Chapter 12

Path Integral for Fermion Fields

After introducing path integrals in quantum mechanics we now turn to the path integral representation of field theories. In this chapter we discuss the fermionic sector of the *Schwinger model*, which is probably the simplest non-trivial field theory. The Schwinger model is just QED for massless fermions in 2 dimensions [42]. This model shows at least two (related) striking features. First the classically massless 'photon' acquires a mass due to its interaction with the massless fermions and second the operator $\bar{\psi}(x)\psi(x)$ has a non-vanishing vacuum expectation [43]. Clearly, since this model contains fermions we first must discuss the path integral for fermionic, and in particular the path integral representation of the n-point functions.

The zero-temperature Schwinger model has been solved some time ago by using operator methods [44] and more recently in the path integral formulation [45]. Some properties of the model (e.g. the non-trivial vacuum structure) are more transparent in the operator approach and others (e.g. the role of the chiral anomaly) are better seen in the path integral approach. More recently the Schwinger model has been solved in the path integral approach on the 2-dimensional sphere and the role of the fermionic zero modes has been emphasized [46].

12.1 Dirac fermions

To arrive at the path integral for Dirac fermions (e.g. electrons). we generalize the above results to field theory, that is, we replace

$$\bar{\alpha}_i(t) \to \bar{\psi}(\boldsymbol{x}, t) \quad \text{and} \quad \alpha_i(t) \to \psi(\boldsymbol{x}, t).$$

The discrete index i becomes the continuous position in space and the summation is to be replaced by an integration over space.

For the Dirac fermions minimally coupled to a gaute field A_μ the action reads

$$S = \int_\Omega \mathcal{L} \quad \text{with} \quad \mathcal{L} = \bar{\psi}(i\not{D} - m)\psi. \tag{12.1}$$

The canonical momentum density is proportional to the field,

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger, \tag{12.2}$$

and not to the time-derivative of the field, since the Lagrangian density only contains first order derivatives. The Hamiltonian is given by a Legendre transform,

$$H = \int \mathcal{H}, \qquad \mathcal{H} = \pi\dot{\psi} - \mathcal{L} = -i\bar{\psi}\gamma^j D_j\psi + m\bar{\psi}\psi. \tag{12.3}$$

Inserting this into the field-theoretical generalization of (10.25) we obtain the functional integral representation

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{iS[\bar{\psi},\psi]}, \tag{12.4}$$

where S is the action for fields on a space-time region Ω. The boundary conditions for the fields on the boundary $\partial\Omega$ must be specified. Here we choose for Ω the Minkowski space to avoid boundary effects.

We are primarily interested in the generating functional in the presence of external currents, which now is constructed by using two *anticommuting sources* $\bar{\eta}(x)$ and $\eta(x)$:

$$Z[\bar{\eta},\eta] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; \exp\Big(iS[\bar{\psi},\psi] + i\int [\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)]d^dx\Big). \tag{12.5}$$

We can simplify this path integral by expanding the exponent about its extremum. The exponent is extreme for

$$\psi_{\rm cl} = -(i\not{D} - m)^{-1}\eta \quad \text{and} \quad \bar{\psi}_{\rm cl} = -\bar{\eta}(i\not{D} - m)^{-1}.$$

Shifting the variables according to $\psi \to \psi_{\rm cl} + \psi$ etc. the exponent becomes

$$iS_{\rm cl} + iS[\bar{\psi},\psi], \quad \text{where} \quad S_{\rm cl} = -\bar{\eta}\frac{1}{i\not{D} - m}\eta = -\int \bar{\eta}(x)G_F(x,y)\eta(y), \tag{12.6}$$

and G_F denotes the Feynman propagator

$$(i\not{D}_x - m)G_F(x,y) = \delta(x-y). \tag{12.7}$$

For example, for the free field ($A = 0$) one has

$$G_F(\xi) = -(i\not{\partial} + m)\Delta_F(\xi), \tag{12.8}$$

where Δ_F is the Feynman propagator of the Klein-Gordon field:

$$\Delta_F(\xi) = -\frac{1}{(4\pi)^2}\int d^4x e^{-ip\xi}\frac{1}{p^2 - m^2 + i\epsilon} \Longrightarrow (\partial_\mu\partial^\mu + m^2)\Delta_F = \delta^4(\xi). \tag{12.9}$$

