Chapter 4

Perturbation Theory

In conventional perturbation theory one assumes that the coupling constant λ in

$$H = H_0 + \lambda V \tag{4.1}$$

is small and expands the eigenvalues and eigenfunctions of H in a power series in λ. Here we perform an expansion of the evolution kernel in powers of the coupling constant. For H_0 one usually takes the Hamiltonian of the free particle or the harmonic oscillator such that for $\lambda = 0$ the problem is soluble. This way one obtains a non-convergent series which (at least in quantum mechanics) has a good chance of being asymptotic.

4.1 Perturbation expansion for the propagator

We consider a particle with mass m in a given external potential V. We decompose the action into a term S_0 belonging to the free particle with mass m and a term S_I describing the interaction of the particle with the potential,

$$S = S_0 + S_I = \frac{m}{2}\int_0^t \dot{w}^2 ds - \lambda \int_0^t V(w(s))ds. \tag{4.2}$$

The coupling constant λ measures the strength of the interaction. It is introduced for an easy identification of terms contributing to a given order in the perturbative expansion. In order to find this expansion for the propagator we use its path integral representation

$$K(t, q, q') = \int\limits_{w(0)=q'}^{w(t)=q} \mathcal{D}w \, e^{iS[w]/\hbar}, \tag{4.3}$$

where one integrates over all paths with fixed endpoints q' and q. Inserting the decomposition (4.2) one immediately obtains a power series expansion for K in powers of λ. We assume a small coupling and expand $\exp(iS_I/\hbar)$ in powers of λ with the result

$$\begin{aligned} K(t,q,q') &= \int \mathcal{D}w\; e^{iS_0/\hbar}\, e^{iS_I/\hbar} \\ &= \int \mathcal{D}w\, e^{iS_0/\hbar} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{i\hbar}\right)^n \left(\int V(w(s))ds\right)^n . \end{aligned} \tag{4.4}$$

The leading term is just the propagator of the free particle (2.22). The sub-leading term of order $O(\lambda)$ is given by by the path integral

$$K_1(t,q,q') = \frac{\lambda}{i\hbar}\int_0^t ds \int\limits_{w(0)=q'}^{w(t)=q} \mathcal{D}w\, e^{iS_0[w]/\hbar}\, V(w(s)), \tag{4.5}$$

where we have interchanged the order of integrations and first did the path integral and then the time-integration. To calculate the path integral at hand (prior to the s-integration) we first integrate over all path from the initial position q' at time 0 to an intermediate event s,u and then over all path from (s,u) to the final position q at time t. Finally we integrate over all intermediate position u,

$$\int \mathcal{D}w\, e^{iS_0[w]/\hbar}\, V(w(s)) = \int du \int\limits_{w(s)=u}^{w(t)=q} \mathcal{D}w\, e^{iS_0[w]/\hbar}\, V(u) \int\limits_{w(0)=q'}^{w(s)=u} \mathcal{D}w\, e^{iS_0[w]/\hbar}. \tag{4.6}$$

The two path integrals are given by the propagator K_0 of the free particle (2.22). Hence we arrive at the following expression for the first order perturbation K_1,

$$K_1(t,q,q') = \frac{\lambda}{i\hbar}\int_0^t ds \int_{-\infty}^{\infty} du\; K_0(t-s,q,u)V(u)K_0(s,u,q'). \tag{4.7}$$

Since $K_0(s,u,v)$ is a Gaussian function of u and v the integral over the intermediate position u can be calculated explicitly for a polynomial potential. This expression for K_1 can be interpreted as follows: first the particle propagates freely from q' to u, where at time s it is 'hit' by the potential. Then it again propagates freely to q during the time interval $t-s$. The total traveling time being t. Then the amplitudes for all intermediate positions u and times s of possible hits are summed. One of Feynman's big achievements was to provide a pictorial representation of the amplitude by a so-called Feynman diagram. The contribution of order $O(\lambda^2)$ to the propagator reads

$$K_2(t,q,q') = \frac{1}{2}\left(\frac{\lambda}{i\hbar}\right)^2 \int \mathcal{D}w\, e^{iS_0[w]/\hbar} \int_0^t ds ds'\, V(w(s))V(w(s')) \tag{4.8}$$

$$= \left(\frac{\lambda}{i\hbar}\right)^2 \int\limits_0^t ds \int\limits_0^s ds' \int dudv\, K_0(t-s,q,u)V(u)K_0(s-s',u,v)V(v)K_0(s',v,q'),$$

