Chapter 5

Particles in electromagnetic fields

In this section study the dynamics of a charged particle in a given external electromagnetic field. In reality the field is modified by a moving charge, for example by the radiation emitted by the particle. But here we shall neglect this backreaction. This is a reasonable approximation for strong or/and almost constant fields.

5.1 Charged scalar particle

In classical physics we use the concept of an idealized point particle with mass m and electric charge e. Such a particle moves along a trajectory and its position at a given time is determined by its initial conditions and the equation of motion. On a particle at a position $\boldsymbol{x}$ with velocity $\dot{\boldsymbol{x}}$ acts the Lorenz force

$$\boldsymbol{F} = e\left(\boldsymbol{E}(t,\boldsymbol{x}) + \frac{1}{c}\,\dot{\boldsymbol{x}} \wedge \boldsymbol{B}(t,\boldsymbol{x})\right). \tag{5.1}$$

To write down a Lagrangian or Hamiltonian function which lead to the corresponding equation of motion one introduces the *electromagnetic potentials* φ and $\boldsymbol{A}$ in

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} \quad , \quad \boldsymbol{B} = \nabla \wedge \boldsymbol{A}. \tag{5.2}$$

Two potentials related by a *gauge transformation* with gauge function $\lambda(t,\boldsymbol{x})$,

$$\begin{aligned} \boldsymbol{A}(t,\boldsymbol{x}) &\to \boldsymbol{A}(t,\boldsymbol{x}) - \nabla\lambda(t,\boldsymbol{x}) \\ \varphi(t,\boldsymbol{x}) &\to \varphi(t,\boldsymbol{x}) + \frac{1}{c}\frac{\partial}{\partial t}\lambda(t,\boldsymbol{x}) \end{aligned} \tag{5.3}$$

give rise to the same electromagnetic field. The non-relativistic *Lorentz equation* $m\ddot{\boldsymbol{x}} = \boldsymbol{F}$ is the Euler-Lagrange equation for the Lagrangian

$$L = \frac{m}{2}\dot{\boldsymbol{x}}^2 + \frac{e}{c}\,\dot{\boldsymbol{x}} \cdot \boldsymbol{A}(t,\boldsymbol{x}) - e\varphi(t,\boldsymbol{x}). \tag{5.4}$$

A Legendre transformation leads to the classical Hamiltonian function

$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(t,\boldsymbol{x})\Big)^2 + e\varphi(t,\boldsymbol{x}), \tag{5.5}$$

and with the help of the correspondence principle we arrive at the Hamiltonian operator $\hat{H}$ and time-dependent Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle\,, \quad \hat{H} = \frac{1}{2m}\Big(\hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}(t,\hat{\boldsymbol{x}})\Big)^2 + e\varphi(t,\hat{\boldsymbol{x}}). \tag{5.6}$$

The operator-ordering is chosen such that $\hat{H}$ gives rise to a unitary time evolution. Under a gauge transformation (5.3) the wave function transforms as

$$\psi(t,\boldsymbol{x}) \longrightarrow e^{-ie\lambda(t,\boldsymbol{x})/\hbar c}\psi(t,\boldsymbol{x}). \tag{5.7}$$

If ψ fulfills the time-dependent Schrödinger equation with potentials φ and $\boldsymbol{A}$ then the gauge-transformed wave function fulfills the Schrödinger equation with gauge-transformed potentials. According to the general rules we expect that the path integral representation for the propagation of a charged particle from $(t',\boldsymbol{x}')$ to $(t,\boldsymbol{x})$ in an electromagnetic field is given by

$$K(t,\boldsymbol{x},t',\boldsymbol{x}') = \int \mathcal{D}\boldsymbol{w}\, e^{iS[\boldsymbol{w},\boldsymbol{A}]/\hbar}, \quad S = \int_{t'}^{t} ds\left(\frac{m}{2}\dot{\boldsymbol{w}}^2 + \frac{e}{c}\dot{\boldsymbol{w}}\cdot\boldsymbol{A} - e\varphi\right), \tag{5.8}$$

