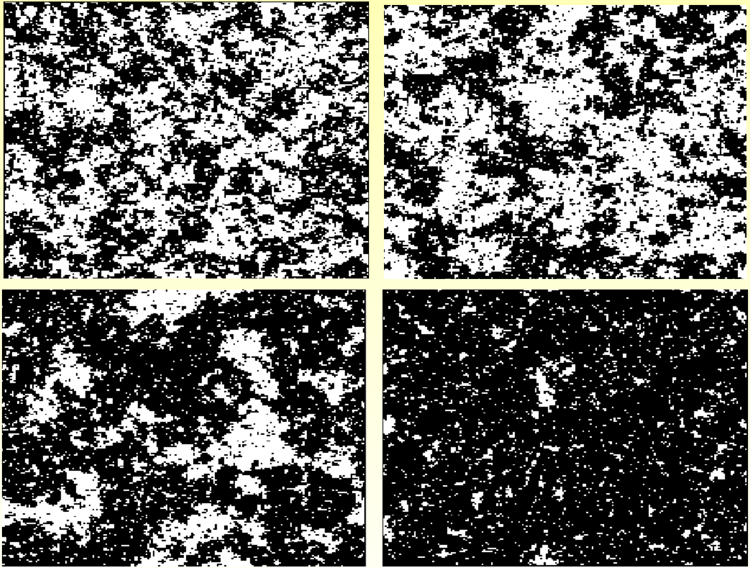
The Functional Renormalization Group Method – An Introduction

A. Wipf

Theoretisch-Physikalisches Institut, FSU Jena

Juiz de Fora

30. August 2013

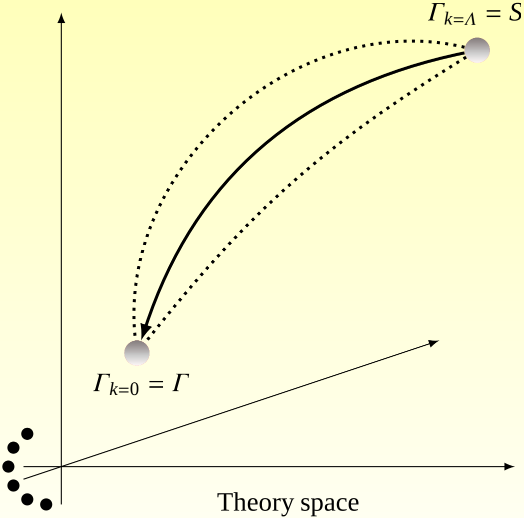


9

Introduction

- implementation of the renormalization group concept
- for continuum field theory in momentum-space
- functional methods + renormalization group idea
- known microscopic laws $\longrightarrow$ complex macroscopic phenomena
- scale-dependent Schwinger functional or effective action
- conceptionally simple, technically demanding flow equations
- scale parameter k = adjustable screw of microscope
- large values of a momentum scale k: high resolution
- lowering k: decreasing resolution of the microscope
- flow microscopic $\to$ macroscopic scales
- non-perturbative

- flow of Schwinger functional $W_k[j]$: Polchinski equation
- flow of effective action $\Gamma_k[\varphi]$: Wetterich equation
- flow from classical action $S[\varphi]$ to effective action $\Gamma[\varphi]$
- applied to many physical systems
  - strong interaction
  - electroweak phase transition
  - asymptotic safety scenario
  - condensed matter theory to a unified description
  - e.g. Hubbard model, liquid He^4, frustrated magnets, superconductivity . . .
  - effective models in nuclear physics
  - ultra-cold atoms


$\Gamma_{k=\Lambda} = S$
$\Gamma_{k=0} = \Gamma$
Theory space

1. K. Aoki, *Introduction to the nonperturbative renormalization group and its recent applications*, Int. J. Mod. Phys. **B14** (2000) 1249.
2. C. Bagnus and C. Bervillier, *Exact renormalization group equations: An introductiory review*, Phys. Rept. **348** (2001) 91.
3. J. Berges, N. Tetradis and C. Wetterich, *Nonperturbative renormalization flow in quantum field theory and statistical physics*, Phys. Rept. **363** (2002) 223.
4. J. Polonyi, *Lectures on the functional renormalization group methods*, Central Eur. J. Phys. **1** (2003) 1.
5. J. Pawlowski, *Aspects of the functional renormalisation group*, Annals Phys. **322** (2007) 2831.

1. H. Gies, *Introduction to the functional RG and applications to gauge theories*,
Lect. Notes Phys. **162**, Renormalization group and effective field theory approaches to many-body systems, ed. by A. Schwenk, J. Polonyi (2012).
2. P. Kopietz, L. Bartosch and F. Schütz, *Introduction to the functional renormalization group*,
Lecture Notes in Physics, Vol. 798, Springer, Berlin (2010).
3. A. Wipf, *Statistical Approach to Quantum Field Theory*,
Lecture Notes in Physics 864, Springer 2013

Scale-dependent functionals

- generating functional or (Euclidean) correlation functions

$$Z[j] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]+(j,\phi)}, \quad (j,\phi) = \int \mathrm{d}^d x \, j(x)\phi(x)$$

- Schwinger functional $W[j] = \log Z[j] \to$ connected correlation functions
- effective action = Legendre transform of $W[j]$

$$\Gamma[\varphi] = (j,\varphi) - W[j] \quad \text{with} \quad \varphi(x) = \frac{\delta W[j]}{\delta j(x)} \tag{1}$$

$\to$ one-particle irreducible correlation functions

- last equation in (1) $\to j[\varphi]$
- insert into first equation in (1)
- Γ: all properties of QFT in a most economic way

- add scale-dependent IR-cutoff term ΔS_k to classical action in functional integral $\to$ scale-dependent generating functional

$$Z_k[j] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]+(j,\phi)-\Delta S_k[\phi]}$$

