Chapter 16

Effective potentials

We have already pointed out the difficulty with (local) mass terms in pure gauge theories. Explicit mass terms spoil the crucial gauge invariance of the massless theory (however, as we have seen in the Schwinger model, non-local gauge invariant mass terms are possible). The problem of generating masses in a manner consistent with gauge invariance was solved by Weinberg and Salam. For a historical account and references see [53]. They used the idea of spontaneous symmetry breaking. A familiar example of this mechanism is the magnetisation of a ferro-magnetic material below its Curie temperature.

In field theory the symmetry breaking is implemented by scalar fields which minimally couple to gauge fields and interact with themselves. The electro-weak Lagrangian for the gauge, scalar and fermion fields has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D})\psi + (D_\mu\phi)^\dagger D^\mu\phi - \Gamma\bar{\psi}\phi\psi - V(\phi), \tag{16.1}$$

where $F_{\mu\nu}$ is the field strength tensor (8.71), $\not{D}$ the Dirac operator (8.66) acting on quarks and leptons, $D_\mu\phi = (\partial_\mu - iA_\mu)\phi$ the covariant derivative of the scalar field, $\Gamma\bar{\psi}\phi\psi$ the Yukawa interaction between the fermions and scalars and $V(\phi)$ the self-interaction of the scalars. All fields transform under certain representation of the electro-weak gauge group $SU(2)_L \times U(1)$.

If the scalar field acquires a non-vanishing vacuum expectation value, $\langle\phi\rangle = v$, then both the gauge bosons and fermions may become massive, $m_A = ev$, $m_\psi = \Gamma v$, due to the third and fourth term on the right hand side of (16.1). In what follows we shall concentrate on the scalar sector to understand how ϕ can acquire a non-vanishing vacuum expectation value. The proper quantities to describe the spontaneous symmetry breaking mechanism are effective potentials.

16.1 Legendre transformation

First we study these effective potential in quantum mechanics. We have already seen that the Schwinger function

$$W(\beta,j)=\frac{1}{\beta}\log\operatorname{tr}e^{-\beta(H-jq)}=\frac{1}{\beta}\log\Big[c\cdot\int\mathcal{D}x\exp\Big(-S+j\int\limits_0^\beta x(\tau)\Big)\Big], \tag{16.2}$$

where j is a constant external current, has the property that

$$W(j)=\lim_{\beta\to\infty}W(\beta,j)=-E_0(j). \tag{16.3}$$

Here $E_0(j)$ denotes the ground state energy of the shifted Hamiltonian $H-jq$. The conventional effective potential is obtained from the Schwinger function by a Legendre transformation

$$\Gamma(\beta,\bar\phi)=(LW)(\bar\phi)=\sup_j\Big[j\bar\phi-W(\beta,j)\Big]. \tag{16.4}$$

The maximizing current (if it exists) is called the current conjugate to $\bar\phi$.
Since Legendre transformations play an important role in the classical mechanics, thermodynamics and quantum field theory, let us first collect some relevant properties of these transformations. In the following $\bar\phi$ and j are elements of a convex set in $\mathbb{R}^n$.

1. The Legendre transform of a function which is convex for sufficient large arguments (here we are not concerned with domain problems) is *always convex.*
   To see that let

   $$\bar\phi_\alpha=(1-\alpha)\bar\phi_1+\alpha\bar\phi_2,\quad 0\le\alpha\le 1 \tag{16.5}$$

   be a point between $\bar\phi_1$ and $\bar\phi_2$. Then

   $$\begin{aligned}\Gamma(\bar\phi_\alpha)&=\sup_j\Big[(1-\alpha)(j,\bar\phi_1)+\alpha(j,\bar\phi_2)-\{(1-\alpha)+\alpha\}W(j)\Big]\\&\le(1-\alpha)\sup_j\Big[(j,\bar\phi_1)-W(j)\Big]+\alpha\sup_j\Big[(j,\bar\phi_2)-W(j)\Big]\\&=(1-\alpha)\Gamma(\bar\phi_1)+\alpha\Gamma(\bar\phi_2),\end{aligned}$$

   where we have used that the supremum of the sum is less or equal to the sum of the suprema. The last expression is just the linear interpolation between the points $(\bar\phi_i,\Gamma(\bar\phi_i))$. Thus we have shown that the graph of Γ is always below the segment between two points on this graph and this proves the convexity of Γ.

