Prof. Andreas Wipf, MSc. Marc Steinhauser January 14, 2020

Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 11

Problem 26: Gamma Matrices 2+1+1+1+2 = 7 points

In the chiral representation the Dirac matrices have the form

$$\gamma^0 = \sigma_1 \otimes \sigma_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^k = -\mathrm{i}\sigma_2 \otimes \sigma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

and in the Dirac representation

$$\gamma^0 = \sigma_3 \otimes \sigma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma^k = \mathrm{i}\sigma_2 \otimes \sigma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}.$$

1. Show that these two sets of matrices obey the anti-commutation relations (ACR) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Hint: You may use the rules for tensor products, e.g. $(A \otimes B)(C \otimes D) = AC \otimes BD$.
2. What are the hermiticity properties of the γ^μ? Why can γ^1 not be hermitean?
3. Calculate $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ for both representations.
4. Use (only) the ACR for the γ^μ to prove that γ_5 anti-commutes with the γ^μ, $\{\gamma_5, \gamma^\mu\} = 0$.
5. With the help of the ACR prove the identities

$$\gamma^\mu \gamma_\mu = 4\mathbb{1}$$
$$\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$$
$$\gamma^\mu \not{p}\not{q} \gamma_\mu = 4p \cdot q\mathbb{1},$$

with $\not{p} = p^\mu\gamma_\mu$ and $p \cdot q = p^\mu q_\mu$ (split here: 0.5+0.5+1 points).

Problem 27: Lagrangian of the free Dirac equation 1+1+2+1+1 = 6 points

The Lagrangian of the free Dirac field theory (describing a charged particle with spin 1/2) is given by

$$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi\,.$$

Given that the component fields ψ_a of ψ are complex valued it will be convenient to treat ψ_a and $\bar{\psi}_a$ as independent field variables (instead of considering real and imaginary parts of ψ_a as the independent).

1. Show, that $\mathcal{L}_D$ is Lorentz-invariant, $\mathcal{L}_D(x') = \mathcal{L}_D(x)$.
2. Derive the field equations for ψ and $\bar{\psi}$ and show that they are related by complex conjugation and multiplication by γ^0.
3. Consider the symmetry transformation $\psi(x) \mapsto e^{i\alpha}\psi(x)$ and the related transformation of $\bar{\psi}(x)$. Show that this $U(1)$ transformation leaves the Lagrangian $\mathcal{L}_D$ invariant and compute the associated Noether current density $j^\mu(x)$.
4. Show that j^μ is covariantly conserved, $\partial_\mu j^\mu(x) = 0$, given that ψ and $\bar{\psi}$ solve the Dirac equation.
5. What is the Lagrangian density for a charged spin-1/2 particle in an external electromagnetic field described by a 4-potential $A_\mu(x)$?

Submission date: Thursday, 23.01.2020, before the lecture