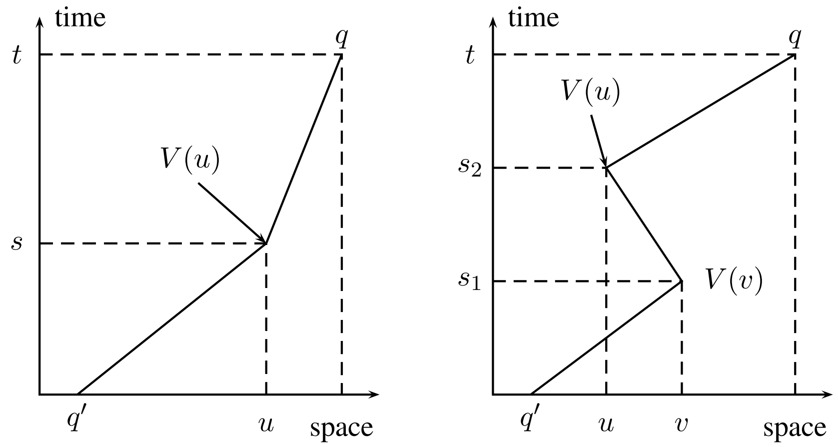

Figure 4.1: *The Feynman graphs associated to first and second order perturbation theory.*

which can be visualized as a particle propagating freely from q' to v, where at time s' it is hit by V, then moving freely to u, where it is hit by V at time s and finally propagates freely to q. It arrives at the final position at time t. Then the amplitudes for all intermediate positions u and v and intermediate times s' and s are superimposed.

The perturbative expansion can easily be calculated with the help of the generating functional for the Greenfunctions of the *free particle*. According to our result (3.45) this functional reads

$$K_0(t,q,q';j) = \int \mathcal{D}w\; e^{iS_{0j}[w]/\hbar} = K_0(t,q,q')\, e^{iW_0[j]/\hbar}, \tag{4.9}$$

where $K_0(t,q,q')$ denotes the propagator without source and the Schwinger functional $W_0[j]$ depends quadratically on the source. Because of

$$\left(\frac{\hbar}{i}\frac{\delta}{\delta j(s)}\right)^n \int \mathcal{D}w\, e^{iS_{0j}/\hbar} = \int \mathcal{D}w\, e^{iS_{0j}/\hbar}\, w^n(s) \tag{4.10}$$

we may calculate the path integrals appearing in the perturbative expansion (4.4) as follows,

$$V\left(\frac{\hbar}{i}\frac{\delta}{\delta j(s)}\right) \int \mathcal{D}w\, e^{iS_{0j}/\hbar} = \int \mathcal{D}w\, e^{iS_{0j}/\hbar}\, V(w(s)). \tag{4.11}$$

The final expansion for the kernel can be written in the concise form

$$K(t,q,q') = K_0(t,q,q')\, \exp\left[\frac{\lambda}{i\hbar}\int ds\, V\left(\frac{\hbar}{i}\frac{\delta}{\delta j(s)}\right)\right]\; e^{iW_0[j]/\hbar}\big|_{j=0}. \tag{4.12}$$

To calculate the moments in (4.10) we define the 'normalized' n-point correlation functions of the free theory with action S_0,

$$G_0^{(n)}(q,q';t_1,\dots t_n) = \frac{\int \mathcal{D}w\, e^{iS_0/\hbar} w(t_1)\cdots w(t_n)}{\int \mathcal{D}w\, e^{iS_0/\hbar}}. \tag{4.13}$$