- Scale-dependent Schwinger functional

$$W_k[j] = \log Z_k[j] \tag{2}$$

- regulator: quadratic functional with a momentum-dependent mass,

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \phi^*(p) R_k(p)\phi(p) \equiv \frac{1}{2}\int_p \phi^*(p) R_k(p)\phi(p) \,,$$

$\to$ one-loop structure of flow equation

conditions on cutoff function $R_k(p)$

- should recover effective action for $k \to 0$:

$$R_k(p) \xrightarrow{k\to 0} 0 \quad \text{for fixed} \quad p$$

- should recover classical action at UV-scale Λ:

$$R_k \xrightarrow{k\to\Lambda} \infty$$

- regularization in the IR:

$$R_k(p) > 0 \quad \text{for} \quad p \to 0$$

Possible cut-offs are

the exponential regulator: $R_k(p) = \dfrac{p^2}{e^{p^2/k^2} - 1}$,

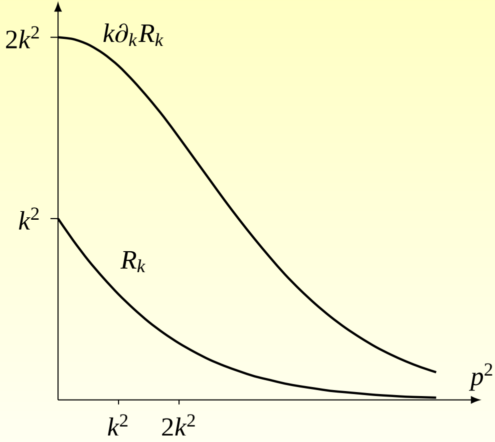
the optimized regulator: $R_k(p) = (k^2 - p^2)\,\theta\left(k^2 - p^2\right)$,

the quartic regulator: $R_k(p) = k^4/p^2$,

the sharp regulator: $R_k(p) = \dfrac{p^2}{\theta\left(k^2 - p^2\right)} - p^2$,

the Callan-Symanzik regulator: $R_k(p) = k^2$

exponential cutoff function and its derivative



Polchinski equation

- partial derivative of W_k in (2) is given by

$$\partial_k W_k[j] = -\frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \, \langle \phi(x)\partial_k R_k(x,y)\phi(y)\rangle_k$$

- relates to connected two-point function

$$G_k^{(2)}(x,y) \equiv \frac{\delta^2 W_k[j]}{\delta j(x)\delta j(y)} = \langle \phi(x)\phi(y)\rangle_k - \varphi(x)\varphi(y)$$

- Polchinski equation

$$\begin{aligned} \partial_k W_k[j] &= -\frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \, \partial_k R_k(x,y) G_k^{(2)}(y,x) - \partial_k \Delta S_k[\varphi] \\ &= -\frac{1}{2}\mathrm{tr}\left(\partial_k R_k \, G_k^{(2)}\right) - \partial_k \Delta S_k[\varphi] \end{aligned}$$

scale dependent effective action

- average field of the cutoff theory with j

$$\varphi(x) = \frac{\delta W_k[j]}{\delta j(x)} \qquad (3)$$

- fixed source $\to$ average field depends on cutoff
- fixed average field $\to$ source depends on cutoff
- modified Legendre transformation:

$$\Gamma_k[\varphi] = (j, \varphi) - W_k[j] - \Delta S_k[\varphi] \qquad (4)$$

- solve (3) for $j = j[\varphi] \to$ use solution in (4)
- Γ_k not Legendre transform of $W_k[j]$ for $k > 0$!
- Γ_k need not to be convex, but $\Gamma_{k\to 0}$ is convex

Wetterich equation

- vary effective average action

$$\frac{\delta\Gamma_k}{\delta\varphi(x)} = \int \frac{\delta j(y)}{\delta\varphi(x)}\varphi(y) + j(x) - \int \frac{\delta W_k[j]}{\delta j(y)}\frac{\delta j(y)}{\delta\varphi(x)} - \frac{\delta\Delta S_k[\varphi]}{\delta\varphi(x)}$$

- terms chancel $\rightarrow$ effective equation of motion

$$\frac{\delta\Gamma_k}{\delta\varphi(x)} = j(x) - \frac{\delta}{\delta\varphi(x)}\Delta S_k[\varphi] = j(x) - (R_k\varphi)(x)$$

- flow equation: φ fixed, j depends on scale, differentiate Γ_k

$$\partial_k\Gamma_k = \int \mathrm{d}^d x\, \partial_k j(x)\varphi(x) - \partial_k W_k[j] - \int \frac{\partial W_k[j]}{\partial j(x)}\partial_k j(x) - \partial_k\Delta S_k[\varphi]$$

- variation of W_k two contributions
  - term $\partial_k W_k[j]$: only scale dependence of the parameters
  - third term: scale dependence of argument j

$$\begin{aligned}\partial_k \Gamma_k &= -\partial_k W_k[j] - \partial_k \Delta S_k[\varphi] \\ &= -\partial_k W_k[j] - \frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y\, \varphi(x)\partial_k R_k(x,y)\varphi(y)\end{aligned}$$

- use Polchinski equation $\rightarrow$

$$\partial_k \Gamma_k = \frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y\, \partial_k R_k(x,y)\, G_k^{(2)}(y,x) \tag{5}$$

second derivative of W_k vs. second derivative of Γ_k:

$$\varphi(x) = \frac{\delta W_k[j]}{\delta j(x)} \quad \text{and} \quad j(x) = \frac{\delta \Gamma_k}{\delta \varphi(x)} + \int \mathrm{d}^d y\, R_k(x,y)\varphi(y)$$

- chain rule $\rightarrow$

$$\delta(x-y) = \int \mathrm{d}^d z \, \frac{\delta\varphi(x)}{\delta j(z)} \frac{\delta j(z)}{\delta\varphi(y)} = \int \mathrm{d}^d z \, G_k^{(2)}(x,z) \left\{ \Gamma_k^{(2)} + R_k \right\} (z,y)$$

