Friedrich-Schiller-Universität Jena

Prof. Dr. Andreas Wipf

Dr. Luca Zambelli

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Problems in Advanced Quantum Mechanics

Problem Sheet 9

Problem 21: Path integral for a free particle on a circle 4+1 = 5 points

We consider a free particle with mass m and Lagrangian $L = m\dot{\ell}^2/2$, which is constrained to stay on the circle $S^1 = \mathbb{R}/(2\pi R\,\mathbb{Z})$ with Radius R. The curvilinear coordinate ℓ is the distance along the circle from an arbitrary reference point. Two different distances ℓ and ℓ' correspond to the same point if $\ell \simeq \ell' \pmod{2\pi R}$. In fact we can always write $\ell = x + k\,(2\pi R)$, with $x \in [-\pi R, +\pi R]$, and $k \in \mathbb{Z}$. We are interested in the amplitude

$$\langle \ell_f, t_f | \ell_i, t_i \rangle_{S^1} = \langle \ell_f, T | \ell_i, 0 \rangle_{S^1}, \quad T = t_f - t_i\,,$$

for the transition from an initial point to a final point on the circle. To set up a path integral representation of this transition amplitude, we fix ℓ_i and we consider paths that end up at all possible values ℓ'_f provided $\ell'_f \simeq \ell_f \pmod{2\pi R}$. Thus, every path contributing to this amplitude is associated to a certain winding number $k = (\ell'_f - \ell_i)/(2\pi R)$.

1. Compute the ratio between the transition amplitude on the circle S^1 and the corresponding transition amplitude on the straight line $\mathbb{R}$

$$\frac{\langle \ell_f, t_f | \ell_i, t_i \rangle_{S^1}}{\langle x_f, t_f | x_i, t_i \rangle_{\mathbb{R}}}\,, \qquad \ell_f - \ell_i \equiv x_f - x_i \pmod{2\pi R}\,.$$

   with extrema $x_i, x_f \in [-\pi R, +\pi R]$.
   Hint: Any path from x_i to x_f on $\mathbb{R}$ can be split into a constant-speed trajectory between these extrema plus an arbitrary periodic function $x_\mathrm{p}(t)$, with $x_\mathrm{p}(T) = x_\mathrm{p}(0) = 0$:

$$x(t) = x_i + \frac{x_f - x_i}{T}\,t + x_\mathrm{p}(t)\,.$$

   The transition amplitude on $\mathbb{R}$ is given by the sum over all periodic functions $x_\mathrm{p}(t)$. On the circle, the same reasoning can be applied to $\ell(t)$, such that the integral is a sum over periodic functions and over winding numbers. The latter two sums factorize.
2. Express your answer in terms of the ϑ function

$$\vartheta(z, \tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi\tau n^2 + 2\pi\mathrm{i}nz}\,.$$

Remark: though it is not needed in the solution of this problem, it is possible to explicitly work out the path integral by expanding x_p in the functions

$$\sqrt{\frac{2}{T}}\cos\frac{2\pi n t}{T} \quad \text{and} \quad \sqrt{\frac{2}{T}}\sin\frac{2\pi n t}{T}, \quad n \in \mathbb{N}\,,$$

which are orthogonal to the constant functions. These trigonometric functions together with the constant function $1/\sqrt{T}$ form an orthonormal basis of periodic functions. Thus one concludes that the linear mapping from $x_{\mathrm{p}} \to \{\alpha_n, \beta_n\}$, where α_n and β_n are the expansion coefficients, is orthogonal and thus has Jacobian determinant 1.

Problem 22: Phase space path integral 3+1+2 = 6 points

As in the lecture we consider the transition amplitude (the propagator)

$$\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | U(t_f, t_{N-1}) U(t_{N-1}, t_{N-2}) \cdots U(t_1, t_i) | x_i \rangle \,.$$

1. Insert the resolution of the identity $\mathbb{1} = \int \mathrm{d}x_n |x_n\rangle\langle x_n|$ with $n = 1, \ldots, N$ between each pair of U's,

   $$\langle x_f, t_f | x_i, t_i \rangle = \int \prod_{n=1}^{N-1} \mathrm{d}x_n \prod_{n=1}^{N} \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$$

   with the identifications

   $$(x_N, t_N) \equiv (x_f, t_f) \quad \text{and} \quad (x_0, t_0) \equiv (x_i, t_i) \,.$$

   Let us assume that the Hamiltonian has the form $H(t, p, x) = T(t, p) + V(t, x)$ and use the Baker-Campbell-Hausdorff formula

   $$\mathrm{e}^{\mathrm{i}\epsilon(T+V)/\hbar} = \mathrm{e}^{-\mathrm{i}\epsilon V/\hbar} \mathrm{e}^{-\mathrm{i}\epsilon T/\hbar} \mathrm{e}^{-\mathrm{i}\epsilon^2 X/\hbar^2} \,.$$

   By neglecting the term proportional to ϵ^2 in the exponent prove that

   $$\langle x_f, t_f | x_i, t_i \rangle \approx \int \prod_{n=0}^{N-1} \mathrm{d}x_n \prod_{n=1}^{N} \frac{\mathrm{d}p_n}{2\pi\hbar} \exp\left(\frac{\mathrm{i}}{\hbar} \mathcal{A}^N\right)$$

   where $\mathcal{A}^N$ is the sum

   $$\mathcal{A}^N = \sum_{n=1}^{N+1} \big(p_n(x_n - x_{n-1}) - \epsilon H(t_n, p_n, x_n)\big) \,.$$

   Hint: you may need the inner product of the position and momentum eigenstates:

   $$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{\mathrm{i}px/\hbar} \,.$$

2. For $\epsilon \to 0$ the Riemann sum in the exponent turns into an integral. Identify the Riemann sum and write down the (formal) phase-space path integral in the continuum limit $\epsilon \to 0$.
3. Assume $T(t, p) = p^2/2m$ and perform the integration of the momentum variables. What do you get?
   Hint: perform the integration over the momenta in the discrete version of the phase-space path integral with variables $p_1, \ldots, p_n$, and not in the formal continuum version.

Submission date: Thursday, 21. December 2017, before the lecture begins.