where the values of the potentials along the particle path enter, for example $\varphi = \varphi(t,\boldsymbol{w}(t))$. To prove that this propagator satisfies the time dependent Schrödinger equation we proceed similarly as in section 2.3 and replace the time-integral (5.8) by a Riemann sum. In the discretisation of the integral $\int ds\,\dot{\boldsymbol{w}}\cdot\boldsymbol{A}$ we must choose the *midpoint rule*,

$$\int ds\,\dot{\boldsymbol{w}}(s)\cdot\boldsymbol{A}(s,\boldsymbol{w}(s)) \longrightarrow \sum_{j=0}^{n-1}\epsilon\left\{\frac{\boldsymbol{w}_{j+1}-\boldsymbol{w}_j}{\epsilon}\cdot\boldsymbol{A}\left(\frac{s_{j+1}+s_j}{2},\frac{\boldsymbol{w}_{j+1}+\boldsymbol{w}_j}{2}\right)\right\} \tag{5.9}$$

with $\boldsymbol{w}_j = \boldsymbol{w}(s_j)$. This corresponds to the socalled *Ito-calculus* in the theory of stochastic differential equations. If we would take the potential at $\boldsymbol{w}_j$ instead of the midpoint between $\boldsymbol{w}_j$ and $\boldsymbol{w}_{j+1}$ then we would obtain a gauge non-invariant propagator.

Now we take a wave function at time $t-\epsilon$ and let it be propagated toward t. If $\boldsymbol{u} = \boldsymbol{x}-\boldsymbol{y}$ denotes the difference between the final and initial position then we obtain up to terms of $O(\epsilon^2)$

$$\begin{aligned}\psi(t,\boldsymbol{x}) &\approx \lim_{\epsilon\to 0} A_\epsilon^3 \int d^3u \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{u}^2\right)\exp\left(\frac{i\epsilon}{\hbar}L_{\text{int}}\right)\psi(t-\epsilon,\boldsymbol{x}-\boldsymbol{u})\\ L_{\text{int}} &= \frac{e}{c}\frac{\boldsymbol{u}}{\epsilon}\cdot\boldsymbol{A}\left(t-\frac{\epsilon}{2},\boldsymbol{x}-\frac{\boldsymbol{u}}{2}\right) - e\varphi\left(t-\frac{\epsilon}{2},\boldsymbol{x}-\frac{\boldsymbol{u}}{2}\right),\end{aligned} \tag{5.10}$$

As earlier $A_\epsilon = (m/2\pi i\hbar\epsilon)^{1/2}$ enters as normalizing factor. Expanding the two last factors in the first line up to terms linear in ϵ or quadratic in $\boldsymbol{u}$. We obtain

$$\psi(t,\boldsymbol{x}) = \lim_{\epsilon\to 0} A_\epsilon^3 \int d^3u \exp\left\{\frac{im}{2\hbar\epsilon}\boldsymbol{u}^2\right\}\left\{\psi(t-\epsilon) + \frac{1}{2}u_iu_jD_iD_j\psi - \frac{ie\epsilon}{\hbar}\varphi\psi + \dots\right\}, \tag{5.11}$$

where we are lead to the *covariant derivative*

$$\boldsymbol{D} = \nabla - \frac{ie}{\hbar c}\boldsymbol{A}. \tag{5.12}$$

The potentials and wave function between the last curly brackets in (5.11) are taken at the position $\boldsymbol{x}$. With the help of the Gaussian integrals

$$\int d^3u \exp\left\{\frac{im}{2\hbar\epsilon}\boldsymbol{u}^2\right\} = \frac{1}{A_\epsilon^3} \quad \text{and} \quad \int d^3u \exp\left\{\frac{im}{2\hbar\epsilon}\boldsymbol{u}^2\right\} u_i u_j = \frac{1}{A_\epsilon^3}\frac{i\hbar\epsilon}{m}\delta_{ij} \tag{5.13}$$

we obtain in the limit $\epsilon \to 0$ the partial differential equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = -\frac{\hbar^2}{2m}(\boldsymbol{D}^2\psi)(t,\boldsymbol{x}) + e\varphi(t,\boldsymbol{x})\psi(t,\boldsymbol{x}), \tag{5.14}$$