- second functional derivative of Γ_k

$$\Gamma_k^{(2)}(x,y) = \frac{\delta^2 \Gamma_k}{\delta\varphi(x)\delta\varphi(y)}$$

- curly bracket = inverse of the connected two-point function G_k:

$$G_k^{(2)} = \frac{1}{\Gamma_k^{(2)} + R_k}$$

- insert into (5) $\rightarrow$ Wetterich equation

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{tr} \left(\frac{\partial_k R_k}{\Gamma_k^{(2)}[\varphi] + R_k} \right) \tag{6}$$

- non-linear functional integro-differential equation
- full propagator on enters flow equation
- Polchinski and Wetterich equations = exact FRG equations
- Polchinski: simple polynomial structure
  favored in structural investigations
- Wetterich: second derivative in the denominator
  stabilizes flow in (numerical) solution
  mainly used in explicit calculations.
- in practice: truncation = projection onto finite-dim. space
- difficult: error estimate for flow
  $\rightarrow$ improve truncation, optimize regulator, check stability

Quadratic action

- at cutoff

$$\Gamma_\Lambda[\varphi] = \frac{1}{2}\int \mathrm{d}^d x\, \varphi(-\Delta + m_\Lambda^2)\varphi\ ,$$

- solution solution of the FRG-equation

$$\Gamma_k[\varphi] = \Gamma_\Lambda[\varphi] + \frac{1}{2}\log\det\left(\frac{-\Delta + m_\Lambda^2 + R_k}{-\Delta + m_\Lambda^2 + R_\Lambda}\right) \tag{7}$$

- last term for optimized cutoff

3d: $$\frac{1}{6\pi^2}\left(m_\Lambda^3 \arctan\frac{m_\Lambda(k-\Lambda)}{m_\Lambda^2 + k\Lambda} + m_\Lambda^2(\Lambda - k) + \frac{k^3}{3} - \frac{\Lambda^3}{3}\right)\ ,$$

4d: $$\frac{1}{64\pi^2}\left(m_\Lambda^4 \log\frac{m_\Lambda^2 + k^2}{m_\Lambda^2 + \Lambda^2} + m_\Lambda^2\left(\Lambda^2 - k^2\right) + \frac{k^4}{2} - \frac{\Lambda^4}{2}\right)$$

Functional renormalization in QM

- anharmonic oscillator

$$S[q] = \int d\tau \left(\frac{1}{2}\dot{q}^2 + V(q) \right) ,$$

- here LPA (local potential approximation)

$$\Gamma_k[q] = \int d\tau \left(\frac{1}{2}\dot{q}^2 + u_k(q) \right) \tag{8}$$

- low-energy approximation
  leading order in gradient expansion
- scale-dependent effective potential u_k
- neglected: higher derivative terms, mixed terms $q^n \dot{q}^m$

- flow equation contains $\Gamma_k^{(2)} = -\partial_\tau^2 + u_k''(q)$
- LPA: sufficient to consider a constant $q \to$ momentum space

$$\int \mathrm{d}\tau \, \frac{\partial u_k(q)}{\partial k} = \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}\tau' \frac{\partial R_k}{\partial k}(\tau - \tau') \frac{1}{-\partial_\tau^2 + u_k''(q) + R_k}(\tau' - \tau)$$

$$= \frac{1}{2} \int \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\partial_k R_k(p)}{p^2 + u_k''(q) + R_k(p)}$$

- choose optimal regulator function

$$R_k(p) = (k^2 - p^2)\,\theta\left(k^2 - p^2\right) \Longrightarrow \partial_k R_k(p) = 2k\theta(k^2 - p^2)$$

- non-linear partial differential equation for u_k:

$$\partial_k u_k(q) = \frac{1}{\pi} \frac{k^2}{k^2 + u_k''(q)}$$

- minimum of $u_k(q)$ not ground state energy
  differs by q-independent contribution
- free particle limit fixes subtraction in flow equation

$$\partial_k u_k(q) = \frac{1}{\pi}\left(\frac{k^2}{k^2 + u_k''(q)} - 1\right) = -\frac{1}{\pi}\frac{u_k''(q)}{k^2 + u_k''(q)} \qquad (9)$$

- assume $u_\Lambda(q)$ even $\to$ $u_k(q)$ even
- polynomial ansatz

$$u_k(q) = \sum_{n=0,1,2\dots} \frac{1}{(2n)!}\, a_{2n}(k)\, q^{2n}\ ,$$

- scale-dependent couplings a_{2n}
- Insert into (29), compare coefficients of powers of q^2
- $\to$ infinite set of coupled ode's

$$\frac{\mathrm{d}a_0}{\mathrm{d}k} = -\frac{1}{\pi} a_2 \Delta_0, \qquad \Delta_0 = \frac{1}{k^2 + a_2},$$
$$\frac{\mathrm{d}a_2}{\mathrm{d}k} = -\frac{k^2}{\pi} a_4 \Delta_0^2,$$
$$\frac{\mathrm{d}a_4}{\mathrm{d}k} = -\frac{k^2 \Delta_0^2}{\pi} \left(a_6 - 6 a_4^2 \Delta_0 \right),$$
$$\frac{\mathrm{d}a_6}{\mathrm{d}k} = -\frac{k^2 \Delta_0^2}{\pi} \left(a_8 - 30 a_4 a_6 \Delta_0 + 90 a_4^3 \Delta_0^2 \right),$$
$$\vdots$$

- initial condition: a_{2n} at cutoff = parameters in classical potential
- projection onto space of polynomials up to given degree
- e.g. $a_6 = 0$ in system of ode's.standard notation