2. *The Legendre transform is involutive on convex functions.*
For convex W's there is a hyperplane passing through $(j_0, W(j_0))$ and lying below the graph of W. In other words, there is a $\bar\phi$ such that

$$W(j_0) + (\bar\phi, j - j_0) \leq W(j) \qquad \text{for all } j.$$

It follows that

$$(\bar\phi, j) - W(j) \leq (\bar\phi, j_0) - W(j_0) \Longrightarrow \Gamma(\bar\phi) \leq (\bar\phi, j_0) - W(j_0).$$

Since this is true for any j_0, we conclude

$$W(j_0) \leq (\bar\phi, j_0) - \Gamma(\bar\phi) \Longrightarrow W(j_0) \leq (L^2W)(j_0),$$

that is the double-Legendre transform is always greater or equal to the original function. On the other hand

$$\Gamma(\bar\phi) \geq (\bar\phi, j) - W(j) \quad \text{for all } \bar\phi \Longrightarrow W(j) \geq (\bar\phi, j) - \Gamma(\bar\phi).$$

Taking the supremum over all $\bar\phi$ of the last inequality we conclude

$$W(j) \geq (L^2W)(j),$$

or that the double-Legendre transform is always less or equal to the original function. Together with the above inequality we conclude that for any convex function

$$(L^2W)(j) = W(j). \tag{16.6}$$

3. If a continuous Schwinger function is not differentiable and possesses a *cusp*, then $\Gamma = LW$ develops a *plateau*. In the one-component case the width of the plateau is equal to the jump of W' at the cusp. Conversely, a plateau is transformed into a cusp.
This property follows from the graphical representation of the Legendre transformation: $\Gamma(\bar\phi)$ is just $L(0)$, where the linear function $L(j) = \bar\phi j - c$ is uniquely defined by the requirement that its graph (which is a plane) touches $W(j)$. For a given $\bar\phi$ and differentiable and strictly convex Schwinger function the conjugate current is determined by the requirement that $L(j)$ is tangential to the graph of $-W(j)$ at the conjugate current. The constant c in the linear function is then just $c = \bar\phi j + W(j)$ where j denotes the conjugate current.

4. An immediate consequence of the previous properties is that the double-Legendre transform of any function (which is convex for large arguments) is the *convex hull* of this function.

5. In the *differentiable case* the conjugate variables $\bar\phi$ and j are related by

$$\bar\phi = W'(j) \quad \text{and} \quad j = \Gamma'(\bar\phi). \tag{16.7}$$

   If we replace $(j,\bar\phi)\to(p,\dot x)$ and $(W,\Gamma)\to(H,L)$ this is the familiar Legendre transformation in classical mechanics from the Hamiltonian to the Lagrangian formulation.

6. One can prove the following identities

$$\begin{array}{l} LW=\Gamma \Longrightarrow LW_\alpha=\Gamma_\alpha, \quad \text{where} \quad F_\alpha(x)=\alpha F(x/\sqrt{\alpha}) \\ W(j)+\Gamma(\bar\phi)\geq (j,\bar\phi), \qquad = \Longleftrightarrow (j,\bar\phi) \text{ are conjugate} \\ W(j)=\dfrac{1}{\alpha}j^\alpha \Longleftrightarrow \Gamma(\bar\phi)=\dfrac{1}{\beta}\bar\phi^\beta, \quad \text{where} \quad \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1. \end{array}$$

After this excursion to the property of Legendre transformation we note that the Schwinger function is always convex, since

$$\frac{d^2}{dj^2}W(\beta,j)=\frac{1}{\beta}\int\limits_0^\beta ds d\tau \Big\langle \Big[x(s)-\langle x(s)\rangle_j\Big]\Big[x(\tau)-\langle x(\tau)\rangle_j\Big]\Big\rangle_j \geq 0,$$

where the expectation values are taken with respect to the shifted action $S-j\int x$, and thus are current-dependent. To get a better intuition for its Legendre transform Γ we note that for $\beta\to\infty$

$$W(j)=\sup_\psi\langle jq-H\rangle=\sup_\rho \text{ tr } \rho[jq-H]=\sup_{\bar\phi}\Big[j\bar\phi-\inf_{\text{tr}\,\rho q=\bar\phi}\text{tr}\,(\rho H)\Big], \tag{16.8}$$

where we have used that the set of density matrices $\{\rho|\text{tr}\,\rho=1,\rho=\rho^\dagger>0\}$ is a convex and compact set, and hence the infimimum of the linear functional $\text{tr}\,\rho(jq-H)$ is attained for pure states, $\rho=P_\psi$.

On the other hand, the constraints (there may be more than one q and thus several constraints) $\text{tr}\,\rho q=\bar\phi$ define a plane and thus the density matrices obeying these constraints form again a convex (and compact) set. It follows that the infimum of $\text{tr}\,\rho H$ on the constraint plane is attained on the intersection of this plane with the boundary of the set of density matrices. Let us assume that

$$\inf_{\text{tr}\,\rho=\bar\phi_i}\text{tr}\,\rho H=\text{tr}\,\rho_i H, \quad i=1,2$$

that is, ρ_1 and ρ_2 are the densities which minimize $\text{tr}\,\rho H$ under constraints $\text{tr}\,\rho_i q=\bar\phi_i$. Defining $\rho_\alpha=(1-\alpha)\rho_1+\alpha\rho_2$ one easily sees that $\text{tr}\,\rho_\alpha q=\bar\phi_\alpha$ (see 16.5) and hence

$$\begin{aligned} \inf_{\text{tr}\,\rho q=\bar\phi_\alpha}\text{tr}\,\rho H &\leq \text{tr}\,\rho_\alpha H=(1-\alpha)\text{tr}\,\rho_1 H+\alpha\text{tr}\,\rho_2 H \\ &= (1-\alpha)\inf_{\text{tr}\,\rho q=\bar\phi_1}\text{tr}\,\rho H+\alpha\inf_{\text{tr}\,\rho q=\bar\phi_2}\text{tr}\,\rho H. \end{aligned}$$

