Friedrich-Schiller-Universität Jena Winter term 2017/18
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Problems Advanced Quantum Mechanics

Blatt 1

Aufgabe 1: Relativistic Effects 1+4 = 5 points

This problem relates to the stationary perturbation theory taught in introductory Quantum Mechanics. Please consult your notes or a text book to recall this chapter of Quantum Theory.

The Hamilton operator of a non-relativistic electron with mass m in the spherically Coulomb potential reads

$$H_0 = \frac{p^2}{2m} - \frac{e^2}{r}\,.$$

Now we include relativistic effects in the most simple fashion by considering the relativistic expression for the kinetic energy, which leads to

$$H' = \sqrt{m^2c^4 + p^2c^2} - \frac{e^2}{r}\,.$$

- Expand the kinitic term $\sqrt{m^2c^4 + p^2c^2}$ in powers of $x = p^2/(m^2c^2)$ up to second order and consider the terms not appearing in the non-relativistic Hamiltonian as perturbation (hint: neglect constant terms).
- Calculate the change of the ground state energy in first order perturbation theory (hint: use your knowledge about the non-relativistic hydrogen atom).

Aufgabe 2: Perturbation of harmonic oscillator 1+3+2 = 6 points

Now we consider a harmonic oscillator with Hamiltonian

$$H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2 = \hbar\omega\Big(a^\dagger a + \frac{1}{2}\Big)\,.$$

We perturb the oscillator with an attractive force $-4\lambda x^3$ with is derived from a quartic potential

$$V(x) = \lambda x^4 = \frac{\lambda}{16\zeta^4}\Big(a + a^\dagger\Big)^4 \equiv \frac{\lambda}{16\zeta^4}\Delta\,.$$

- Determine the constant ζ?
- Multiply out the quartic polynomial $\Delta = (a + a^\dagger)^4$ and collect terms which change the occupation number $N = a^\dagger a$ by the same amount. With the help of $aa^\dagger = a^\dagger a + 1 = N + 1$ bring the operator Δ into the form

$$\Delta = P_1(N) + aP_2(N)a + a^\dagger P_2(N)a^\dagger + a^4 + a^{\dagger 4}$$

Determine the terms P_1 und P_2.

- Calculate the change of energies $E_n = \hbar\omega(n+1/2)$ of the harmonic oscillator in first order perturbation theory.

Submission date: Tuesday, 24. October 2017, before the lecture begins

Solutions of problems in series 1

Aufgabe 1: Relativistic Effects

(a) By using the Taylor expansion: $\sqrt{1+x} = 1 + x/2 - x^2/8 + O(x^3)$ one obtains:

$$\sqrt{m^2c^4 + p^2c^2} = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + O\left(\frac{p^6}{m^6c^6}\right).$$

Dropping the rest energy, the perturbation reads:

$$H' - H_0 = \lambda V = -\frac{1}{2mc^2}\left(\frac{p^2}{2m}\right)^2 = -\frac{1}{2mc^2}\left(H_0 + \frac{e^2}{r}\right)^2.$$

(b) We will use the last expression for the perturbation, in terms of H_0 and $1/r$, in the computation of the matrix element

$$E_0^{(1)} = \langle 100|\lambda V|100\rangle = -\frac{1}{2mc^2}\left(E_0^2 + 2E_0\langle\frac{e^2}{r}\rangle + \langle\frac{e^4}{r^2}\rangle\right)$$

where $E_0 = -e^2/2a$ is the ground state energy for H_0, and $a = \hbar^2/me^2$ is the Bohr radius. For the ground state, with wave function

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

one has

$$\langle\frac{e^2}{r}\rangle = \frac{e^2}{a} = -2E_0\ , \quad \langle\frac{e^4}{r^2}\rangle = 2\frac{e^4}{a^2} = 8E_0^2$$

so in conclusion

$$E_0^{(1)} = -\frac{5E_0^2}{2mc^2} = \frac{5}{4}\alpha^2 E_0.$$

Here $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant, hence $E_0^{(1)}/E_0 \ll 1$.

Aufgabe 2: Störung des harmonischen Oszillators

This solution follows my notes on QM I, 11.1.1 and 11.1.2.
(a) The annihilation operator reads

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{ip}{\sqrt{2m\hbar\omega}} = \zeta x + \frac{ip}{2\hbar\zeta}$$

so

$$\zeta = \sqrt{\frac{m\omega}{2\hbar}}$$

that has dimension of an inverse length, and $x = (a + a^\dagger)/2\zeta$.
(b) First of all we expand

$$\begin{aligned}(a+a^\dagger)^4 &= a^4 + {a^\dagger}^4 + \left(a^3a^\dagger + a^2a^\dagger a + aa^\dagger a^2 + a^\dagger a^3\right)\\ &+ \left({a^\dagger}^3 a + {a^\dagger}^2 a a^\dagger + a^\dagger a {a^\dagger}^2 + a{a^\dagger}^3\right)\\ &+ \left(a^2{a^\dagger}^2 + {a^\dagger}^2a^2 + aa^\dagger aa^\dagger + a^\dagger a a^\dagger a + a{a^\dagger}^2 a + a^\dagger a^2 a^\dagger\right)\end{aligned}$$

Then we need to use the following relations:

$$\begin{aligned} aa^\dagger &= N+1 \\ aN &= (N+1)a\,, \quad Na = a(N-1) \\ a^\dagger N &= (N-1)a^\dagger\,, \quad Na^\dagger = a^\dagger(N+1) \end{aligned}$$

where $N = a^\dagger a$. The first bracket on the right hand side (rhs) of Eq.(1) consists of

$$\begin{aligned} a^3 a^\dagger &= a(N+2)a \\ a^2 a^\dagger a &= a(N+1)a \\ a a^\dagger a^2 &= aNa \\ a^\dagger a^3 &= a(N-1)a \end{aligned}$$

and these sum up to: $aP_2(N)a = 2a(2N+1)a$. The second bracket on the rhs of Eq.(1) gives the Hermitean conjugate of this result. The last bracket on the rhs of Eq.(1) consists of

$$\begin{aligned} a^2 {a^\dagger}^2 &= N^2 + 3N + 2 \\ {a^\dagger}^2 a^2 &= N^2 - N \\ aa^\dagger aa^\dagger &= N^2 + 2N + 1 \\ a^\dagger a a^\dagger a &= N^2 \\ aa^\dagger a^\dagger a &= N^2 + N \\ a^\dagger a a a^\dagger &= N^2 + N \end{aligned}$$

and these sum up to: $P_1(N) = 3(2N^2 + 2N + 1)$.
(c) Each eigenstate $|n\rangle$ is nondegenerate, hence the first-order correction to the energies is

$$E_n^{(1)} = \frac{\lambda}{16\zeta^4}\langle n|\Delta|n\rangle.$$

Recalling that

$$\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}\;|n+1\rangle\,, \quad a|n\rangle = \sqrt{n}\;|n-1\rangle \\ \langle n|a^\dagger &= \langle n-1|\sqrt{n}\,, \quad \langle n|a = \langle n+1|\sqrt{n+1} \end{aligned}$$

one sees that only the first contribution to Δ, namely $P_1(N)$, has a nonvanishing diagonal matrix element. Therefore

$$E_n^{(1)} = \frac{\lambda P_1(n)}{16\zeta^4} = \frac{3\lambda}{16\zeta^4}(2n^2 + 2n + 1) = \hbar\omega\frac{3}{8}\frac{\lambda}{\lambda_0}(2n^2 + 2n + 1)$$

where $\lambda_0 = m^2\omega^3/2\hbar$.