In our notation we made the dependence on the end points for the path over which one integrates explicit. Inserting the result (4.9) for the generating functional the normalized correlation functions take the simple form

$$G_0^{(n)}(q,q';t_1,\dots,t_n)=\left(\frac{\hbar}{i}\frac{\delta}{\delta j(t_1)}\cdots\frac{\hbar}{i}\frac{\delta}{\delta j(t_n)}\right)\Big|_{j=0}\,e^{iW_0[j]/\hbar}. \tag{4.14}$$

Using the explicit form of $W_0[j]$ in (3.44) the $G_0^{(n)}$ can be calculated *explicitly*. Actually, since W_0 is a quadratic functional of j they can be expressed in terms of the 1 and 2-point correlation functions. The formulas expressing the higher n-point functions in terms of the 1 and 2-point functions is the celebrated *Theorem of Wick*.

In case $q'=q=0$ the homogeneous solution w_h vanishes for all times and the theorem takes a much simpler form, since the generating functional simplifies to

$$e^{iW_0[j]/\hbar}=\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{i}{2\hbar}\int_0^t j(s)\,G_D(s,s')\,j(s')\right)^n. \tag{4.15}$$

To simplify our notation we denote the Greenfunctions with $q'=q=0$ by $G_0^{(n)}(0,t_1,\dots,t_n)$. Since the functional contains even powers of j only, the $G_0^{(n)}$ vanish for odd n. The first non-vanishing correlation function is

$$G_0^{(2)}(0,t_1,t_2)=\frac{\hbar}{i}\,G_D(t_1,t_2). \tag{4.16}$$

For general even n the Greenfunction is given by a sum of products of the two-point function,

$$G_0^{(2n)}(0,t_1,\dots,t_n)=\sum_{\text{pairs }(i_1i_2)\cdots(i_{2n-1}i_{2n})}G_0^{(2)}(0,t_{i_1},t_{i_2})\cdots G_0^{(2)}(0,t_{i_{2n-1}},t_{i_{2n}}), \tag{4.17}$$

where two indices in the sum are unequal and the pairs are ordered. This is the *Wick theorem* found in most text books and it holds for all theories with quadratic actions. For example, the 4-point function contains 3 terms

$$\begin{aligned}G_0^{(4)}(0,t_1,\dots,t_4) &= G_0^{(2)}(0,t_1,t_2)G_0^{(2)}(0,t_3,t_4)\\ &+ G_0^{(2)}(0,t_1,t_3)G_0^{(2)}(0,t_2,t_4)\\ &+ G_0^{(2)}(0,t_1,t_4)G_0^{(2)}(0,t_2,t_3).\end{aligned} \tag{4.18}$$

For all theories with quadratic action the generating functional $W[j]$ is quadratic in j and the *truncated* or *connected correlation functions*

$$G_c^{(n)}(q,q';t_1,\dots,t_n)=\frac{i}{\hbar}\prod_{k=1}^n\left(\frac{\hbar}{i}\frac{\delta}{\delta j(t_k)}\right)W[j]|_{j=0} \tag{4.19}$$

vanish for $n>2$. This simple observation then just proves the theorem of Wick.

4.2 Quartic potentials

In order to calculate the corrections to the evolution kernel in first order perturbation theory (4.5) for a quartic potential $V = q^4$ we must determine

$$K_1(t, q, q') = \frac{\lambda}{i\hbar} \int ds \int \mathcal{D}w\, e^{iS_0[w]/\hbar} w^4(s), \tag{4.20}$$

where one integrates over paths with $q(0) = q'$ and $q(t) = q$. The last path integral is generated by $K_0(t, q, q', j)$ in (3.45) such that

$$\int \mathcal{D}w\, e^{iS_0/\hbar}\, w^4(s) = \left(\frac{\hbar}{i} \frac{\delta}{\delta j(s)}\right)^4 e^{iW_0[j]/\hbar}\Big|_{j=0} K_0(t, q, q'). \tag{4.21}$$