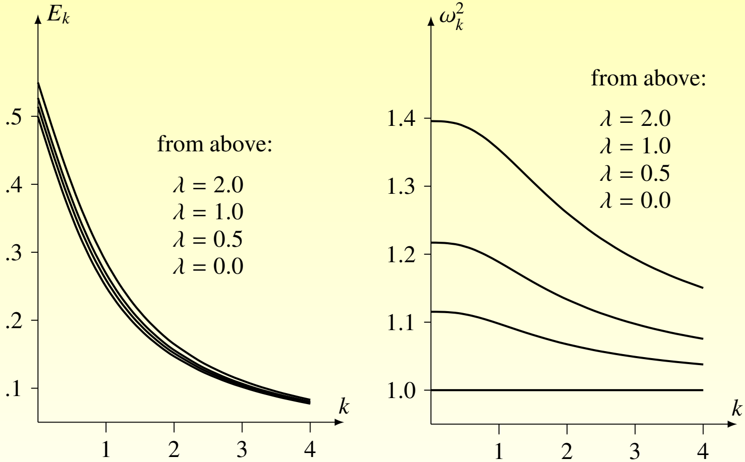
- notation

$$a_0 = E, \quad a_2 = \omega^2 \quad \text{and} \quad a_4 = \lambda \,,$$

- $\to$ truncated system of flow equations

$$\frac{\mathrm{d}E_k}{\mathrm{d}k} = -\frac{\omega_k^2}{\pi}\Delta_0, \quad \frac{\mathrm{d}\omega_k^2}{\mathrm{d}k} = -\frac{k^2\lambda_k}{\pi}\Delta_0^2, \quad \frac{\mathrm{d}\lambda_k}{\mathrm{d}k} = \frac{6k^2\lambda_k^2}{\pi}\Delta_0^3$$

- solve numerically (eg. with `octave`)
- initial conditions $E_\Lambda = 0,\ \omega_\Lambda = 1$, varying λ at the cutoff scale
- $\to$ scale-dependent couplings E_k and ω_k^2
- hardly change for $k \gg \omega$
- variation for $k \approx \omega$



The flow of the couplings E_k and ω_k^2 ($E_\Lambda = 0$, $\omega_\Lambda^2 = 1$).

- $\omega = \omega_{k=0} > 0 \;\Rightarrow$ effective potential minimal at origin
- ground state energy: $E_0 = \min(u_{k=0})$
- energy of *first excited state*

$$E_1 = E_0 + \sqrt{u''_{k=0}(0)} = E_0 + \omega_{k=0}$$

- already good results with simple truncation

Energies with varying λ
different truncations und regulators
units of $\hbar\omega$

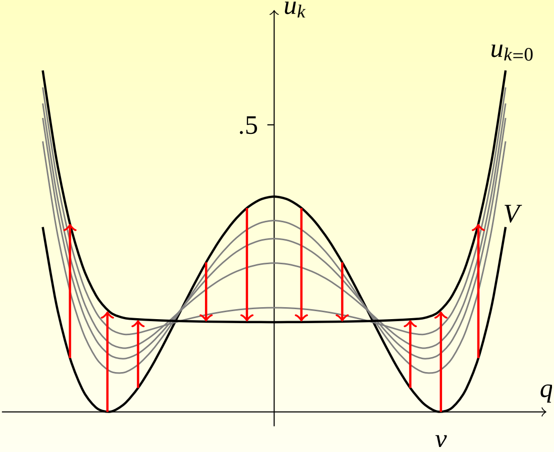
	ground state energy				energy of first excited state			
cutoff	optimal order 4	optimal order 12	Callan order 4	exact result	optimal order 4	optimal order 12	Callan order 4	exact result
$\lambda = 0$	0.5000	0.5000	0.5000	0.5000	1.5000	1.5000	1.5000	1.5000
$\lambda = 1$	0.5277	0.5277	0.5276	0.5277	1.6311	1.6315	1.6307	1.6313
$\lambda =2$	0.5506	0.5507	0.5504	0.5508	1.7324	1.7341	1.7314	1.7335
$\lambda = 3$	0.5706	0.5708	0.5703	0.5710	1.8177	1.8207	1.8159	1.8197
$\lambda = 4$	0.5885	0.5889	0.5882	0.5891	1.8923	1.8968	1.8898	1.8955
$\lambda = 5$	0.6049	0.6054	0.6045	0.6056	1.9593	1.9652	1.9562	1.9637
$\lambda = 6$	0.6201	0.6207	0.6196	0.6209	2.0205	2.0278	2.0168	2.0260
$\lambda = 7$	0.6343	0.6350	0.6336	0.6352	2.0771	2.0857	2.0728	2.0836
$\lambda = 8$	0.6476	0.6484	0.6469	0.6487	2.1299	2.1397	2.1250	2.1374
$\lambda = 9$	0.6602	0.6611	0.6594	0.6614	2.1794	2.1905	2.1741	2.1879
$\lambda = 10$	0.6721	0.6732	0.6713	0.6735	2.2263	2.2385	2.2205	2.2357
$\lambda = 20$	0.7694	0.7714	0.7679	0.7719	2.5994	2.6209	2.5898	2.6166

Recall flow equation in LPA:

$$\partial_k u_k(q) = -\frac{1}{\pi} \frac{u_k''(q)}{k^2 + u_k''(q)}$$

- negative ω^2 in V: local maximum at 0 and two minima
- denominator minimal where u_k'' minimal (maximum of u_k)
- denominator positive for large scales
  $\Rightarrow$ remains positive during the flow
- flow equation $\Rightarrow$
  $u_k(q)$ increases toward infrared if $u_k''(q)$ is positive
  $u_k(q)$ decreases toward infrared if $u_k''(q)$ is negative
- $\Rightarrow$ double-well potential flattens during flow, becomes convex
- expected on general grounds

solution of partial differential equation, $\omega^2 = -1$, $\lambda = 1$



- energies of ground state and first excited state:
  less good, less stable
- fourth-order polynomials $\rightarrow$ inaccurate results for weak couplings
- numerical solution of the flow equation ok
- decreasing λ (increasing barrier) $\rightarrow$ increasingly difficult
- to detect splitting induced by instanton effects:
  must go beyond leading order LPA

energies for $\omega^2 = -1$ and varying λ
optimized regulator, units of $\hbar\omega$

