Friedrich-Schiller-Universität Jena Winter term 2017/18
Prof. Dr. Andreas Wipf
Dr. Luca Zambelli

Problems in Advanced Quantum Mechanics

Problem Sheet 12

Problem 27: Gauge principle for SU(2) 1+1+1+1+1+1+1 = 7 points

In the lecture we discussed the gauge principle for the gauge group $U(1) = \{U = \mathrm{e}^{\mathrm{i}\lambda} | \lambda \in \mathbb{R}\}$ (electromagnetism) in detail. In this exercise we generalize this principle to the gauge group of 2×2 complex unitary matrices

$$\mathrm{SU}(2) = \{U \in \mathrm{Mat}(2) | U^\dagger = U^{-1},\, \det(U) = 1\}\,.$$

In contrast to U(1) this group is non-Abelian (non-commutative). This means, that in general $U_1 U_2 \neq U_2 U_1$. The group shows up in the discussion of the electron spin in non-relativistic quantum mechanics. It is (almost) the gauge group of the electroweak interaction in particle physics.
Assume that we have a scalar field $\boldsymbol{\phi}$ with two complex components (like the Higgs-boson)

$$\boldsymbol{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$$

which under local gauge transformation transforms as

$$\boldsymbol{\phi}(x) \longrightarrow \boldsymbol{\phi}'(x) = U(x)\boldsymbol{\phi}(x)\,.$$

The covariant derivative of $\boldsymbol{\phi}$ is (we use natural units with $e = \hbar = c = 1$)

$$(D_\mu \boldsymbol{\phi})(x) = \partial_\mu \boldsymbol{\phi}(x) + \mathrm{i} A_\mu(x) \boldsymbol{\phi}(x)\,,$$

where now each component of the gauge potential A_μ is a traceless and hermitean 2×2 matrix and thus can be expanded in terms of the 3 Pauli matrices: $A_\mu(x) = A^a_\mu(x)\sigma_a$ (sum over a).

1. D_μ should be a covariant derivative, which means that $D_\mu \boldsymbol{\phi}$ should transform exactly in the same way as $\boldsymbol{\phi}$:

   $$(D'_\mu \boldsymbol{\phi}')(x) = U(x)(D_\mu \boldsymbol{\phi})(x), \qquad D'_\mu = \partial_\mu + \mathrm{i} A'_\mu\,.$$

   What is the gauge transformation for $A_\mu \to A'_\mu$ such that this is true?
2. Argue that the components A'_μ are traceless and hermitian if the A_μ have this property. Hint: For unitary matrix $U^\dagger = U^{-1}$ and in addition one can use $\partial_\mu(UU^{-1}) = 0$
3. As in electrodynamics we define the covariant components of the field strength tensor $F_{\mu\nu}$ according to

   $$[D_\mu, D_\nu] = \mathrm{i} F_{\mu\nu}\,.$$

   Write $F_{\mu\nu}$ explicitly in terms of the vector potential A_μ.
4. Argue that $F_{\mu\nu}$ is traceless an hermitian.

5. How does $F_{\mu\nu}$ transform under gauge transformations?
6. How does the Klein-Gordon (KG) equation look for $\boldsymbol{\phi}$ in an external SU(2) gauge field A_μ?
   Hint: As in the lecture we begin with the free KG equation $(\partial_\mu \partial^\mu + m^2)\boldsymbol{\phi} = 0$ for the two-component field $\boldsymbol{\phi}$ and make this equation gauge covariant by replacing...
7. Show that $(\boldsymbol{\phi}', A'_\mu)$ solves the equation if $(\boldsymbol{\phi}, A_\mu)$ does.
   Hint: repeat the steps given on the blackboard during the lecture.

Problem 28: Gamma matrices 2+1+1+1+2 = 7 points

In the chiral representation the gamma matrices have the form

$$\gamma^0 = \sigma_1 \otimes \sigma_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^k = -\mathrm{i}\sigma_2 \otimes \sigma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

and in the Dirac representation

$$\gamma^0 = \sigma_3 \otimes \sigma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma^k = \mathrm{i}\sigma_2 \otimes \sigma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}.$$

1. Show that these matrices fulfill the anti-commutation rules $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.
   Hint: The calculations simplify when you use the tensor product rules, e.g. $(A \otimes B)(C \otimes D) = AC \otimes BD$.
2. What are the hermiticity properties of the γ^μ? Why can γ^1 not be hermitean?
3. Calculate $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ for both representations.
4. Use the anti-commutation rules of the γ^μ to show that γ_5 anti-commutes with all γ-matrices: $\{\gamma_5, \gamma^\mu\} = 0$.
5. In addition, prove the identities

   $$\begin{aligned} \gamma^\mu \gamma_\mu &= 4\mathbb{1} \\ \gamma^\mu \not{p} \gamma_\mu &= -2\not{p} \\ \gamma^\mu \not{p} \not{q} \gamma_\mu &= 4p \cdot q \mathbb{1}, \end{aligned}$$

   where $\not{p} = p^\mu \gamma_\mu$ and $p \cdot q = p^\mu q_\mu$. (0.5+0.5+1 points)

Submission date: Thursday, 25. January 2018, before the lecture begins.