This implies that the function

$$\Gamma(\bar{\phi}) = \inf_{\operatorname{tr}\rho q=\bar{\phi}} H \tag{16.9}$$

is convex. From (16.8) it follows that W is the Legendre transform of the convex potential Γ and from our general consideration about Legendre transformations we conclude that the Legendre transform of Γ must be W:

$$\Gamma(\bar{\phi}) = \sup_j \Big[j\bar{\phi} - W(j)\Big] \quad \text{and} \quad W(j) = \sup_{\bar{\phi}} \Big[j\bar{\phi} - \Gamma(\bar{\phi})\Big]. \tag{16.10}$$

The infimum of Γ is

$$\inf_{\bar{\phi}} \Gamma(\bar{\phi}) = \inf_{\bar{\phi}} \inf_{\operatorname{tr}\rho q=\bar{\phi}} \operatorname{tr}\rho H = \inf_{\rho} H = \inf_{\psi} \langle\psi|H|\psi\rangle = E_0(j=0) \tag{16.11}$$

and thus just the vacuum energy of the (un-shifted) Hamiltonian.

The field $\bar{\phi}$ which minimizes Γ is then the expectation value $\bar{\phi} = \operatorname{tr}\rho q$ of q in the minimizing state ρ. If ρ is a pure state, then $\bar{\phi}$ is the unique vacuum expectation value of q. Else ρ can be written as convex combination of two pure states ρ_1 and ρ_2 with the same energy. It follows that for all $\bar{\phi}$ between $\bar{\phi}_1$ and $\bar{\phi}_2$, where $\bar{\phi}_i = \operatorname{tr}\rho_i q$, the value $\Gamma(\bar{\phi})$ is the same. In particular we conclude that Γ need not be strictly convex. More precisely, if the boundary of the set of states contains a "plane part" then any convex combination of two states on this plane is on the boundary. Thus the inequality above (16.9) becomes an equality and Γ develops a plateau. According to what we have said earlier, the Schwinger function is non-differentiable if Γ develops a plateau.

16.2 Effective potentials in field theory

Consider a field theory described by a Lagrangian density

$$\mathcal{L}(\phi(x)) = \int \Big\{\frac{1}{2}\partial_i\phi(x)\partial_i\phi(x) + V(\phi(x))\Big\}, \tag{16.12}$$

where $\phi(x)$ is a Higgs field which generally transforms non-trivially under the action of a symmetry group G. The classical vacuum is defined by the minimum of the classical action and thus is given by a constant field which minimizes the classical potential $V(\phi)$. This value is not necessarily the vacuum expectation value of the quantum field $\langle\phi(x)\rangle$. To study the quantum corrections to the classical value one introduces effective potentials.

Similarly to the quantum mechanical situation we begin with the partition function

$$Z(\Omega, j) = \int \mathcal{D}\phi \exp\Big(-S[\phi] + j\int_\Omega \phi(x)\Big) \tag{16.13}$$

in the presence of a constant external current j. The current is chosen constant so as to preserve the translational invariance of $Z(j)$. For finite volumes Ω, translational invariance is understood to be with respect to periodic boundary conditions. Again the Schwinger function

$$W(\Omega, j) = \frac{1}{\Omega} \log\, Z(\Omega, j) \tag{16.14}$$

is strictly convex since its second derivative is (Ω times) the expectation value of the positive quantity $(M - \langle M\rangle_j)^2$, where $M = (1/\Omega)\int \phi(x)d^dx$. The current-dependent expectation values are to be computed with the shifted action as in (16.13). $W(j)$ allows one to compute the effective field, defined as

$$\langle\phi(x)\rangle_j = \frac{\int \mathcal{D}\phi\,\phi(x)\,e^{-S[\phi]+j\int\phi}}{\int \mathcal{D}\phi\,e^{-S[\phi]+j\int\phi}} = \frac{dW}{dj}. \tag{16.15}$$

Of course, in cases where W is non-differentiable (or equivalently Γ shows at least one plateau) we must be cautious what we mean by formulae like (16.15). We shall come back to this point later on.