	ground state energy				energy of first excited state			
	optimal order 4	optimal order 12	pde	exact	optimal order 4	optimal order 12	pde	exact
$\lambda = 1$			-0.8732	-0.8556			-0.7887	-0.8299
$\lambda = 2$		-0.2474	-0.2479	-0.2422		0.0049	0.0063	-0.0216
$\lambda = 3$	0.2473	-0.0681	-0.0679	-0.0652	-0.2241	0.3514	0.3500	0.3307
$\lambda = 4$	-0.0186	0.0286	0.0290	0.0308	0.3511	0.5753	0.5755	0.5598
$\lambda = 5$	0.0654	0.0949	0.0953	0.0967	0.5835	0.7455	0.7462	0.7324
$\lambda = 6$	0.1234	0.1457	0.1461	0.1472	0.7509	0.8842	0.8851	0.8723
$\lambda = 7$	0.1688	0.1871	0.1876	0.1885	0.8851	1.0021	1.0030	0.9909
$\lambda = 8$	0.2063	0.2223	0.2228	0.2236	0.9987	1.1052	1.1061	1.0944
$\lambda = 9$	0.2671	0.2530	0.2535	0.2543	1.1863	1.1972	1.1981	1.1866
$\lambda = 10$	0.2386	0.2803	0.2808	0.2816	1.0978	1.2805	1.2814	1.2701
$\lambda = 20$	0.4536	0.4632	0.4639	0.4643	1.7866	1.8638	1.8648	1.8538

Scalar Field Theory

- QM = 1-dimensional field theory
- Now: Euclidean scalar field theory in d dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)$$

- first local potential approximation

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\partial_\mu \varphi)^2 + u_k(\varphi) \right)$$

- second functional derivative: $\Gamma_k^{(2)} = -\Delta + u_k''(\varphi)$
- flow of effective potential already from constant average field

$$\partial_k u_k(q) = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\partial_k R_k(p)}{p^2 + u_k''(q) + R_k(p)} \tag{10}$$

- optimized regulator:
  result *to* volume of the d-dimensional ball divided by $(2\pi)^d$,

$$\mu_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$$

- p-integration can be done $\to$ flow equation

$$\partial_k u_k(\varphi) = \mu_d \frac{k^{d+1}}{k^2 + u_k''(\varphi)}\;, \qquad (11)$$

- dimensions enters via k^{d+1} and μ_d
- nonlinear partial differential equation
- polynomial ansatz for even potential

- flow equations for infinite set of couplings

$$\frac{\mathrm{d}a_0}{\mathrm{d}k} = -\mu_d k^{d+2}\Delta_0, \qquad \Delta_0 = \frac{1}{k^2+a_2}\,,$$

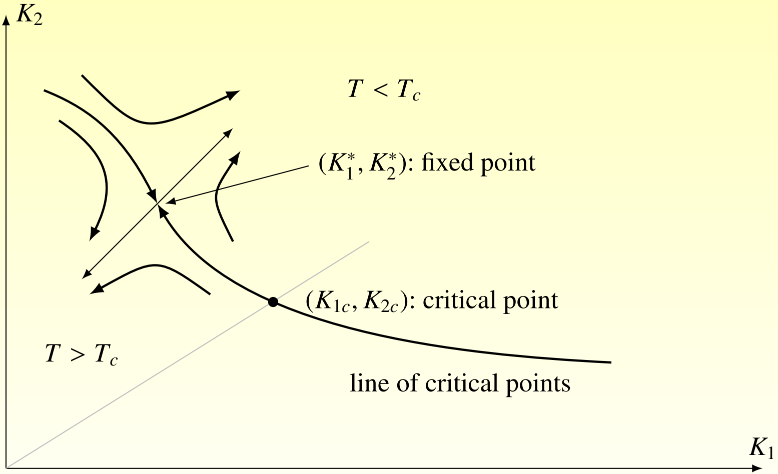
$$\frac{\mathrm{d}a_2}{\mathrm{d}k} = -\mu_d k^{d+2}\Delta_0^2 a_4\,,$$

$$\frac{\mathrm{d}a_4}{\mathrm{d}k} = -\mu_d k^{d+2}\Delta_0^2\left(a_6 - 6a_4^2\Delta_0\right)\,,$$

$$\frac{\mathrm{d}a_6}{\mathrm{d}k} = -\mu_d k^{d+2}\Delta_0^2\left(a_8 - 30a_4a_6\Delta_0 + 90a_4^3\Delta_0^2\right)\,,$$

$$\vdots \qquad \vdots$$

Fixed points



- critical hyper-surfaces on which $\xi = \infty$
- RG trajectory moves away from critical surface
- If flow begins on critical surface $\rightarrow$ stays on surface
- most critical points are not fixed point
- $d \geq 3$: expect a finite set of isolated fixed points
- fixed point $K^* = (K_1^*, K_2^*, \dots)$
- RG flow in the vicinity of fixed point $K = K^* + \delta K$
- linearize flow around fixed point

$$K_i' = K_i^* + \delta K_i' = R_i\big(K_j^* + \delta K_j\big) = K_i^* + \frac{\partial R_i}{\delta K_j}\Big|_{K^*} \delta K_j + O(\delta K^2)$$

- $\to$ linearized RG transformation,

$$\delta K_i' = \sum_j M_i^{\ j} \delta K_j, \qquad M_i^{\ j} = \frac{\partial R_i}{\partial K_j}\Big|_{K^*}$$

- eigenvalues and left-eigenvectors of matrix M

$$\sum_j \Phi_\alpha^j M_j^{\ i} = \lambda_\alpha \Phi_\alpha^i = b^{y_\alpha} \Phi_\alpha^i$$

- subset of $\{\Phi_\alpha\}$ span space tangential to critical surface at K^*
- every λ_α defines a critical exponent y_α
- consider the new variables

$$g_\alpha = \sum_i \Phi_\alpha^i \delta K_i \ .$$

- We have

$$g'_\alpha = \sum_i \Phi^i_\alpha \delta K'_i = \sum_{ij} \Phi^i_\alpha M_i^{\,j} \delta K_j = \sum_j b^{y_\alpha} \Phi_{\alpha j} \delta K_j = b^{y_\alpha} g_\alpha \,. \qquad (12)$$