which is just the Schrödinger equation (5.6) in the position representation. It is a useful exercise to show that if we do not take the midpoint rule in (5.9) then we would get a different result. Actually for the scalar potential and for the time-integration no midpoint rule is needed. We would still get the correct propagator in the continuum limit if we would take

$$L_{\text{int}} = \frac{e}{c}\frac{\boldsymbol{u}}{\epsilon}\cdot\boldsymbol{A}\left(t,\boldsymbol{x}-\frac{\boldsymbol{u}}{2}\right) - e\varphi\left(t,\boldsymbol{x}\right), \tag{5.15}$$

instead of L_{int} in (5.10). But with the choice (5.10) the convergence to the continuum limit is faster. Under a gauge transformation (5.3) with gauge function $\lambda(t,\boldsymbol{x})$ the action changes by path independent boundary terms,

$$\Delta S[\boldsymbol{w},A,\varphi] = -\frac{e}{c}\int_{t'}^{t} ds\left(\dot{\boldsymbol{w}}\cdot\nabla\lambda + \frac{\partial}{\partial s}\lambda\right) = -\frac{e}{c}\left\{\lambda(t,\boldsymbol{x}) - \lambda(t',\boldsymbol{x}')\right\} \tag{5.16}$$

such that the propagator transforms covariantly under gauge transformations,

$$K(t,\boldsymbol{x};t',\boldsymbol{x}') \longrightarrow e^{-ie\lambda(t,\boldsymbol{x})/\hbar c}\,K(t,\boldsymbol{x},t',\boldsymbol{x}')\,e^{ie\lambda(t',\boldsymbol{x}')/\hbar c}. \tag{5.17}$$

This agrees with the transformation rule (5.7) for the solutions of the Schrödinger equation under gauge transformations.

5.1.1 The Aharonov-Bohm effect

The Aharonov-Bohm effect demonstrates that in quantum mechanics a charged particle passing through a space region without electric and magnetic field can be influenced by electric and magnetic fields *outside* of this region [16, 17]. In quantum mechanics the motion is described by the Feynman path integral for the propagator (5.8) in which the potentials and not the field strength enter. Even if $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish in some region of space, $\boldsymbol{A}$ need not vanish there due to the presence of a magnetic field outside of the region.

Here we consider the Aharonov-Bohm effect due to a magnetic flux Φ confined to a solenoid. We assume that the solenoid is straight and very long and choose the coordinate system such that the z-axis is the symmetry axis of the solenoid. Outside the solenoid there in no magnetic field and for an infinitely long solenoid the magnetic potential has the form

$$\boldsymbol{A}\cdot d\boldsymbol{x}=\frac{\Phi}{2\pi}\frac{xdy-ydx}{\rho^2},\qquad \rho^2=x^2+y^2. \tag{5.18}$$

We assume that the particle can not penetrate into the solenoid. Let us consider a particle trajectory $\boldsymbol{w}(s)$ defining a curve $\mathcal{C}$. The term containing the magnetic vector potential in the action (5.8) is proportional to

$$\int_{t'}^{t}\boldsymbol{A}(\boldsymbol{w}(s))\cdot\frac{d\boldsymbol{w}(s)}{ds}\,ds=\int_{\mathcal{C}}\boldsymbol{A}(\boldsymbol{x})\cdot d\boldsymbol{x}=\frac{\Phi}{2\pi}\int_{\mathcal{C}}\frac{xdy-ydx}{\rho^2}. \tag{5.19}$$

Transforming to cylinder coordinates $(x,y,z)=(\rho\cos\varphi,\rho\sin\varphi,z)$ the line integral becomes

$$\int_{\mathcal{C}}\boldsymbol{A}\cdot d\boldsymbol{x}=\frac{\Phi}{2\pi}\int_{\mathcal{C}}d\varphi. \tag{5.20}$$

A path $\mathcal{C}_n:\boldsymbol{x}'\to\boldsymbol{x}$ outside the solenoid is characterized by its *winding number* $n\in\mathbb{Z}$. For its definition one takes some standard contour $C_0:\boldsymbol{x}'\to\boldsymbol{x}$ and counts the number of times that the closed curve $\mathcal{C}_n-\mathcal{C}_0$ winds around the solenoid. In figure 5.1 we have depicted a reference