The conventional effective potential $\Gamma(\Omega, \bar\phi)$ in (16.4) is the Legendre transform of W. If the minimum of Γ occurs at a unique point $\bar\phi = \bar\phi_0$, the point $\bar\phi_0$ defines the vacuum state of the theory, and the semiclassical expansion around $\bar\phi_0$ generates the one-particle-irreducible Feynman graphs [54]. The minimum is unique if either the volume is finite, or the classical potential is convex (or both). When the classical potential is not convex, as happens in particular for spontaneous broken potentials, the minimal points $\bar\phi_0$ of $\Gamma(\bar\phi) = \Gamma(\infty, \bar\phi)$ are not unique but lie on a plane in $\bar\phi$-space, as pointed out above. In this case the vacuum is not determined by $\Gamma(\bar\phi)$ but by $\Gamma(\bar\phi)$ plus the direction from which a trigger current j approaches the value zero. Such a trigger current forces the system into a pure state. As we have seen, the expectation value $\bar\phi$ in a pure state lies on the edge of the plane of Γ. Furthermore, in the degenerate case the naive semiclassical expansion for the effective potential breaks down and must be replaced by some alternative approximation.

Since for $\Omega = \infty$, V non-convex, the loop expansion (semiclassical expansion) has problems, a computational approach is more desirable, and in that case Γ may not be the best quantity to consider. Also note that we haven't been able to write down an explicit path integral representation for the conventional effective potential Γ. A much more suitable and direct (at least in the path integral approach) quantity is the effective potential defined by

$$\exp\Big(-\Omega U(\Omega,\bar\phi)\Big) = \int \mathcal{D}\phi\,\delta(M-\bar\phi)\,e^{-S[\phi]}, \qquad M = \frac{1}{\Omega}\int d^dx\,\phi(x), \tag{16.16}$$

which we called *constraint effective potential* in [55]. Clearly, if the classical potential is invariant under the action of the symmetry group, then $U(\Omega, \bar\phi)$ is invariant as well. The constraint effective potential is relates to similar definitions in statistical mechanics and spin systems and

$$P(\bar\phi) \equiv \frac{e^{-\Omega U(\Omega,\bar\phi)}}{\int d\bar\phi\, e^{-\Omega U(\Omega,\bar\phi)}} \tag{16.17}$$

it to be interpreted as the probability density for the system to be in the state of "magnetization" $\bar{\phi}$. The probability for the occurrence of a state whose averaged field is not a minimum of U then becomes less and less as $\Omega \to \infty$. Also, the constraint effective potential is a more direct quantity to compute with Monte Carlo simulations, since an external current need not be introduced.

Multiplying both sides of (16.16) by $\exp(\Omega j\bar{\phi})$ and integrating over $\bar{\phi}$, yields

$$\int e^{\Omega[j\bar{\phi}-U(\Omega,\bar{\phi})]}d\bar{\phi} = e^{\Omega W(\Omega,j)}. \tag{16.18}$$

Hence W is related to U by a Laplace transformation. Note that since Γ is the Legendre transform of W, the function $\Gamma(\Omega,\bar{\phi})$ is uniquely defined by $U(\Omega,\bar{\phi})$. Conversely, since W is the Legendre transform of Γ, U can be recovered from Γ by an inverse Laplace transformation. Thus there is a one-to-one correspondence between $U(\Omega,\bar{\phi})$ and $\Gamma(\Omega,\bar{\phi})$.
Now let us discuss what happens in the infinite-volume limit $\Omega \to \infty$. In this limit the saddle-point approximation to the ordinary integral (16.18) becomes exact. Then

$$W(j) = \sup_{\bar{\phi}} \left(j\bar{\phi} - U(\bar{\phi})\right) = (LU)(j). \tag{16.19}$$

It follows that $\Gamma = LW = L^2U$. Thus Γ is the convex hull of U. Although $U(\Omega,\bar{\phi})$ is in general not convex for finite volumes, one can prove that it becomes convex for $\Omega \to \infty$ [55] so that

$$\Gamma(\bar{\phi}) = U(\bar{\phi}). \tag{16.20}$$

Thus in the infinite-volume limit the two potentials become identical. However, in a finite volume the two potentials are not identical and $U(\Omega,\bar{\phi})$ need not necessarily be convex.

The constraint effective potential is also useful for extracting information directly about the gross properties of the system such as whether is suffers a spontaneous symmetry breakdown or whether it has a finite correlation length. To see this ones notes that

$$\frac{\int \bar{\phi}^p \exp\left[\Omega(j\bar{\phi} - U(\Omega,\bar{\phi}))\right] d\bar{\phi}}{\int \exp\left[\Omega(j\bar{\phi} - U(\Omega,\bar{\phi}))\right] d\bar{\phi}} = \frac{1}{\Omega^p}\int d^dx_1 \dots d^dx_p \langle\phi(x_1)\cdots\phi(x_d)\rangle_j^\Omega, \tag{16.21}$$

i.e. that the moments of $N^{-1}\exp\left[\Omega(j\bar{\phi} - U(\Omega,\bar{\phi}))\right]$ are the averaged Schwinger (correlation) functions. For $p = 1$ this gives the vacuum expectation value

$$\langle\phi(x)\rangle_j^\Omega = N^{-1}\int \bar{\phi} e^{\Omega[j\bar{\phi}-U(\Omega,\bar{\phi})]}d\bar{\phi}. \tag{16.22}$$

For any finite volume the symmetry of U leads to $\langle\phi(x)\rangle_0^\Omega = 0$ To get a non-trivial result one must keep a trigger current, and only after the infinite-volume limit has been taken can the trigger be removed. If there remains a non-trivial expectation value after setting $j = 0$ then there is a spontaneous symmetry breaking.

