Prof. Andreas Wipf, MSc. Marc Steinhauser November 27, 2019

Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 7

Problem 15: Harmonic oscillator coupled to 2-state system 2+1 = 3 points

A one-dimensional harmonic oscillator and a 2-state system are described by $H_0 = \hbar\omega_0 a^\dagger a + \varepsilon\sigma_z$. The two systems are coupled by the interaction $V(t) = g(\sigma_+ a + \sigma_- a^\dagger)e^{\eta t}$. The total Hamiltonian is given by $H = H_0 + V(t)$.

1. Calculate in first order perturbation theory the transition rate from the state $|n\rangle \otimes |\uparrow\rangle$ into the state $|n+1\rangle \otimes |\downarrow\rangle$ of the unperturbed Hamiltonian H_0. Choose as lower integration limit $t = -\infty$.
2. What happens to the transition rate in the limit $\eta \to 0$?

Problem 16: H-atom between the plates of a capacitor 1+3=4 points

A hydrogen atom in its ground state is placed between two parallel plates of a capacitor. An impulse voltage produces a spatially homogeneous electric pulse

$$\boldsymbol{E}(t) = -E_0\,\theta(t)\,\mathrm{e}^{-t/\tau}\,\boldsymbol{e}_z, \quad \tau > 0,$$

between the plates, and orthogonal to them (parallel to the z-axis with unit vector $\boldsymbol{e}_z$). Calculate in first order perturbation theory the transition probability, that the atom at $t > 0$ is

1. in the $2s$ state
2. in one of the $2p$ states.

What happens for $\tau \to \infty$?
Hint: You might need the explicit form of some of the following wave functions of the Hydrogen atom $\langle \boldsymbol{r}|n\ell m\rangle = R_{n\ell}(r)Y_{\ell m}(\theta,\varphi)$:

$$R_{10}(r) = \frac{2}{a^{3/2}}\mathrm{e}^{-r/a}, \quad R_{20}(r) = \frac{2}{(2a)^{3/2}}(1 - r/2a)\mathrm{e}^{-r/2a}, \quad R_{21}(r) = \frac{1}{\sqrt{3}(2a)^{3/2}}\frac{r}{a}\mathrm{e}^{-r/2a},$$

$$Y_{00}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}, \qquad Y_{10}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta, \qquad Y_{1\pm1}(\theta,\varphi) = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\varphi}.$$

Problem 17: Rabi oscillations 3+2+1=6 points

Given is the Hamilton operator $H(t) = H_0 + V(t)$ with

$$H_0 = E_1\,|1\rangle\langle 1| + E_2\,|2\rangle\langle 2|\,, \qquad V(t) = \hbar\omega_0\mathrm{e}^{\mathrm{i}\omega t}|1\rangle\langle 2| + \hbar\omega_0\mathrm{e}^{-\mathrm{i}\omega t}|2\rangle\langle 1|\,,$$

with positive $\omega, \omega_0 > 0$ and with $E_2 > E_1$. The two states $|1\rangle, |2\rangle$ form an orthonormal basis of the Hilbert space. Find the state $|\psi(t)\rangle$, which solves the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

with initial condition $|\psi(t=0)\rangle = |1\rangle$.

1. Find the exact solution of the problem. To find the solution you may
   - Study the time evolution of the state $|\psi(t)\rangle$ in the base $|n\rangle$: $|\psi(t)\rangle = c_1(t)\mathrm{e}^{-\mathrm{i}E_1 t/\hbar}|1\rangle + c_2(t)\mathrm{e}^{-\mathrm{i}E_2 t/\hbar}|2\rangle$. Which initial conditions fulfill the coefficients $c_1(t)$ and $c_2(t)$?
   - Insert the state vector $|\psi(t)\rangle$ into the Schrödinger equation. You will obtain two coupled differential equations for $c_1(t)$ and $c_2(t)$.
   - Solve these equations.
2. Solve the problem in first order perturbation theory.
3. Compare the perturbative result with the exact solution.

Submission date: Thursday, 05.12.2019, before the lecture