Friedrich-Schiller-Universität Jena Summer Term 2020
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Problems Quantum Field Theory

Sheet 1

Problem 1: Natural units

In natural units $c = \hbar = 1$: speed is measured in units of c and actions in units of $\hbar$. In this system of units we only have powers of a length or equivalently powers of an energy. For example, the Compton wave length $\lambda = \hbar/mc \sim 1/m \sim 1/mc^2$ has the dimension of a length or of an inverse mass or of an inverse energy. To find the Compton wave length of an electron in units of energy one multiplies λ_e with suitable powers of c and $\hbar$ to arrive at its rest energy $1/m_e c^2$. Natural units are common in particle physics and cosmology.
Express the gravitational constant $G_N = 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ in units of GeV. What is the value of the Planck length $m_{\mathrm{pl}} = G_N^{-1/2}$.
Which length, time, energy and mass (in SI-units) correspond to 1 GeV (in natural units).

Problem 2: Intervals in Minkowski spacetime

Two events P_1 and P_2 in an inertial system (IS) can be space-like, time-line or light-like separated. We use standard coordinate for which free particles move with constant speed along straight lines. Show that

1. there exists an IS, in which two space-like separated events are simultaneous and that their time order can be reversed by a suitable change of the IS,
2. there is always an IS in which two time-like separated events happen a the same point in space.
3. Find the hypersurface in spacetime, on which a light-like separated event P_2 can be with respect to a given even P_2

Hint: Choose P_1 as origin of the coordinate system.

Problem 3: Lorentz group

Show that the homogeneous Lorentz transformations form a group. Use, that the matrix Λ in the transformation $x \to \Lambda x$ obeys

$$\Lambda^T \eta \Lambda = \eta, \qquad \eta = \mathrm{diag}(1, -1, -1, -1)\,.$$

Find two non-trivial subgroups of the Lorentz group. Repeat this for the Poincaré transformations $x \to \Lambda x + a$, where the vector a characterizes the translation of time ans space and Λ is the matrix of a Lorentz transformation.

Problem 4: Anti-commutation relations

The interpretation of the operators $a_i, a_i^\dagger$ (i denotes the particle species or the a quantum number of the particle) as annihilation and creation operators relies on their commutation relations with the number operators $N_i = a_i^\dagger a_i$:

$$[N_i, a_j] = -\delta_{ij} a_j \quad , \quad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger$$

and the existence of a vacuum state $|0\rangle$ with $a_i|0\rangle = 0$ for all i.

1. proof the operator identity $[AB, C] = A\{B, C\} - \{A, C\}B$.
2. consider the operators $a_i, a_i^\dagger$, which fulfil the anti-commutation relations

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \text{alle anderen} \quad \{.,,\} = 0$$

   and show, that these operators fulfil the above commutation relations with the number operators. erfüllen.

Problem 5: Maxwell equations

Maxwell's equations are the Euler-Lagrange equations for the action

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

whereby the components A_μ of the 4−potential are the dynamical variables. Proof this statement.

Problem 6: *D*-function

In the lecture the following D-function plays an important role. Proof the identity

$$D(t, \boldsymbol{x}) = -\frac{c}{(2\pi)^3}\int \frac{d\boldsymbol{k}}{2\omega_k} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sin\omega_k t = \frac{1}{8\pi r}(\delta(ct + r) - \delta(ct - r))$$