Friedrich-Schiller-Universität Jena Summer Term 2020
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Problems Quantum Field Theory

Sheet 3

Problem 10: Lorentz-invariant measure

Show that the integral measure

$$d\mu(\mathbf{p}) = \frac{d^3p}{2\omega_{\mathbf{p}}} \quad ,$$

with $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ is Lorentz invariant on the mass shell $p^2 = m^2$ and for $p_0 > 0$.

Problem 11: Commutation relations for creation and annihilation operators

Consider the decomposition of a real scalar field $\phi(x)$ and its conjugate momentum $\pi(x)$ into normal modes $u_{\mathbf{p}}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}}$ introduced in the lecture:

$$\begin{aligned} \phi(\mathbf{x}) &= \int d\mu(\mathbf{p}) \left(a_{\mathbf{p}} u_{\mathbf{p}}(\mathbf{x}) + a_{\mathbf{p}}^\dagger u_{\mathbf{p}}^*(\mathbf{x}) \right) \quad , \\ \pi(\mathbf{x}) &= \frac{1}{i} \int d\mu(\mathbf{p}) \omega_{\mathbf{p}} \left(a_{\mathbf{p}} u_{\mathbf{p}}(\mathbf{x}) - a_{\mathbf{p}}^\dagger u_{\mathbf{p}}^*(\mathbf{x}) \right) \quad . \end{aligned}$$

Compute the commutation relations for the creation and annihilation operators $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$ from the known relations for $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$.

Problem 12: Quantization of the complex scalar field

For problem **9** on exercise sheet **2** we considered a classical complex scalar field,

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi \quad ,$$

and showed that its Hamiltonian density is given by

$$\mathcal{H} = \pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi \quad .$$

In order to quantize the theory, first replace the canonical variables by operators, i.e., in particular, $\phi^*, \pi^* \to \phi^\dagger, \pi^\dagger$.

1. Quantize the field operators by introducing the representation of the real field components ϕ_1 and ϕ_2 in $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ in terms of ladder operators. Define

$$a(\mathbf{p}) = \frac{1}{\sqrt{2}}(a_1(\mathbf{p}) + i a_2(\mathbf{p})), \quad b(\mathbf{p}) = \frac{1}{\sqrt{2}}(a_1(\mathbf{p}) - i a_2(\mathbf{p})),$$

   and show that these complex ladder operators obey two independent ladder operator algebras (Note that we use the notation $a(\mathbf{p})$ instead of $a_{\mathbf{p}}$ here).

2. Express the complex field and momentum operators in terms of $a, a^\dagger, b$ and $b^\dagger$.

3. Show that the Hamiltonian of the theory can be written as

$$H = \int d\mu(\mathbf{p})\, \omega_{\mathbf{p}} \left(a^\dagger(\mathbf{p}) a(\mathbf{p}) + b^\dagger(\mathbf{p}) b(\mathbf{p}) \right) + \text{zero point energies.}$$

4. Consider the Noether charge

$$Q = i \int d^3x (\phi^\dagger \partial^0 \phi - \phi \partial^0 \phi^\dagger)$$

and show that the creation operators $a^\dagger$ and $b^\dagger$ generate field excitations whose charges differ in sign.

Problem 13: Linear chain of coupled oscillators

Consider a system of N particles with equal masses m on a one-dimensional chain with lattice constant (separation of equilibrium positions) a. Let each particle move in a harmonic potential (Ω_0) and couple nearest neighbors harmonically (Ω) as well. For the n-th particle of the chain, denote its displacement from equilibrium q_n and its momentum p_n. Then, the Hamiltonian of the system reads

$$H = \sum_{n=0}^{N-1} \frac{p_n^2}{2m} + \frac{m\Omega^2}{2}(q_n - q_{n-1})^2 + \frac{m\Omega_0^2}{2} q_n^2 \quad .$$

To diagonalize H, we introduce the normal coordinates and momenta Q_k and P_k:

$$q_n = \frac{1}{\sqrt{mN}} \sum_k e^{ikan} Q_k \quad , \quad p_n = \sqrt{\frac{m}{n}} \sum_k e^{-ikan} P_k \quad ,$$

which inherit the canonical commutation relations form q_n and p_n, i.e., $[Q_k, P_{k'}] = i\delta_{k,k'}$, while the other commutators vanish.

1. Choose periodic boundary conditions, i.e., $q_0 = q_N$ and determine the possible values that k can take (1^{st} Brillouin zone (BZ)) for even or odd N respectively.

2. Prove the orthogonality relation

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{ian(k-k')} = \delta_{k,k'} \quad ,$$

which holds as long as k and k' are both within the 1^{st} BZ, use it to show that the Hamiltonian can be written as

$$H = \frac{1}{2} \sum_k (P_k P_k^\dagger + \omega_k^2 Q_k Q_k^\dagger) \quad ,$$

and determine ω_k^2. Plot $\omega_k > 0$ as a function of k for $\Omega_0 = 0$ and for $\Omega_0 \neq 0$ (choose the other parameters as you please). Interpret the result.

3. Now go back to the initial definition of H above and write down the Hamiltonian equations of motion for each q_n and p_n. Then, after introducing

$$q(x,t) = q_n(t)\sqrt{\frac{m}{a}} \quad , \quad p(x,t) = p_n(t)\sqrt{ma} \quad ,$$

perform the continuum limit $a \to 0$, $N \to \infty$, keeping $L = aN$, $\rho = \frac{m}{a}$ and $v = \Omega a$ constant and show that in this limit the equations of motion assume the form of a one-dimensional Klein-Gordon equation for the field $q(x,t)$.