For $p = 2$ and $j = 0$ the formula (16.21) reads

$$N^{-1}\int \bar\phi^2 e^{\Omega[j\bar\phi - U(\Omega,\bar\phi)]}d\bar\phi = \frac{1}{\Omega^2}\int d^dx_1 d^dx_2 \langle\phi(x_1)\phi(x_2)\rangle_0^\Omega.$$

The expectation value of the r.h.s. is the $2-$point Schwinger function $S_2(x_2 - x_1)$ which only depends on the difference of the coordinates because of translational invariance. So we end up with the explicit formula for the susceptibility

$$\chi = \int S_2(x)d^dx = \Omega\frac{\int \bar\phi^2 e^{-\Omega U(\Omega,\bar\phi)}}{\int e^{-\Omega U(\Omega,\bar\phi)}}. \tag{16.23}$$

16.3 Lattice approximation

For the above formal manipulations to make sense we need to define the functional integrals for scalar fields. In the previous chapters we have dealt with functional integrals fermions (see 12.4) and gauge bosons (see 14.32). Fermionic integrals are Gaussian integrals for Grassmann-valued variables (at least for fermions without explicit self-interaction as in the Thirring model). Thus fermionic path integrals always lead to determinants and can be given a precise meaning by defining determinants ”properly”. Similarly, for the Schwinger model the integral over all gauge fields lead to a functional determinant as well and the $\mathcal{D}A_\mu$ integral can be defined via the corresponding determinant. For a genuinely self-interacting field this ins not possible anymore, at least if we go beyond perturbation theory.

One of the more popular, non-perturbative definition uses the lattice regularization. As in quantum mechanics (see 6.21) one first puts the field theory on a $d-$dimensional space-time lattice discretizing the euclidean space-time by a hypercubic lattice with lattice spacing a. The action for a scalar field becomes

$$S[\phi] = \sum_{\langle ij\rangle} a^{d-2}\frac{1}{2}\left(\phi_i - \phi_j\right)^2 + \sum_i a^d V(\phi_i), \tag{16.24}$$

where $\phi_i = phi(x_i), (i = 1, 2, \ldots, N = \Omega/a^d)$ and $\sum_{\langle ij\rangle}$ is the sum over all nearest neighbour pairs. We take periodic boundary conditions. By introducing a dimensionless lattice field $\phi^L = a^{d/2-1}\phi$, (16.24) can be rewritten as

$$S[\phi] = S^L[\phi^L] = \sum_{\langle ij\rangle}\frac{1}{2}\left(\phi_i^L - \phi_j^L\right)^2 + \sum_i V^L(\phi_i^L), \tag{16.25}$$

where the lattice potential V^L is equal to the classical potential, but with rescaled parameters. The masses and coupling constants are rescaled according to their dimensions, e.g. $m_L = a^2 m$ etc. By using the lattice field as new integration variable the constraint effective lattice potential, which is related to the continuum potential as

$$\Omega U(\Omega,\bar\phi) = NU^L(N,\bar\phi^L) + \text{const}(a), \tag{16.26}$$

where $\bar{\phi}^L = a^{d/2-1}\bar{\phi}$ is dimensionless, is easily found to be

$$e^{-NU^L(N,\bar{\phi}^L)} = \int \prod d\phi_i^L \delta\left(\frac{1}{N}\sum \phi_i^L - \bar{\phi}^L\right) e^{-S^L[\phi^L]}. \tag{16.27}$$

This lattice version (or rather the corresponding lattice version for the partition function) should be compared with the analog expression (6.21) in quantum mechanics. For a finite a one recovers $U(\Omega,\bar{\phi})$ from $U^L(N,\bar{\phi}^L)$ by a trivial rescaling of U^L and $\bar{\phi}^L$. In what follows the subscript L will mostly be dropped. Note that in terms of dimensionless quantities the theory is defined only on a unit lattice of size N. For a fixed lattice constant a the volume is proportional to N. Hence, studying the volume dependence of $U(\Omega,\bar{\phi})$ is equivalent to studying the N-dependence of the corresponding lattice potential.

Let us first consider models in which there are no kinetic terms, which we shall call incoherent models since the field on different lattice points then behave independently. At first sight these models may appear to be trivial, but there are good reasons for studying them. First, they show properties which one encounters in the full theory and secondly we can extract the influence of the kinetic term on the effective potentials by comparing the incoherent models with those of the full theory. In addition, the incoherent models deliver upper and lower bounds for the true effective potentials.