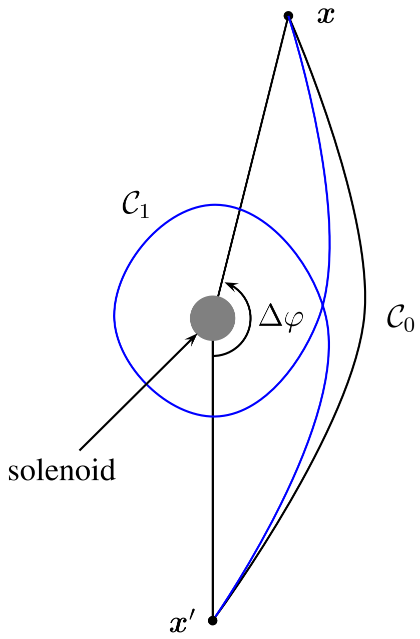

Figure 5.1: *A reference path $\mathcal{C}_0$ and a path $\mathcal{C}_1$ with relative winding* 1.

path $\mathcal{C}_0$ and a path $\mathcal{C}_1$ with winding number one. For a path with winding n one has

$$\int_{\mathcal{C}_n}\boldsymbol{A}\cdot d\boldsymbol{x}=n\Phi+\int_{\mathcal{C}_0}\boldsymbol{A}\cdot d\boldsymbol{x}=n\Phi+\frac{\Phi}{2\pi}\Delta\phi, \tag{5.21}$$

where $\Delta\Phi$ is the angle shown in figure 5.1. In the path integral one admits all paths connecting $\boldsymbol{x}'$ with $\boldsymbol{x}$. We do the integration in two steps: first we integrate over the set paths $\{\mathcal{C}_n\}$ with winding number n and then sum over all winding numbers. This yields

$$K(t,\boldsymbol{x},\boldsymbol{x}') = \sum_n \int_{\{\mathcal{C}_n\}} \mathcal{D}\boldsymbol{w}\, e^{iS[\boldsymbol{w},\boldsymbol{A}]/\hbar} = e^{ie\Phi\Delta\phi/hc} \sum_n e^{ine\Phi/\hbar c} K_n(t,\boldsymbol{x},\boldsymbol{x}'), \tag{5.22}$$

where K_n is the $\boldsymbol{A}$-independent topologically constrained Feynman path integral

$$K_n(t,\boldsymbol{x},\boldsymbol{x}') = \int_{\{\mathcal{C}_n\}} \mathcal{D}\boldsymbol{w} \exp\left\{\frac{i}{\hbar}\int_0^t ds\,\Big(\frac{m}{2}\dot{\boldsymbol{w}}^2 - e\varphi(\boldsymbol{x})\Big)\, ds\right\} \tag{5.23}$$

in which one integrates over trajectories which (when completed into a closed loop by continuing them with $-\mathcal{C}_0$) wind n-times around the solenoid. We see that no Aharonov-Bohm effect will occur if the magnetic flux in the solenoid obeys the quantization condition

$$\frac{e\Phi}{hc} = 0, \pm 1, \pm 2, \dots \tag{5.24}$$

In this cases the phase factors containing n in (5.22) are unity and the summation over n gives

$$K(t,\boldsymbol{x},\boldsymbol{x}') = \exp\Big(\frac{ie\Phi}{hc}\Delta\phi\Big)\, K_0(t,\boldsymbol{x},\boldsymbol{x}') \tag{5.25}$$

where K_0 denotes the full, unconstrained, propagator for a particle in the absence of the magnetic vector potential. If the magnetic flux does not fulfill the quantization condition (5.24) then the contributions of the various toplogical sectors to the propagator will interfere, and when a screen is placed behind the solenoid the interference pattern on the screen will change when Φ is increased. This is the Aharonov-Bohm effect.

We have seen that the Aharonov-Bohm effect originates in the interaction between the electron and the external gauge potential $\boldsymbol{A}$ whose $\boldsymbol{B}$-field vanishes locally. One can show that the effect can equally well be regarded as originating in the interaction of the magnetic field of the electron with the distant $\boldsymbol{B}$-field inside the solenoid. From this point of view the effect is seen to have a natural classical origin and loses much of its mystery [18].