- $y_\alpha > 0$: deviation g_α increases, flow moves point $K^* + g_\alpha$ away from the fixed point $K^* \to$ relevant perturbation
- $y_\alpha < 0$: deviation g_α decreases, flow carries point $K^* + g_\alpha$ towards the fixed point $K^* \to$ irrelevant perturbation
- $y_\alpha = 0$: marginal coupling
- relevant couplings determine important scaling laws
- all TD critical exponent functions of relevant exponents
- relevant couplings and exponents determine IR-physics

Fixed point analysis for scalar models

- introduce the dimensionless field and potential,

$$\varphi = k^{(d-2)/2}\sqrt{\mu_d}\,\chi \quad \text{and} \quad u_k(\varphi) = k^d \mu_d v_k(\chi)$$

- flow equation in terms of dimensionless quantities

$$k\partial_k v_k + d v_k - \frac{d-2}{2}\chi v_k' = \frac{1}{1+v_k''}, \quad v_k' = \frac{\partial v_k}{\partial \chi}\dots$$

- at a fixed point: $\partial_k v_k = 0 \Rightarrow$
- fixed point equation for effective potential:

$$d v_* - \frac{d-2}{2}\chi v_*' = \frac{1}{1+v_*''}$$

- constant solution $dv_* = 1 \to$ trivial Gaussian fixed point
- are there non-Gaussian fixed points?
- answer depends on the dimension d of spacetime
- even classical potential $\to v_k$ even as well:

$$v_k(\chi) = w_k(\varrho), \quad \text{with} \quad \varrho = \frac{\chi^2}{2}$$

- flow equation for $w_k(\varrho)$

$$k\partial_k w_k(\varrho) + dw_k(\varrho) - (d-2)\,\varrho w_k'(\varrho) = \frac{1}{1 + w_k'(\varrho) + 2\varrho w_k''(\varrho)}$$

- fixed point equation

$$dw_*(\varrho) - (d-2)\,\varrho w_*'(\varrho) = \frac{1}{1 + w_*'(\varrho) + 2\varrho w_*''(\varrho)}$$

- 2d theories: ∞ many fixed-point solutions [Morris 1994]
- also true for 2d Yukawa theories [Synatschke et al.]
- polynomial truncation to order m:

$$w^{(m)} = \sum_{n=0}^{m} c_n \varrho^n$$

- flow equation for couplings

$$\begin{aligned}
k\partial_k c_0 &= -dc_0 + \Delta_0, \qquad \Delta_0 = (1 + c_1)^{-1} \,, \\
k\partial_k c_1 &= -2c_1 - 6c_2\Delta_0^2 \,, \\
k\partial_k c_2 &= (d-4)c_2 - 15c_3\Delta_0^2 + 36c_2^2\Delta_0^3 \\
k\partial_k c_3 &= (2d-6)c_3 - 28c_4\Delta_0^2 + 180c_2c_3\Delta_0^3 - 216c_2^3\Delta_0^4 \,, \\
k\partial_k c_4 &= (3d-8)c_4 - 45c_5\Delta_0^2 + (336c_2c_4 + 225c_3^2)\Delta_0^3 \\
&\quad - 1620c_2^2c_3\Delta_0^4 + 1296c_2^4\Delta_0^5 \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

Scalar fields in three dimensions

- expect non-trivial fixed point
- no solution to full fixed point equation

  polynomial truncation: above equation with $d = 3$ and lhs = 0
- m algebraic equations for the $m + 1$ fixed-point couplings,

$$0 = f_0(c_0^*, c_1^*) = f_1(c_1^*, c_2^*) = \cdots = f_{m-1}(c_1^*, \ldots, c_m^*)$$

- polynomials in $c_0^*, c_2^*, \ldots, c_m^*$ and $\Delta_0 = 1/(1 + c_1^*)$
- non-polynomial c_1^* (= slope at origin)
- solve the system for $c_0^*, c_2^*, c_3^*, \ldots, c_m^*$ in terms of c_1^*
- algebraic program $\rightarrow$ solution for m up to 42

- explicit expression for the lowest fixed-point couplings

$$
\begin{aligned}
c_0^* &= \frac{1}{3}\frac{1}{1+c_1^*} \\
c_2^* &= -\frac{c_1^*(1+c_1^*)^2}{3} \\
c_3^* &= \frac{c_1^*(1+c_1^*)^3(1+13c_1^*)}{45} \\
c_4^* &= -\frac{c_1^{*2}(1+c_1^*)^4(1+7c_1^*)}{21}\,, \\
&\vdots \\
c_m^* &= c_1^{*2}(1+c_1^*)^m P_{m-3}(c_1^*)\,,
\end{aligned}
$$

- P_k polynomial of order k

- trivial solution (Gaussian fixed point $w'_* = 1$)

$$c_0^* = \frac{1}{3}, \quad 0 = c_2^* = c_3^* = c_4^* = \dots$$

- search for other fixed points:
- set $c_m^* = 0 \to P_{m-3}(c_1^*) = 0$
- polynomials P_k has several real roots c_1^*
- for each m choose c_1^* such that for large m they converge
- the approximating polynomials converge to a power series with maximal radius of convergence
- for example, $m = 20$ and $m = 42$ we find $c_1^* = -.186066$ and $c_1^* = -.186041$
- insert solution $\Rightarrow$

