Friedrich-Schiller-Universität Jena Winter term 2017/18
Prof. Dr. Andreas Wipf
Dr. Luca Zambelli

Problems in Advanced Quantum Mechanics

Problem Sheet 11

Problem 24: Lorentz transformation of $F^{\mu\nu}$ 2+2 = 4 points

The contravariant components of the field strength tensor transform under a change of the inertial systems $I \to I'$ according to

$$F^{\mu\nu}(x) \mapsto F'^{\mu\nu}(x') = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}(x)\,.$$

Consider the Lorentz boost

$$x'^0 = \gamma x^0 - \beta\gamma x^1, \qquad x'^2 = x^2\,,$$
$$x'^1 = \gamma x^1 - \beta\gamma x^0, \qquad x'^3 = x^3\,.$$

How does the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ (which make up the field strength tensor) transform under this Lorentz-transformation? Use the same notation and conventions as in the lecture.

Problem 25: the scalar field 2+2 = 4 Punkte

In the lectures we defined the current density 4-vector j^μ for a Klein-Gordon field ϕ in presence of an external electromagnetic field with potential A_μ as follows:

$$j^\mu = \frac{\mathrm{i}\hbar}{2m}\left(\phi^* D^\mu \phi - \phi (D^\mu \phi)^*\right)$$

where the covariant derivative is given by

$$D_\mu \phi = \left(\partial_\mu + \frac{\mathrm{i}e}{\hbar c} A_\mu\right)\phi\,.$$

1. Show that the current density is gauge invariant, i.e. invariant under the transformation

   $$A_\mu \mapsto A_\mu - \partial_\mu \lambda, \quad \phi \mapsto \mathrm{e}^{\mathrm{i}e\lambda/\hbar c}\phi$$

   for any arbitrary gauge function λ.
2. Show that the current is conserved

   $$\partial_\mu j^\mu = 0,$$

   if ϕ solves the Klein-Gordon equation

   $$\left(D_\mu D^\mu + \frac{m^2 c^2}{\hbar^2}\right)\phi = 0\,.$$

Submission date: Thursday, 18. January 2018, before the lecture begins.