5.2 Spinning particles

In the non-relativistic limit the wave function of a spin-$\frac{1}{2}$ particle has two components, it is a spinor, and correspondingly is the Schrödinger operator, called *Pauli-Hamiltonian* after Wolfgang Pauli, a 2-dimensional matrix differential operator

$$H = \frac{1}{2m}\Big\{\boldsymbol{\sigma}\cdot\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(t,\boldsymbol{x})\Big)\Big\}^2 + e\varphi(t,\boldsymbol{x})\mathbb{1}_2. \tag{5.26}$$

Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-tuple of Pauli matrices. The Pauli-Hamiltonian contains a coupling of the electron spin to a magnetic field with the correct g-factor of 2. Indeed, with the help of $\sigma_i\sigma_j = i\epsilon_{ikj}\sigma_k + \mathbb{1}_2$ the Pauli-Hamiltonian can be rewritten as

$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(t,\boldsymbol{x})\Big)^2 + e\varphi(t,\boldsymbol{x}) - \frac{e}{mc}\boldsymbol{B}(t,\boldsymbol{x})\cdot\boldsymbol{s}, \quad \boldsymbol{s} = \frac{\hbar}{2}\boldsymbol{\sigma}, \tag{5.27}$$

where the two first terms act as identity operator in spin space. The corresponding matrix-valued Lagrange function

$$L = \frac{m}{2}\dot{\boldsymbol{x}}^2 + \frac{e}{c}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A}(t,\boldsymbol{x}) - e\varphi(t,\boldsymbol{x}) + \frac{e}{mc}\boldsymbol{B}(t,\boldsymbol{x})\cdot\boldsymbol{s} \tag{5.28}$$

should enter the path integral for a non-relativistic spin-$1/2$ particle. Although L is matrix valued we could proceed as in the previous section and would end up with the result (5.10) with interaction Lagrangian

$$L_{\text{int}}(t,\boldsymbol{x},\boldsymbol{u}) = \left(\frac{e}{c}\frac{\boldsymbol{u}}{\epsilon}\cdot\boldsymbol{A} - e\varphi + \frac{e}{mc}\boldsymbol{B}\cdot\boldsymbol{s}\right)_{\text{midpoint}}. \tag{5.29}$$

If the propagation is from $(t-\epsilon, \boldsymbol{x}-\boldsymbol{u}) \to (t,\boldsymbol{x})$ as it is in (5.10), then the midpoint rule means evaluation of the potentials and magnetic field at time $t - \frac{1}{2}\epsilon$ and position $\boldsymbol{x} - \frac{1}{2}\boldsymbol{u}$. This way one obtains for the propagator the representation

$$\begin{aligned} K(t,\boldsymbol{x},t',\boldsymbol{x}') &= \lim_{n\to\infty} A_\epsilon^{3n}\int d^3w_1\cdots d^3w_{n-1}\, e^{i\epsilon L_{n-1}/\hbar}\cdots e^{i\epsilon L_0/\hbar}, \\ L_j &= \frac{m}{2}\frac{\boldsymbol{u}_j^2}{\epsilon^2} + \frac{e}{c}\frac{\boldsymbol{u}_j}{\epsilon}\cdot\boldsymbol{A}(\bar{s}_j,\bar{\boldsymbol{w}}_j) - e\varphi(\bar{s}_j,\bar{\boldsymbol{w}}_j) + \frac{e}{mc}\boldsymbol{B}(\bar{s}_j,\bar{\boldsymbol{w}}_j)\cdot\boldsymbol{s}, \end{aligned} \tag{5.30}$$

where $\boldsymbol{w}_0 = \boldsymbol{x}'$, $\boldsymbol{w}_n = \boldsymbol{x}$ and we have used the abbreviations

$$\boldsymbol{u}_j = \boldsymbol{w}_{j+1} - \boldsymbol{w}_j, \quad \bar{\boldsymbol{w}}_j = \frac{\boldsymbol{w}_{j+1}+\boldsymbol{w}_j}{2}, \quad \bar{s}_j = \frac{s_{j+1}+s_j}{2}. \tag{5.31}$$

