Friedrich-Schiller-Universität Jena Summer Term 2020
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Problems Quantum Field Theory

Sheet 5

Problem 16: Coherent states

Consider a hermitian scalar field $\phi(\mathbf{x})$ given by

$$\phi(\mathbf{x}) = \int d\mu(\mathbf{p}) \left(a_{\mathbf{p}} u_{\mathbf{p}}(\mathbf{x}) + a^\dagger_{\mathbf{p}} u^*_{\mathbf{p}}(\mathbf{x}) \right) \quad ,$$

where we use notation established on previous exercise sheets. Now consider the coherent state $|\eta\rangle$, defined by

$$|\eta\rangle = \exp\left[\int d\mu(\mathbf{p}) \eta(\mathbf{p}) a^\dagger_{\mathbf{p}} \right] |0\rangle \quad ,$$

where $\eta(\mathbf{p})$ is a normalizable function defined on the mass shell $p^2 = m^2$, $p^0 > 0$:

$$\int d\mu(\mathbf{p}) |\eta(\mathbf{p})|^2 < \infty \quad .$$

1. Show that $|\eta\rangle$ is an eigenstate of the annihilation operator $a(\mathbf{p})$ and therefore diagonalizes the annihilation part of $\phi(\mathbf{x})$.

2. Show that coherent states $|\eta_1\rangle$ and $|\eta_2\rangle$ are non-orthogonal for $\eta_1(\mathbf{p}) \neq \eta_2(\mathbf{p})$. Are they normalized?

3. Let $|\{n, \mathbf{p}\}\rangle = |n_1, \mathbf{p}_1; n_2, \mathbf{p}_2; ...; n_N, \mathbf{p}_N\rangle$ denote a state in Fock space with n_1 particles of momentum $\mathbf{p}_1$, n_2 particles of momentum $\mathbf{p}_2$, etc. Show that $|\eta\rangle$ can be used as a generating function for $|\{n, \mathbf{p}\}\rangle$.

4. Now define ϕ_f as the smearing of $\phi(\mathbf{x})$ (which is an operator-valued distribution) with some suitable test function $f(\mathbf{x})$:

$$\phi_f = \int d^3x f(\mathbf{x}) \phi(\mathbf{x}) \quad .$$

   One may study the probability distribution of ϕ_f in a state $|\{n, \mathbf{p}\}\rangle$, defined as

$$\rho_{\{n,\mathbf{p}\}}(\alpha) = \langle \{n, \mathbf{p}\} | \delta(\phi_f - \alpha) | \{n, \mathbf{p}\} \rangle = \int \frac{ds}{2\pi} e^{-i\alpha s} \langle \{n, \mathbf{p}\} | e^{is\phi_f} | \{n, \mathbf{p}\} \rangle \quad .$$

   Since $|\eta\rangle$ generates the $|\{n, \mathbf{p}\}\rangle$, the problem of finding $\rho_{\{n,\mathbf{p}\}}(\alpha)$ reduces to that of finding $\langle \eta_1 | e^{is\phi_f} | \eta_2 \rangle$. Thus, compute $\langle \eta_1 | e^{is\phi_f} | \eta_2 \rangle$. Then, show explicitly that the vacuum distribution $\rho_0(\alpha) = \langle 0 | \delta(\phi_f - \alpha) | 0 \rangle$ is a Gaussian distribution centered around $a = 0$ with a variance $\sigma^2 = \int d\mu(\mathbf{p}) |\tilde{f}(\mathbf{p})|^2$, where

$$\tilde{f}(\mathbf{p}) = \int d^3x \, u_{\mathbf{p}}(\mathbf{x}) f(\mathbf{x})$$

   is the Fourier transform of $f(\mathbf{x})$.

Problem 17: Heisenberg picture

Consider a charged scalar field in the Schrödinger picture:

$$\phi(\mathbf{x}) = \int d\mu(\mathbf{p}) \left(a_{\mathbf{p}} u_{\mathbf{p}}(\mathbf{x}) + b_{\mathbf{p}}^{\dagger} u_{\mathbf{p}}^{*}(\mathbf{x}) \right) \quad .$$

Now transition to the Heisenberg picture by introducing time dependent operators

$$a_{\mathbf{p}}(t) = e^{iHt} a_{\mathbf{p}} e^{-iHt} \quad ,$$

etc., where H denotes the Hamiltonian of the theory. Show that, in the Heisenberg picture,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d\mu(\mathbf{p}) \left(a_{\mathbf{p}} e^{-ipx} + b_{\mathbf{p}}^{\dagger} e^{ipx} \right),$$

where $px = x_{\mu} p^{\mu}$.

Problem 18: Helicity

The angular momentum of the radiation field is given by

$$\mathbf{J} = \int d^3x \, \mathbf{x} \wedge (\mathbf{E} \wedge \mathbf{B}) \quad .$$

Define the corresponding operator. Then, consider the component of $\mathbf{J}$ in the direction of propagation, i.e., the projection

$$\mathbf{J} \cdot \hat{\mathbf{k}} \quad ,$$

where $\hat{\mathbf{k}}$ denotes the normalized vector. For a state

$$|\mathbf{k}, \epsilon\rangle = (\alpha \, a_{\mathbf{k},1}^{\dagger} + \beta \, a_{\mathbf{k},2}^{\dagger})|0\rangle \quad ,$$

determine the coefficients α und β, such that the (normalized) state is an eigenstate of $\mathbf{J} \cdot \hat{\mathbf{k}}$. (Here, the expansion of the field in terms of momentum modes and the operators $a_{\mathbf{k},1/2}^{\dagger}$ are defined as in the lecture).