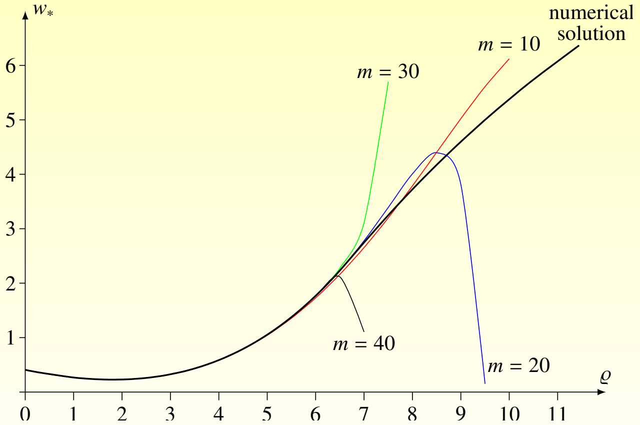
- With n! multiplied fixed-point coefficients c_n^*

	c_0^*	c_1^*	c_2^*	c_3^*	c_4^*	c_5^*	c_6^*
$m=20$	0.409534	-0.186066	0.082178	0.018981	0.005253	0.001104	-0.000255
$m=42$	0.409533	-0.186064	0.082177	0.018980	0.005252	0.001104	-0.000256
	c_7^*	c_8^*	c_9^*	c_{10}^*	c_{11}^*	c_{12}^*	c_{13}^*
$m=20$	-0.000526	-0.000263	0.000237	0.000632	0.000438	-0.000779	-0.002583
$m=42$	-0.000526	-0.000263	0.000236	0.000629	0.000431	-0.000799	-0.002643
	c_{14}^*	c_{15}^*	c_{16}^*	c_{17}^*	c_{18}^*	c_{19}^*	c_{20}^*
$m=20$	-0.002029	0.007305	0.028778	0.034696	-0.077525	-0.381385	0.000000
$m=42$	-0.002216	0.006677	0.026544	0.026320	-0.110498	-0.517445	-0.587152

- lowest coefficients do not change much when m increases

Polynomial approximations vs. numerical solution

numerics: shooting method with seventh-order Runge-Kutta



- fine-tune slope at origin $\rightarrow w_*'(0) \approx -0.186064249376$
- Polynomial of degree 42 $\rightarrow w_*'(0) \approx -0.186064279993$

Critical exponents

- flow equation in the vicinity of fixed-point solution w_*
- set $w_k = w_* + \delta_k$, linearize the flow in small δ_k
- $\rightarrow$ linear differential equation for the small fluctuations

$$
\begin{aligned}
k\partial_k\delta_k = &- d\delta_k + (d-2)\,\varrho\delta_k' \\
&- \left(dw_* - (d-2)\,\varrho w_*'\right)^2 \left(\delta_k' + 2\varrho\delta_k''\right)
\end{aligned}
$$

- insert the polynomial approximation for fixed-point solution
- polynomial ansatz for the perturbation $\rightarrow$

$$
\delta_k(\varrho) = \sum_{n=0}^{m-1} d_n \varrho^n \qquad \varrho = \frac{\chi^2}{2}
$$

- linear system for the coefficients d_m

$$k\partial_k \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{m-1} \end{pmatrix} = M(c_0^*) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{m-1} \end{pmatrix}$$

- critical exponents = eigenvalues of m-dimensional matrix M
- $\rightarrow$ up to order $m = 46$ with algebraic program

m	$\nu = -1/\omega_1$	ω_2	ω_3	ω_4	ω_5
10	0.648617	0.658053	2.985880	7.502130	17.913494
14	0.649655	0.652391	3.232549	5.733445	9.324858
18	0.649572	0.656475	3.186784	5.853987	9.141093
22	0.649554	0.655804	3.170538	5.977066	8.522811
26	0.649564	0.655629	3.182910	5.897290	8.844632
30	0.649562	0.655791	3.180847	5.903039	8.907607
34	0.649561	0.655749	3.178636	5.922910	8.702583
38	0.649562	0.655731	3.180577	5.908885	8.814225
42	0.649562	0.655755	3.180216	5.909910	8.847386
46	0.649562	0.655746	3.179541	5.915754	8.738608

- convergence
- two negative exponents $\omega_0 = -3$ and $\omega_1 = -1/\nu$
- ω_0 ground state energy, unrelated to critical behavior
- $\omega_2, \omega_3, \omega_4, \ldots$ all positive
- LPA-prediction: $\nu = 0.649562$ (high-T expansion: $\nu = 0.630$)

Wave function renormalization

- next-to-leading in derivative expansion $\rightarrow$ wave function renormalization $Z_k(p,\varphi)$
- difficult non-linear parabolic partial differential RG-equations
- first step: neglect field and momentum dependence $\rightarrow$

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2} Z_k \, (\partial_\mu \varphi)^2 + u_k(\varphi) \right) \; .$$

- second functional derivative $\Gamma_k^{(2)} = -Z_k \Delta + u_k''(\varphi)$
- flow equation (simplification for $R_k \rightarrow Z_k R_k$):

$$\int \mathrm{d}^d x \left(\frac{1}{2} (\partial_k Z_k) \, (\partial_\mu \varphi)^2 + \partial_k u_k(\varphi) \right) = \frac{1}{2} \mathrm{tr} \left(\frac{\partial_k (Z_k R_k)}{Z_k (p^2 + R_k) + u_k''(\varphi)} \right)$$

- simple: flow of effective potential:

$$\partial_k u_k = \frac{\mathcal{Z}_k}{Z_k k^2 + u_k''}, \quad \mathcal{Z}_k = \frac{\mu_d}{d+2}\,\partial_k\left(k^{d+2} Z_k\right) .$$

- more difficult: flow of Z_k
- project flow on operator $(\partial\phi)^2$
- must admit non-homogeneous fields $\rightarrow [p^2, u_k''(\varphi)] \neq 0$
- final answer

$$k\partial_k Z_k = -\mu_d\, k^{d+2}\left(Z_k a_3 \Delta_0^2\right)^2, \quad \Delta_0 = \frac{1}{Z_k k^2 + a_2}$$

see A. Wipf, Lecture Notes in Physics 864

- anomalous dimension

$$\eta = -k\partial_k \log Z_t$$

Linear O(N) models

- scalar field $\phi \in \mathbb{R}^N$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi)$$