As earlier the propagation time interval $[t',t]$ is divided into n intervals of length $\epsilon = (t-t')/n$ and $s_j = t' + j\epsilon$. For a time and/or space dependent magnetic field two L_j in (5.30) with different j do not commute due to the $\boldsymbol{B}\cdot\boldsymbol{s}$-term in the Lagrangian. In the (formal) continuum limit we identify $\boldsymbol{w}_j$ with the position $\boldsymbol{w}(s_j)$ of the particle at time s_j. Then L_j is the value of the Lagrangian at time s_j. We see that the factors in (5.30) are time ordered: on the right we have the factor $\exp(i\epsilon L_0/\hbar)$ at earliest time and on the left the factor $\exp(i\epsilon L_{n-1}/\hbar)$ at latest time. Thus we are lead to the path ordered integral

$$\begin{aligned} K(t,\boldsymbol{x},t',\boldsymbol{x}') &= \int \mathcal{D}\boldsymbol{w}\,\mathcal{P}\exp\left(\frac{i}{\hbar}\int_{t'}^t ds\, L(s)\right), \\ L(s) &= L\Big(\boldsymbol{w}(s), \boldsymbol{A}(s,\boldsymbol{w}(s)), \varphi(s,\boldsymbol{w}(s))\Big), \end{aligned} \tag{5.32}$$

where the time is ordered along the path $\boldsymbol{w}(s)$. The path ordered integral satisfies the differential equation

$$\frac{\partial}{\partial t}\mathcal{P}\exp\left(\frac{i}{\hbar}\int_{t'}^{t} ds\, L(s)\right) = \frac{i}{\hbar}\, L(t)\,\mathcal{P}\exp\left(\frac{i}{\hbar}\int_{t'}^{t} ds\, L(s)\right), \tag{5.33}$$

and this equation together with the initial condition

$$\mathcal{P}\exp\left(\frac{i}{\hbar}\int_{t'}^{t'} ds\, L(s)\right) = \mathbb{1} \tag{5.34}$$

determines the path ordered integral.

5.2.1 Spinning particle in constant B-field

Let us consider a uniform magnetic field pointing in the direction of the z-axis,

$$\boldsymbol{A} = \frac{B}{2}(x\boldsymbol{e}_y - y\boldsymbol{e}_x) \Rightarrow \boldsymbol{B} = B\boldsymbol{e}_z. \tag{5.35}$$

For a uniform magnetic field the action (5.28) for the spinning particle simplifies to

$$S = \frac{m}{2}\int_0^t \dot{\boldsymbol{w}}^2 + \frac{\omega}{2}\int_0^t \hat{\boldsymbol{B}}\cdot(\boldsymbol{L}+2\boldsymbol{s}), \quad \boldsymbol{L} = m\boldsymbol{w}\wedge\dot{\boldsymbol{w}}, \tag{5.36}$$

with cyclotron frequency $\omega = eB/mc$. The particle moves freely in the z-direction and only the propagation in the xy-plane is affected by the external field. Thus we may assume that $\boldsymbol{x}'$ and $\boldsymbol{x}$ are both in the plane with $z=0$ such that the whole tracetory $\boldsymbol{w}(s)$ lies in this plane. Without loss of information we may study the two-dimensional dynamics in the xy-plane and in the following we assume that all vectors lie in the plane, for example $\boldsymbol{w} = w_x\boldsymbol{e}_x + w_y\boldsymbol{e}_y$.

For a uniform magnetic field the spin-term does not depend on the trajectory and hence does not enter the equation of motion. With the help of the rotation matrix

$$R(\omega t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \tag{5.37}$$

the solution of the classical equation of motion can be written as

$$\boldsymbol{w}_{\rm cl}(s) = \boldsymbol{x}' + \frac{\sin(\hat{\omega}s)}{\sin(\hat{\omega}t)}R\left(\hat{\omega}(s-t)\right)(\boldsymbol{x}-\boldsymbol{x}'), \quad \hat{\omega} = \frac{\omega}{2}, \tag{5.38}$$

and its action is given by

$$S[\boldsymbol{w}_{\rm cl}] = \frac{m\hat{\omega}}{2}\cot(\hat{\omega}t)(\boldsymbol{x}-\boldsymbol{x}')^2 - m\hat{\omega}(xy'-yx') + \omega t\, s_3. \tag{5.39}$$

The kinetic energy and the term containing the orbital angular momentum diverge if the propagation time is a multiple of $2\pi/\omega$. Both contributions to the action contain a term proportional to $t/\sin^2(\hat{\omega}t)$ and since they have different signs they cancel in the sum.