In order to factorize the functional integral (16.27) in the absence of the kinetic term we replace the constraint

$$\delta(M-\bar{\phi}) = N\delta\left(\sum \phi_i - \bar{\phi}\right) = \frac{N}{2\pi}\int dp \exp\left[ip(N\bar{\phi} - \sum \phi_i)\right].$$

As a consequence

$$e^{-NU_0(N,\bar{\phi})} = \frac{N}{2\pi}\int dp\, e^{N\left[ip\bar{\phi}+\log f(p)\right]},$$

where $f(p) = \int \exp[-ip\phi - V(\phi)]d\phi$. For large N this integral approaches its saddle-point value. The saddle point of $ip\bar{\phi} + \log f(p)$ in the complex p-plane is the point $p = ij$, where j is a solution of $\bar{\phi} = dW_0/dj$ and $\exp(W_0(j)) = \int d\phi \exp[j\phi - V(\phi)]$. Hence we find that in the limit $N \to \infty$

$$U_0(\bar{\phi}) = \Gamma_0(\bar{\phi}) = (LW_0)(\bar{\phi}), \tag{16.28}$$

where W_0 is the Schwinger function of the zero-dimensional theory with potential V, i.e.

$$e^{W_0(j)} = \int e^{j\phi - V(\phi)} d\phi. \tag{16.29}$$

Clearly, since we have neglected the positive kinetic energy in the action the incoherent potential Γ_0 yields a lower bound on the exact potential,

$$U(\bar{\phi}) \geq \Gamma_0(\bar{\phi}). \tag{16.30}$$

Also note that since W_0 is a differentiable and strictly convex function the constraint potential Γ_0 is strictly convex as well.

Next we derive an upper bound for U. Since $\frac{1}{2}(\phi_i-\phi_j)^2 \le \phi_i^2+\phi_j^2$ we find

$$T[\phi]=\frac{1}{2}\sum_{\langle ij\rangle}(\phi_i-\phi_j)^2 = T[\phi-\bar\phi] \le 2d\sum\phi_i^2 - 4d\bar\phi\sum_i\phi_i + 2dN\bar\phi^2, \tag{16.31}$$

where we have taken into account that in d dimensions every site has $2d$ nearest neighbours. Inserting this inequality into (16.27) one obtains

$$e^{-NU(N,\bar\phi)} \ge e^{2dN\bar\phi^2}\int \delta(M-\bar\phi)e^{-V_{2d}[\phi]},$$

where $V_{2d}[\phi]=2d\sum\phi_i^2+\sum_i V(\phi_i)$. This yields the upper bound

$$U(\bar\phi)\le -2d\bar\phi^2+\Gamma_{2d}(\bar\phi), \tag{16.32}$$

where

$$\Gamma_{2d}(\bar\phi)=(LW_{2d})(\bar\phi) \quad \text{and} \quad W_{2d}=\log\int d\phi\, e^{j\phi-V_{2d}(\phi)} \tag{16.33}$$

is the incoherent constraint effective potential which corresponds to the classical potential with shifted mass $V_{2d}(\phi)$. In general the function on the r.h.s. of (16.32) is not convex. However, since $U(\phi)$ is known to be convex, (16.32) actually implies that

$$U(\bar\phi)\le L^2\left(-2d\bar\phi^2+\Gamma_{2d}(\phi)\right). \tag{16.34}$$

We conclude our discussion of the analytic properties of $U(\bar\phi)$ by deriving an Ehrenfest equation which is very useful for Monte-Carlo simulations. For that purpose we shift the field by a constant $\phi_i\to\phi_i+\bar\phi$ in (16.27). Because of the translational invariance of the measure $\mathcal{D}\phi$ we obtain

$$e^{-NU(N,\bar\phi)}=\int\mathcal{D}\phi\,\delta(M)e^{-S[\phi+\bar\phi]}.$$

Only the potential term in the action is affected by the shift and therefore

$$\frac{d}{d\bar\phi}U(N,\bar\phi)=\frac{1}{N}\left\langle V'[\phi]\right\rangle_{\bar\phi}, \tag{16.35}$$

where

$$\langle O[\phi]\rangle_{\bar\phi}=\frac{\int\mathcal{D}\phi\,\delta(M-\bar\phi)O[\phi]e^{-S[\phi]}}{\int\mathcal{D}\phi\,\delta(M-\bar\phi)e^{-S[\phi]}} \tag{16.36}$$

and this is the required Ehrenfest equation which relates the derivative of the quantum potential to the expectation value of the derivative of the classical potential. For example, for the Higgs-model

$$V(\phi) = \lambda(\phi^2 - \sigma^2)^2 \tag{16.37}$$

the Ehrenfest equation reads

$$U'(\bar{\phi}) = 4\lambda \left[\langle \phi^3(x) \rangle_{\bar{\phi}} - \sigma^2 \bar{\phi}^2 \right], \tag{16.38}$$

where we have used the translational invariance, i.e. that $\langle \sum \phi_i^3 \rangle_{\bar{\phi}} = N \langle \phi_i^3 \rangle_{\bar{\phi}}$. For the Higgs model this equation can be used and has been used for the MC simulations. The following figures show the two effective potentials Γ and U for the Higgs models (16.37) in various dimensions and for different "volumes" N. Also the lower and upper bounds (16.30) and (16.34) are plotted in the figures. The calculation has been done with a modified Metropolis algorithm (see section 9.2).

