Prof. Andreas Wipf, MSc. Marc Steinhauser December 18, 2019

Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 10

Problem bonus task: : Intervals in Minkowski spacetime 1+1+1+1=4 points

Let P and Q denote two events in Minkowski space with coordinates $x = (x^\mu)$ and y.

1. Show that for spacelike separated events there exist inertial systems in which they are simulatenous and show that there are inertial systems in which the time ordering of the two events is reversed.
2. Show that for timelike separated events there is an inertial system, in which P and Q are at the same point in space (assume that Q is at the origin of the coordinate system).
3. For lightlike separated events: determine the hyper surface in spacetime, on which Q is lightlike separated from P.

Hint: No lengthy calculation are required. Try to argue geometrically. Discuss the image set $\{\eta = \Lambda\xi\}$ for spacelike, timelike and lightlike difference vectors $\xi = y - x$ (connecting P with Q) when Λ runs thru the set of Lorentztransformations. For example, you may chose as space-, time- and lightlike vectors $\xi = (0, 1, 0, 0), (1, 0, 0, 0)$ and $\xi = (1, 1, 0, 0)$.

Problem 23: Solutions of the wave equation in $d = 1 + 1$ 2+1+1+2=6 points

Consider the wave equation $\Box\phi = 0$ in 2 space-time dimensions. It is the Klein-Gordon equation for a massless particle (in 2 space-time dimensions).

1. Characterized the general solution of
$$\Box\phi = \left(\frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi = 0\,.$$
Hint: Introduce light-cone coordinates $x^- = ct - x$ and $x^+ = ct + x$.
2. A solution is uniquely given by the initial field $\phi|_{t=0} = \phi_0(x)$ and the initial "velocity field" $\partial_t\phi|_{t=0} = c\phi_1(x)$. Express the general solution in terms of the initial fields ϕ_0 and ϕ_1.
3. Let
$$\phi_0 = \mathrm{e}^{-x^2/2\sigma^2} \quad \text{and} \quad \phi_1 = \phi_0(x)\cdot\sin(kx)\,.$$
How does the solution with this initial condition look like?
Hint: After an appropriate substitution you will encounter the error-function
$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}z\ \mathrm{e}^{-z^2}\ .$$
4. Plot the solutions for $ct = 0, 1, 2, 3, 4$ and 5. Choose $\sigma = 1$ (or phrased differently, give x and ct in multiples of σ) and $k\sigma = 1$.

Problem 24: Klein Paradox 1,5+2+1+1 = 5,5 points

A particle with positive charge e, momentum p and energy E (in $1+1$ dimensions with coordinates (t,x)) hits a electrostatic barrier $A_0(x) = U\theta(x)$ with $U > 0$ at the origin,

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0\,. \end{cases}$$

Find the scattering solutions of the Klein-Gordon equation

$$\left\{ \frac{1}{c^2}\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - eA_0\right)^2 + \hbar^2\frac{\partial^2}{\partial x^2} - m^2c^2 \right\}\phi = 0\,.$$

Assume, that the particles are moving from the left towards the barrier. Then we have for $x < 0$

$$\phi(t,x) = \mathrm{e}^{-\mathrm{i}Et/\hbar}\left(\mathrm{e}^{\mathrm{i}px/\hbar} + \mathcal{R}\mathrm{e}^{-\mathrm{i}px/\hbar}\right)\,,$$

with reflection amplitude $\mathcal{R}$ and for $x > 0$

$$\phi(t,x) = \mathcal{T}\mathrm{e}^{-\mathrm{i}Et/\hbar}\mathrm{e}^{\mathrm{i}qx/\hbar}$$

with transmission amplitude $\mathcal{T}$.

1. What is the relation between energy and momentum to the left and right of the barrier. What can you say about the qualitative behavior of the solution in the three energy-intervals

$$E \geq eU + mc^2, \qquad eU - mc^2 < E < eU + mc^2, \qquad E < eU - mc^2\,?$$

2. Impose that a solution and its derivative are continuous at $x = 0$ and calculate the coefficients $\mathcal{R}$ and $\mathcal{T}$ for energies in the three intervals given.
3. What can you say (without further calculation) about the probability $|\mathcal{R}|^2$ in the intervals

$$-mc^2 \leq E \leq -mc^2 + eU \quad \text{and} \quad mc^2 \leq E \leq mc^2 + eU\,?$$

4. Calculate the group velocity for $x > 0$ and energies $E < -mc^2$. This velocity should be positive. Which seemingly unphysical property of $\mathcal{R}$ follows?

Submission date: Thursday, 16.01.2020, before the lecture