Here we apply Wick's theorem and obtain

$$\left(\frac{\hbar}{i} \frac{\delta}{\delta j(s)}\right)^4 e^{iW_0[j]/\hbar}\Big|_{j=0} = 3G^{(2)}(s, s)G^{(2)}(s, s) + 6G^{(2)}(s, s)w_h^2(s) + w_h^4(s), \tag{4.22}$$

where the 2-point function $G^{(2)} = \frac{\hbar}{i} G_D$ and the homogeneous solution w_h for the free particle have been calculated earlier in (3.43),

$$G^{(2)}(s, s) = \frac{\hbar}{imt}(s - t)s \quad \text{and} \quad w_h(s) = \frac{1}{t}[sq' + (t - s)q]. \tag{4.23}$$

To compute K_1 we just need to integrate the fourth order polynomial in (4.22) which results in

$$\begin{aligned} K_1(t, q, q') \;&=\; \lambda\, K_0(t, q, q') \left\{ \frac{i\hbar}{m^2} \frac{t^3}{10} + \frac{3}{m} \frac{t^2}{10} (q^2 + {q'}^2 + \frac{4}{3}\, qq') \right. \\ &\qquad \left. - \frac{i}{\hbar} \frac{t}{5} (q^4 + q^3 q' + q^2 {q'}^2 + q{q'}^3 + {q'}^4) \right\}. \end{aligned} \tag{4.24}$$

We can trust the perturbative expansion if $K_1 \ll K_0$, which is the case if

$$\lambda \ll \max\left\{ \frac{m^2}{\hbar t^3}, \frac{m}{t^2 q^2}, \frac{\hbar}{t q^4} \right\}.$$

The expansions becomes reliable for short propagation times t and small q' and q. It breaks down for small particle masses. According to Wicks theorem the higher order contributions in the perturbative series (4.4) reduce to integrals of products of 1 and 2-point functions of the free particle. Hence they can be calculated in closed form. However, the number of terms one must include grows rapidly with increasing order n.

The perturbative expansion for the Greenfunction $\langle q, t|\, \mathbf{T}\hat{q}(t_1) \cdots \hat{q}(t_n)|q\rangle$ is obtained similarly as for the evolution kernel. Again we assume that S is the sum of a free part S_0 and an interaction term S_I, see (4.2). Now we expand the right hand side of (2.48) in powers of the coupling constant λ. This leads to the expansion

$$\begin{aligned}&\langle q,t|\,\mathbf{T}\,\hat{q}(t_1)\cdots\hat{q}(t_n)|q'\rangle\\&=\sum\frac{1}{n!}\left(\frac{\lambda}{i\hbar}\right)^n\int ds_1\dots ds_n\,\langle q,t|\,q(t_1)\cdots q(t_n)V(q(s_1))\cdots V(q(s_n))|q'\rangle_0\,.\end{aligned}$$

The matrix elements on the right hand side are to be evaluated for the system without interaction which means for the system with action S_0. Formally this series can be summarized as follows

$$\begin{aligned}\langle q,t|\,\mathbf{T}\,\hat{q}(t_1)\cdots\hat{q}(t_n)|q'\rangle &= K_0(t,q,q')\,\cdot\\ \prod_{k=1}^{n}\left(\frac{\hbar}{i}\frac{\delta}{\delta j(t_k)}\right)&\exp\left[\frac{\lambda}{i\hbar}\int ds\,V\left(\frac{\hbar}{i}\frac{\delta}{\delta j(s)}\right)\right]\;e^{iW_0[j]/\hbar}\Big|_{j=0},\end{aligned}\tag{4.25}$$

with Schwinger function $W_0[j]$ for the non-interacting system, see (3.44). Since W_0 is quadratic in the source j we may use Wick's theorem to calculate the perturbative expansion on the right hand side.