So far we considered the regularized scalar theories, that is we kept the lattice constant a fixed. At the end we wish to let the lattice constant tend to zero in order to remove the regularization. Then the bar quantities have to be related to physical quantities by renormalization. First one introduces a dimensionless lattice length $\epsilon(a = \epsilon \Lambda^{-1},)$ where Λ is a scale parameter with a mass dimension) and compares the lattices $\mathbb{Z}^d$ and $\epsilon\mathbb{Z}^d$ when ϵ is allowed to take values in the interval $0 < \epsilon \leq 1$. The parameters and field are scaled so that the scaled potential becomes

$$U^\epsilon(\bar{\phi}, m, g) = \epsilon^{-d} U^{\epsilon=1} \left(Z \epsilon^{(d-2)/2} \bar{\phi}, m(\epsilon), g(\epsilon) \right) \tag{16.39}$$

has a continuum limit. We define a "physical" mass μ_p and coupling constant g_p in terms of U^ϵ by some typical equation such as (for $d = 4$)

$$\langle \phi^2 \rangle_\epsilon = \frac{\Lambda^2}{\mu_p^2} \quad \text{and} \quad \langle \phi^4 \rangle_\epsilon - \langle \phi^2 \rangle_\epsilon^2 = \frac{1}{g_p}, \tag{16.40}$$

where

$$\langle O \rangle_\epsilon = \frac{\int d\bar{\phi}\, O(\bar{\phi}) e^{-U^\epsilon}}{\int d\bar{\phi}\, e^{-U^\epsilon}}. \tag{16.41}$$

Of course, μ_p and g_p as defined in (16.40) will not necessarily be the physical mass and quartic coupling constant for the scalar field, but just some related physical quantities.

Now the lattice renormalization consists in letting the bare mass and coupling constant m, g depend on ϵ in such a way that the physical constants μ_p, g_p do not depend on ϵ. Given the dependence of U^ϵ on ϵ, m, g and given $m(1) = m, g(1) = g$ the ϵ-dependence of the bare parameters is then determined implicitly by (16.40). In other words, the renormalization consists of constructing an ϵ-dependent map from (μ_p, g_p) to (m, g) by means of (16.40).

In the broken phase it maybe preferable to choose different renormalization conditions. Since the vacuum expectation value of the Higgs field is related to masses of the fermions and massive gauge bosons and the curvature of the effective potential at its minimum is related to the mass of the Higgs particles one may take the renormalization conditions

$$U^\epsilon(\bar\phi) = \text{ minimal for } \bar\phi = \phi_p \quad \text{and} \quad \frac{d^2}{d\bar\phi^2}U^\epsilon|_{\phi_p} = m_p, \tag{16.42}$$

where the physical quantities (ϕ_p, m_p) are measured w.r. to some mass scale Λ. We have already pointed out that in the broken phase Γ and hence U (recall that $\Gamma = U$ in the thermodynamic limit $N \to \infty$) develop a plateau. As minimizing value ϕ_p in (16.42) we take the maximal ϕ_p which minimizes U since this belongs to a pure phase of the theory. Also we evaluate the second derivative near ϕ_p but a little bit away from the plateau.

16.4 Mean field approximation

Let us see how this renormalization works for the mean field approximation to the exact effective potential. In this approximation one replaces the interaction of ϕ_i with its nearest neighbours in the classical action

$$S[\phi] = d\sum_i \phi_i^2 - \sum_{\langle ij\rangle}\phi_i\phi_j + \sum_i V(\phi_i) \tag{16.43}$$

by its mean interaction with all spins

$$\sum_{\langle ij\rangle}\phi_i\phi_j = \sum_i \phi_i \frac{1}{2}\sum_{j:|i-j|=1}\phi_j \longrightarrow \sum_i \phi_i \frac{d}{N}\sum_j \phi_j = \frac{d}{N}\Big(\sum_i \phi_i\Big)^2 .$$

After this replacement the constraint effective potential simplifies to

$$e^{-NU_{MF}(\bar\phi)} = e^{dN\bar\phi^2}\int \mathcal{D}\phi\,\delta(M-\bar\phi)e^{-\sum V_d(\phi_i)}, \quad \text{where} \quad V_d(\phi) = d\phi^2 + V(\phi), \tag{16.44}$$

and hence becomes an incoherent model. Analog to (16.30) and (16.33) we obtain

$$U_{MF}(\bar\phi) = -d\bar\phi^2 + \Gamma_d(\bar\phi), \quad \text{where} \quad \Gamma_d = LW_d, \quad W_d = \log\int d\phi\, e^{j\phi - V_d(\phi)}. \tag{16.45}$$

Note that $U_{MF}(\bar\phi)$ is half way between the lower bound $\Gamma_0(\bar\phi)$ in (16.30) and the upper bound $-2d\bar\phi^2 + \Gamma_{2d}(\bar\phi)$ in (16.33). One can actually prove that U_{MF} is also an upper bound for the exact potential. Note that U_{MF} is differentiable and in the non convex case (the broken phase of the MF-model) it displays no plateau. Hence the apparent problem with the conditions (16.42) mentioned after (16.42) do not arise and we may take these conditions literally.

