Problems: Quantum Fields on the Lattice

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Sheet 2

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3 Semi-classical expansion of the partition function

In the lecture we discussed the path integral representation of the thermal partition function, given by

$$Z(\beta) = C\int \mathrm{d}q \int_{q(0)=q}^{q(\hbar\beta)=q} \mathcal{D}q\, \mathrm{e}^{-S_{\mathrm{E}}[q]/\hbar}\ .$$

We rescale the imaginary time and the amplitude according to

$$\tau \longrightarrow \hbar\tau \quad \text{and} \quad q(.) \longrightarrow \hbar q(.)\ .$$

After this rescaling the 'time interval' is of length β instead of $\hbar\beta$ and

$$Z(\beta) = C\int \mathrm{d}q \int_{q(0)=q/\hbar}^{q(\beta)=q/\hbar} \mathcal{D}q\, \exp\left\{-\int_0^\beta \left(\frac{1}{2}m\dot{q}^2 + V(\hbar q(.))\right)\mathrm{d}\tau\right\}\ .$$

For a moving particle the kinetic energy dominates the potential energy for small $\hbar$. Thus we decompose each path into its constant part and the fluctuations about the constant part: $q(.) = q/\hbar + \xi(.)$. Show that

$$Z(\beta) = \frac{C}{\hbar}\int \mathrm{d}q \int_{\xi(0)=0}^{\xi(\beta)=0} \mathcal{D}\xi\, \exp\left\{-\int_0^\beta \left(\frac{1}{2}m\dot{\xi}^2 + V(q+\hbar\xi)\right)\mathrm{d}\tau\right\}\ .$$

Determine the constant C by considering the limiting case $V = 0$ with the well-known result $Z(\beta, q, q) = (m/2\pi\beta\hbar^2)^{1/2}$. Then expand the integrand in powers of $\hbar$ and prove the intermediate result

$$Z = \frac{C}{\hbar}\int \mathrm{d}q\, \mathrm{e}^{-\beta V(q)} \int_{\xi(0)=0}^{\xi(\beta)=0} \mathcal{D}\xi\, \mathrm{e}^{-\frac{1}{2}m\int \mathrm{d}\tau \dot{\xi}^2} \times \left\{1 - \hbar V'(q)\int \xi(\tau) - \frac{1}{2}\hbar^2\left(V''(q)\int \xi^2(\tau) - V'^2(q)\int \xi(\tau)\int \xi(s)\right) + \cdots\right\}\ .$$

Conditional expectation values as

$$\langle \xi(\tau_1)\xi(\tau_2)\rangle = \langle \xi(\tau_2)\xi(\tau_1)\rangle = C\int_{\xi(0)=0}^{\xi(\beta)=0} \mathcal{D}\xi\, \mathrm{e}^{-\frac{1}{2}m\int \mathrm{d}\tau\,\dot{\xi}^2}\xi(\tau_1)\xi(\tau_2)$$

are computed by differentiating the generating functional

$$C\int_{\xi(0)=0}^{\xi(\beta)=0} \mathcal{D}\xi\, \mathrm{e}^{-\frac{1}{2}m\int \mathrm{d}\tau\,\dot{\xi}^2 + \int \mathrm{d}\tau\, j\xi} = \sqrt{\frac{m}{2\pi\beta}}\exp\left(\frac{1}{m\beta}\int_0^\beta \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'(\beta-\tau)\tau' j(\tau)j(\tau')\right)\ .$$

Prove this formula for the generating functional and compute the leading and sub-leading contributions in the semi-classical expansion.

4 High-temperature expansion of the partition function

Analyze the temperature dependence of the partition function (set $\hbar = 1$). Repeat the calculation in problem 3 but this time with the rescalings

$$\tau \longrightarrow \beta\tau \quad \text{and} \quad \xi \longrightarrow \sqrt{\beta}\xi \, ,$$

and show that

$$Z(\beta) = \frac{C}{\sqrt{\beta}} \int \mathrm{d}q \int_{\xi(0)=0}^{\xi(1)=0} \mathcal{D}\xi \exp\left\{ -\int_0^1 \left(\frac{m}{2}\dot{\xi}^2 + \beta V\left(q + \sqrt{\beta}\xi\right)\right) \mathrm{d}\tau \right\} \, .$$

Expand $Z(\beta)$ in powers of the inverse temperature and use the generating functional in problem 3 (with $\beta = 1$) to compute the correlation functions. The remaining integrals over correlation functions are easily calculated. Determine the contributions of order $T^{1/2}, T^{-1/2}$ and $T^{-3/2}$ in the high-temperature expansion of $Z(\beta)$.