# Problems: Quantum Fields on the Lattice

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**Sheet 5**

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## 11 Link variables

Typically, the degrees of freedom of lattice gauge theories are matrices $U_{x\mu}$ (link variables) at lattice point $x$ in direction $\mu$ and have a group structure. For the gauge group $\mathrm{SU}(n)$ the link variables obey

$$\forall x, \mu : \quad \det U_{x\mu} = 1, \quad U_{x\mu}^\dagger = U_{x\mu}^{-1} \;\Leftrightarrow\; U_{x\mu} U_{x\mu}^\dagger = \mathbb{1}_{x\mu}. \tag{1}$$

In principle, every group element $U_{x\mu} \in \mathrm{SU}(n)$ could be parametrized in terms of the generators $T_a \in \mathfrak{su}(n)$:

$$U_{x\mu} = \exp\left(\mathrm{i}\omega_{x\mu}^a T^a\right) \quad \text{with} \quad \omega_{x\mu}^a \in \mathbb{R}. \tag{2}$$

However, in numerical calculations the $U$s are often represented by complex matrices $\mathbb{C}^{n\times n}$ such that round-off errors will drive them away from the group.

1. Describe a method how a complex $3 \times 3$ matrix $A$ can be "unitarized" by the Gram-Schmidt method.

2. In the special case of $\mathrm{SU}(2)$ one can alternatively use quaternions

$$U_{x\mu} = a_{x\mu}^0 \mathbb{1} + a_{x\mu}^k \sigma^k \in \mathrm{SU}(2) \tag{3}$$

where $\sigma^k$ are the Pauli matrices.

   (a) Find the condition $0 = f(a^0, \dots, a^3)$ such that $U_{x\mu} \in \mathrm{SU}(2)$.
   (b) Compute the product of two link variables in this parametrization, i.e. compute $c(a, b)$ in $U(c) = U(a)U(b)$.
   (c) Compute the trace of a plaquette as a function of the parametrizing vectors, i.e.

$$\mathrm{tr}(P_{x,\mu\nu}) = p(a_{x\mu}, a_{x+\hat{\mu},\nu}, a_{x+\hat{\nu},\mu}, a_{x\nu}) \tag{4}$$

## 12 Continuum limit of the plaquette

Show that in the classical continuum limit

$$\mathrm{tr}(P_{x,\mu\nu}) = -\frac{a^4}{2}\mathrm{tr}(F_{x,\mu\nu}F_{x,\mu\nu}) - \frac{a^5}{2}\mathrm{tr}(F_{x,\mu\nu}[D_\mu + D_\nu]F_{x,\mu\nu}) \tag{5}$$

where $a$ is the lattice separation and $D_\mu = \partial_\mu + A_\mu$.

## 13 Gauge covariance of parallel transport

Consider the parallel transport along an open path from $x(0)$ to $x(s)$

$$P_C = \mathbf{P}\exp\left(\mathrm{i}g\int_0^s \mathrm{d}s'\ \dot{x}^\mu A_\mu(x)\right) \tag{6}$$

with non-abelian gauge fields $A_\mu \in \mathfrak{su}(n)$. Show that under a gauge transformation on $A_\mu$

$$A'_\mu = \Omega(x)A_\mu(x)\Omega^{-1}(x) - \frac{\mathrm{i}}{g}\partial_\mu\Omega(x)\Omega^{-1}(x) \tag{7}$$

$P_C$ transforms as $P'_C = \Omega(x(s))P_C\Omega^{-1}(x(0))$.

## 14 (Bonus) More On The Ising Model

*(Note: This problem will not be discussed in the exercise class unless explicit questions or the need for discussions arises during your preparations.)*
The given code template for the Ising model from Problem 10 was written in a modular fashion where all the parts are interchangeable. It can be easily generalized to other scenarios. If you want to explore the Ising model and its relatives further, here are some suggestions:

- Without any changes the code should run in arbitrary dimensions (Caution: Not thoroughly tested!).
  1. In 1D, there is no phase transition at finite temperature. This is what Ising originally found.
  2. In 2D, you could look in more detail at the critical behavior. Extract, for example, the critical temperature and critical exponents via the various methods described in W. JANKEs *Monte Carlo Methods in Classical Statistical Physics* (see moodle for a link) and compare with the analytical results.
  3. In 3D, there are no analytical results. However, it is of course well-studied by now. See if you can get reasonable accuracy here.
  4. From 4D on, mean-field theory applies. Research the mean-field results (or do the calculations yourself) and compare.
- Implement further observables, e.g. the correlation length or correlation functions. Take a look into *improved estimators* and compare the expectation value of the cluster size in the WOLFF algorithm with the susceptibility.
- Implement a next-to-nearest neighbor (NNN) coupling term. To do so, implement a `getNNN(x)` method in the `Geometry` and change the `expS` method of the `Updater` class. The most interesting case here is a competing setup between NN and NNN coupling.
- Implement another geometry, e.g. the triangular one from Problem 6, and look at the curious behavior of geometric frustration. It might be necessary here to implement further observables, too.
- The Ising model is actually the $\mathrm{O}(1)$ model and in that sense the simplest of the large class of $\mathrm{O}(N)$ models. Change the `Field` class to an $\mathrm{O}(N)$ field (or another special case of this). By smart use of operator overloading the other parts of the code should not need many changes.