- $O(N)$ invariant potential
- fixed-point analysis: dimensionless quantities χ and ν_k
- invariant composite field

$$\varrho = \frac{1}{2} \sum_{i=1}^{N} \chi_i^2$$

- set $\nu_k(\chi) = w_k(\varrho)$

- flow equation in LPA (optimized regulator)

$$k\partial_k w_k + dw_k - (d-2)\,\varrho w_k' = \frac{N-1}{1+w_k'} + \frac{1}{1+w_k'+2\varrho w_k''}$$

- contribution of the $N-1$ Goldstone modes
- contribution of massive radial mode
- large N: Goldstone modes give main contribution
- linearize about fixed-point solution: $w_k = w_* + \delta_k$
- fluctuation δ_k obeys the linear differential equation

$$k\partial_k\delta_k = -d\delta_k + (d-2)\,\varrho\delta_k' - \frac{(N-1)\,\delta_k'}{(1+w_*')^2} - \frac{\delta_k'+2\varrho\delta_k''}{(1+w_*'+2\varrho w_*'')^2}$$

- proceed as before: polynomial truncation to high order (40)
  $\rightarrow$ slope at origin of fixed-point solution
  three smallest eigenvalues of stability matrix
- $\rightarrow$ Wilson-Fisher fixed point
- eigenvalue $\omega_0 = -3$ of the scaling operator 1 not listed

N	1	2	3	100	1000
$-w'_*(0)$	0.186064	0.230186	0.263517	0.384172	0.387935
$\nu = -1/\omega_1$	0.64956	0.70821	0.76113	0.99187	0.99923
ω_2	0.6556	0.6713	0.6990	0.97218	0.99844
ω_3	3.1798	3.0710	3.0039	2.98292	2.99554

- extract asymptotic formulas

$$w'_*(0) \approx -0.3881 + \frac{0.4096}{N}, \quad \nu \approx 0.9998 - \frac{0.9616}{N}$$

Large N Limit

- rather simple flow equation ($t = \log(k/\Lambda)$)

$$k\partial_k w_k = (d-2)\,\varrho w_k' - d w_k + \frac{N}{1+w_k'}$$

$$\partial_t w'(d-2) = \;\varrho w'' - 2w' - \frac{N}{(1+w')^2} w''$$

- can be solved exactly with methods of characteristics
- analytic relation between fixed point solution and perturbation in

$$s(t,\rho) \approx w_*(\rho) + \mathrm{e}^{\omega t}\delta(\rho)$$

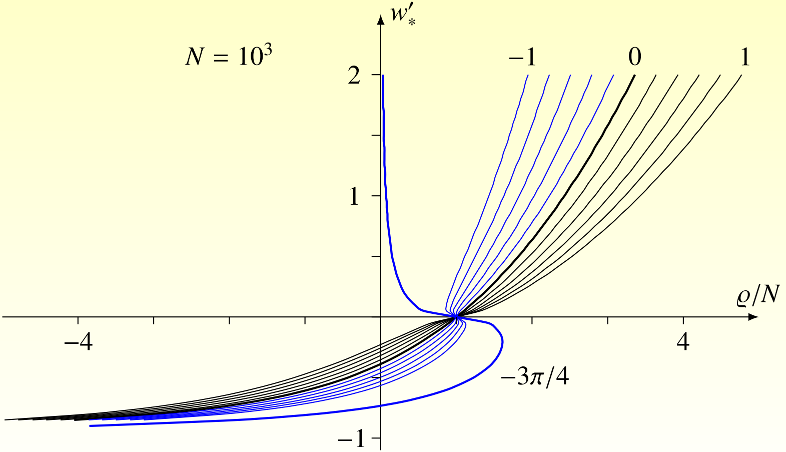
- result:

$$w(t,\varrho) \approx w_*(\varrho) + \text{const} \times \mathrm{e}^{\omega t}\, w_*'(\varrho)^{(\omega+d)/2}\;.$$

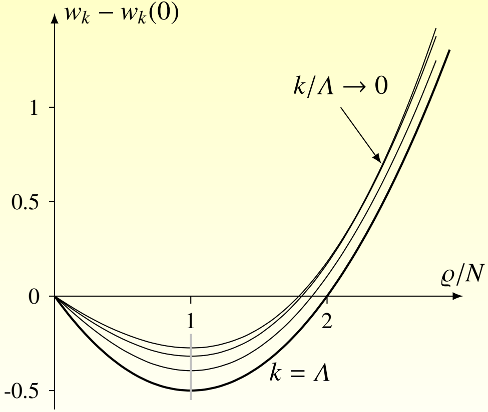
- if perturbation regular → all critical exponents

$$\omega \in \{2n - d \mid n = 0, 1, 2, \dots\}$$

- on-parameter family of fixed point solutions

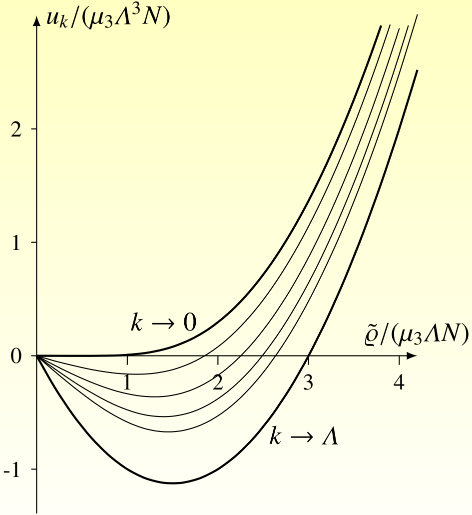


flow of dimensionless potential into
fixed point solution with $c = 0$
$\lambda_\Lambda = 1,\ \kappa_\Lambda = 1$



flow of dimensionful potential below critical temperature

$\lambda_\Lambda = 1,\ \kappa_\Lambda = 1.3\kappa_{\rm crit}$



flow of dimensionful potential above critical temperature

$\lambda_\Lambda = 1,\ \kappa_\Lambda = 0.5\kappa_{\text{crit}}$