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Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 4

Problem 7: Feynman-Hellman Theorem 2 points

A Hamiltonian depends smoothly on a parameter λ (coupling strength, external field, ...). Let us assume, the normalized eigenfunctions are known for all λ

$$H(\lambda)|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle, \qquad \langle\psi_n(\lambda)|\psi_n(\lambda)\rangle = 1 \,.$$

Now prove the Feynman-Hellman Theorem for the expectation value of the Hamilton operator,

$$\frac{d}{d\lambda}E_n(\lambda) = \left\langle\psi_n(\lambda)\right|\frac{dH(\lambda)}{d\lambda}\left|\psi_n(\lambda)\right\rangle .$$

Remark: This theorem can be useful to follow the parameter dependence of energies.

Problem 8: Variational principle applied to the anharmonic oscillator 3+1 = 4 points

As Hamiltonian for the one-dimensional anharmonic oscillator we choose

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \frac{m^2\omega^3}{10\hbar}x^4 \,.$$

1. Apply the Rayleigh-Ritz variational principle with trial functions
$$\psi_\alpha(x) = c\, e^{-\alpha x^2/2}, \quad \alpha > 0$$
to find an approximate value for the ground state energy.
2. Compare with the exact ground state energy $E_0 = 0,559146\,\hbar\omega$ and comment the result.

Hint: Use units with $\hbar = m = \omega = 1$ and use the well-known result

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\alpha x^2} = \left(\frac{\pi}{\alpha}\right)^{1/2} .$$

The relevant moments are obtained by differentiation with respect to α.

Problem 9: Two particles with total angular momentum zero 2+2 = 4 points

Given two particles, each with angular momentum j. Prove, that the wave function of the total two-particle system with vanishing total angular momentum (the singlet) can be written as

$$|\Psi_0^0\rangle = \frac{1}{2j+1}\sum_{m=-j}^{j}(-1)^{m+1/2}|\psi_j^m\rangle\otimes|\psi_j^{-m}\rangle, \quad j \in \mathbb{N}_0 + \frac{1}{2}\,,$$

$$|\Psi_0^0\rangle = \frac{1}{2j+1}\sum_{m=-j}^{j}(-1)^{m}|\psi_j^m\rangle\otimes|\psi_j^{-m}\rangle, \qquad j \in \mathbb{N}_0\,.$$

Hint: Recall how the the step operators $J_\pm = J_x \pm \mathrm{i}J_y$ and the component J_z of the total angular momentum $\boldsymbol{J} = \boldsymbol{J}^{(1)} + \boldsymbol{J}^{(2)}$ act on a product state $|\psi_j^m\rangle\otimes|\psi_j^{m'}\rangle$ and use that, for example, $c_{jm}^+ = c_{jm'}^+$ for certain m, m'. It is sufficient to prove the result for an integer (or a half-integer) value of j.

Submission date: Thursday, 14.11.2019, before the lecture