We shall need the minimum ϕ_0 of U_{MF} and the curvature at this minimum. Using $j(\bar\phi) = \Gamma'_d(\bar\phi)$, which relates the current to its conjugate field, one sees at once that the minimum condition becomes

$$j_0 = j(\phi_0) = 2d\phi_0. \tag{16.46}$$

Since Γ_d is the Legendre transform of W_d the inverse relation reads $\phi(j) = W'_d(j)$. By inserting the minimum condition into that equation we find the self-consistency equation

$$\phi_0 = \frac{\int d\phi\,\phi\, e^{j_0\phi - V_d(\phi)}}{\int d\phi\, e^{j_0\phi - V_d(\phi)}} = \langle\phi\rangle_{j_0}, \quad \text{where} \quad j_0 = 2d\phi_0, \tag{16.47}$$

for the expectation value of the Higgs field. To compute $U''_{MF}(\phi_0)$ we use the relation $\Gamma''(\bar\phi) = W''[j(\bar\phi)]^{-1}$ between the curvatures of two Legendre-related functions. Together with the minimum conditions one obtains

$$m_0 = U''_{MF} = -2d + \Big\langle(\phi-\phi_0)^2\Big\rangle_{j_0}^{-1} \tag{16.48}$$

for the Higgs-boson mass in the broken phase. Clearly the incoherent Schwinger function in (16.45) is strictly convex and symmetric (if V is symmetric) and hence $j(\bar\phi)$ vanishes when $\bar\phi$ does. From (16.48) we conclude that the curvature of U_{MF} at the origin is negative when $\langle\phi^2\rangle_0 > 1/2d$. Consequently the potential (16.45) is spontaneously broken in case where

$$\frac{\int \phi^2 e^{-V_d(\phi)}}{\int e^{-V_d(\phi)}} = \langle\phi^2\rangle_0 > \frac{1}{2d}. \tag{16.49}$$

Suppose, for example, that the mass m in $V(\phi) = m\phi^2 + g\phi^4$ is less than $-d$. Since the m-derivative of the expectation value $\langle\phi^2\rangle_0$ decreases with increasing mass the expectation value becomes smaller when m is replaces by $-d$. However, for $m = -d$ the effective mass $m+d$ in V_d vanishes and the expectation value can be computed explicitly. In this way one finds from (16.49) that U_{MF} is spontaneously broken when

$$m < -d \quad \text{and} \quad g < \left[2d\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]^2. \tag{16.50}$$

The continuum limit for the mean-field theory: As physical parameters we take the expectation value of the Higgs field ϕ_p and the Higgs-boson mass m_p in the broken phase (see 16.42). One can prove that in the mean field approximation the wave function renormalization constant Z in (16.39) is one [56] so that the potential on $(\epsilon\mathbb{Z})^d$ becomes

$$U^\epsilon_{MF}(\bar\phi) = \epsilon^{-d}U_{MF}(\epsilon^{d/2-1}\bar\phi) = -d\epsilon^{-2}\bar\phi^2 + \epsilon^{-d}\Gamma_d\left(m(\epsilon), g(\epsilon), e^{d/2-1}\bar\phi\right),$$

where the scaled bare parameters are to be determined by the renormalization conditions. Here we take the conditions (16.42).

Clearly, when ϕ_p minimizes U^ϵ_{MF} then $\epsilon^{d/2-1}\phi_p$ minimizes U_{MF} and satisfies the self-consistency equation (16.47). Thus, the first renormalization condition reads

$$\epsilon^{d/2-1}\phi_p = \langle\phi\rangle_{j_p}, \quad \text{where} \quad j_p = 2d\epsilon^{d/2-1}\phi_p \tag{16.51}$$

and the expectation values have been defined in (16.47). In the same way, using (16.48), one obtains the second renormalization condition

$$\epsilon^2 m_p = \epsilon^2 \left(U^\epsilon_{MF}\right)''(\phi_p) = -2d + \left\langle \left(\phi - \epsilon^{\frac{d}{2}-1}\phi_p\right)^2 \right\rangle^{-1}_{j_p}. \tag{16.52}$$

The following asymptotic renormalization flows for $\epsilon \to 0$ in 2 and 3 dimensions can be derived:

$$\begin{aligned} d=2: &\qquad g(\epsilon) \sim \frac{m_p}{8\phi_p^2}\epsilon^2, \quad m(\epsilon) \sim -\left(\frac{3}{2} + 2\phi_p^2\right) g(\epsilon) \\ d=3: &\qquad g(\epsilon) \sim \frac{m_p}{8\phi_p^2}\epsilon, \quad m(\epsilon) \sim -g(\epsilon). \end{aligned} \tag{16.53}$$

For details I refer to [56].