Friedrich-Schiller-Universität Jena Winter Term 2017/18
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Problems in Advanced Quantum Mechanics

Problem Sheet 13

Problem 29: Covariantly conserved current 2+2 Points

In presence of an external electromagnetic field with vector potential $A_\mu = (\phi, -\boldsymbol{A})$, the spinor ψ fulfills the Dirac equation

$$\left(\mathrm{i}\gamma^\mu D_\mu - \frac{mc}{\hbar}\right)\psi = 0\,,$$

where $D_\mu = \partial_\mu + \frac{\mathrm{i}e}{\hbar c}A_\mu$. Which equation is then fulfilled by the conjugate Dirac spinor $\bar{\psi}$? Show also that the four-current

$$j^\mu = e\bar{\psi}\gamma^\mu\psi\,,$$

is covariantly conserved, i.e. $\partial_\mu j^\mu = 0$, for such ψ and $\bar{\psi}$.

Problem 30: Antisymmetric tensor Dirac bilinear 2+2 Points

Explicitly derive the transformation of the antisymmetric Dirac bilinear

$$T^{\mu\nu}(x) = \bar{\psi}(x)[\gamma^\mu, \gamma^\nu]\psi(x)$$

under Lorentz transformations with matrix representative $\Lambda^\alpha{}_\beta$, and then under a parity transformation.

Hint: concerning the behavior under parity, it might be useful to compare your result to the corresponding transformation properties of the electromagnetic field-strength tensor $F^{\mu\nu}$.

Problem 31: Plane wave solutions 2+4 Points

In the lecture it was stated that a free particle solution of the Dirac equation takes the following form

$$\psi_p(x) = \frac{1}{\sqrt{2\omega(\boldsymbol{k})}}e^{-\mathrm{i}kx}u_p\,,$$

where $p = \hbar k$ and p is on the mass shell, i.e. $p^2 = m^2c^2$. Let us restrict to positive frequency solutions, and use the chiral representation of Dirac matrices.

- In the rest frame of the particle, where $p = \{mc, \boldsymbol{0}\}$, show that the general solution has

  $$u_p = u_R = \sqrt{mc}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\,,$$

  where ξ is any constant two-components spinor.

- The frame where p is generic can be obtained from the rest frame by a Lorentz boost. Choose the z-axis parallel to $\boldsymbol{p}$ and suppress the other components, such that $p =$

$\{E/c, p_3\}$, and perform a Lorentz boost along the z-direction with rapidity $\alpha = \tanh^{-1}(p^3 c/E)$ (notice that $p^3 = -p_3$). The Dirac spinor changes by $u_p = S(A) u_R$ with

$$S(A) = \exp\left\{ -\frac{\alpha}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right\} .$$

Show that this results in

$$u_p = \begin{pmatrix} \sqrt{p_\mu \tilde{\sigma}^\mu}\, \xi \\ \sqrt{p_\mu \sigma^\mu}\, \xi \end{pmatrix} ,$$

where $\sigma^\mu = \{\sigma_0, -\sigma_k\}$, $\tilde{\sigma}^\mu = \{\sigma_0, \sigma_k\}$, and the square root of a diagonal matrix is the diagonal matrix with square-rooted entries.

Submission date: Thursday, 01.02.2018, before the lecture begins.