As earlier we decompose an arbitrary path as $\boldsymbol{w}(s) = \boldsymbol{w}_{\rm cl}(s) + \boldsymbol{\xi}(s)$, where the fluctuations $\boldsymbol{\xi}$ vanish at initial and final time. With

$$S[\boldsymbol{w}] = S[\boldsymbol{w}_{\rm cl}] + \frac{m}{2}(\boldsymbol{\xi}, M\boldsymbol{\xi}), \quad M = -\frac{d^2}{ds^2} + i\omega\sigma_2\frac{d}{ds}, \tag{5.40}$$

the path integral yields

$$K(t, \boldsymbol{x}, \boldsymbol{x}') = \frac{\mathcal{N}}{\sqrt{\det M}}\, e^{iS[\boldsymbol{w}_{\rm cl}]/\hbar}. \tag{5.41}$$

We remain with calculating the determinant of the matrix differential operator M. This can be achieved by a generalization of the Gelfand-Yaglom initial value problem. One defines a matrix S, the columns of which are linearly independent solutions of $M\boldsymbol{\xi} = 0$ vanishing at $s = 0$,

$$MS = 0 \quad \text{with} \quad S(0) = 0, \quad \dot{S}(0) = \mathbb{1}. \tag{5.42}$$

Any solution of $M\boldsymbol{\xi} = 0,\ \boldsymbol{\xi}(0) = 0$ is a linear combination of the columns of S. Let us now assume that

$$\det S(t) = 0. \tag{5.43}$$

Then there is a linear combination of the columns of S which vanish at the final time t. It is an eigenfunction of the fluctuation operator with zero energy such that $\det M$ must vanish. Since the converse statement is also true, it is not surprising that the ratio of two fluctuation determinants is given by

$$\frac{\det M}{\det M_0} = \frac{\det S}{\det S_0} = \frac{1}{t^2}\det S. \tag{5.44}$$

Here $S_0 = t\mathbb{1}$ is the matrix of solutions of the fluctuation operator M_0 with vanishing ω. In particular for the fluctuation operator in (5.40) we have

$$S(t) = \hat{\omega}^{-1}\sin\hat{\omega}t\,(\cos\hat{\omega}t + i\sin\hat{\omega}t\,\sigma_2) \tag{5.45}$$

and this leads to the following ratio of determinants:

$$\frac{\det M}{\det M_0} = \left(\frac{\sin\hat{\omega}t}{\hat{\omega}t}\right)^2. \tag{5.46}$$

Inserting this result into (5.41) yields the well known propagator for a spinning particle in a uniform magnetic field

$$\begin{aligned} K(t, \boldsymbol{x}, \boldsymbol{x}') &= \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} \frac{\hat{\omega}t}{\sin\hat{\omega}t} \exp\left(\frac{im}{2\hbar t}(z - z')^2 + i\hat{\omega}\sigma_3\right) \\ &\times \exp\left\{\frac{im\hat{\omega}}{\hbar}\left(\frac{\cot\hat{\omega}t}{2}\Big[(x - x')^2 + (y - y')^2\Big] + (x'y - xy')\right)\right\}. \end{aligned} \tag{5.47}$$

To obtain the propagator in 3 dimensions we have multiplied with the propagator for the free motion in the z-direction. Similarly as for the harmonic oscillator the propagator is singular at times $t_n = 2\pi n/\omega$ after which a classical particle returns to its starting point in the plane orthogonal to the $\boldsymbol{B}$-field. Note that the two spin-components acquire different phases in a non-vanishing magnetic field. The above result (without spin-term) has been obtained by GLASSER [20] and by FEYNMAN and HIBBS [4].