Since $S_{\rm cl}$ is independent of the integration variables, the path integral (12.5) reads

$$Z[\bar\eta,\eta]=\det(i\not{D}-m)\exp\Big(-i\int\bar\eta(x)G_F(x,y)\eta(y)d^dxd^dy\Big). \tag{12.10}$$

Differentiating (12.5) with respect to the sources η and $\bar\eta$ yields the correlation function

$$\begin{aligned}&T\langle 0|\psi_{\alpha_1}(x_1)\bar\psi_{\beta_1}(y_1)...\psi_{\alpha_n}(x_n)\bar\psi_{\beta_n}(y_n)|0\rangle\\&=\frac{1}{Z[0]}\frac{\delta^{2n}}{\delta\eta^{\beta_n}(y_n)\delta\bar\eta^{\alpha_n}(x_n)...\delta^{\beta_1}\eta(y_1)\delta\bar\eta^{\alpha_1}(x_1)}Z[\bar\eta,\eta]|_{\bar\eta=\eta=0}.\end{aligned} \tag{12.11}$$

The n-point functions, for n odd, vanish since the source term is even in the current. In particular, for $n=2$ we recover the propagator (Feynman propagator). Using Wick's theorem (which we shall proof later) one shows that the $2n$-point function can be expressed in terms of the two point function only. This shows already the equivalence of the Berezin path integral approach and the canonical approach.

We conclude this section with the proof of Wick's theorem for fermions. This theorem is extensively used in quantum field theory. Originally it was proven using canonical methods. Now we shall see how to derive this theorem using functional integration. What we show is the following representation for the $2n$-point function in terms of the 2-point function:

$$\begin{aligned}T\langle 0|\psi(x_1)\bar\psi(y_1)...\psi(x_n)\bar\psi(y_n)|0\rangle &= \frac{1}{Z[0]}\int\mathcal{D}\psi\mathcal{D}\bar\psi\;\psi(x_1)\bar\psi(y_1)...\psi(x_n)\bar\psi(y_n)e^{iS[\bar\psi,\psi]}\\&=(-i)^n\sum_{\pi\in S_n}\text{sign}(\pi)\prod_{j=1}^n G_F(x_j,y_{\pi(j)}).\end{aligned} \tag{12.12}$$

To prove this identity we use the generating functional (12.5) and expand the exponent containing the source-terms in a power series:

$$\begin{aligned}Z[\bar\eta,\eta]&=\int\mathcal{D}\psi\mathcal{D}\bar\psi\;e^{iS}\sum_n\frac{i^{2n}}{(2n)!}\int dx_1...dx_{2n}\prod_{i=1}^{2n}(\bar\eta(x_i)\psi(x_i)+\bar\psi(x_i)\eta(x_i))\\&=\sum_n\frac{(-)^n}{(2n)!}\int\prod_1^{2n}dx_i\bar\eta^{\alpha_1}(x_1)...\eta^{\beta_n}(x_{2n})\frac{(2n)!}{n!n!}\int\mathcal{D}\psi\mathcal{D}\bar\psi\;e^{iS}\psi_{\alpha_1}(x_1)...\bar\psi_{\beta_n}(x_{2n}),\end{aligned} \tag{12.13}$$

where we have used the anticommutation properties of the fields and sources and the fact that the functional integral is nonzero only if there are as many fields as adjoint fields. On the other hand using (12.10) we may expand the generating functional as

$$\frac{Z[\bar\eta,\eta]}{Z[0]}=\sum_n\frac{(-i)^n}{n!}\int dx_1...dx_ndy_1...dy_n\bar\eta(x_1)\eta(y_1)...\eta(y_n)\prod_{i=1}^n G_F(x_i,y_i) \tag{12.14}$$

and using again the anticommutation properties we can rewrite Z as

$$\frac{Z[\bar\eta,\eta]}{Z[0]}=\sum_n\frac{(-i)^n}{n!n!}\int\prod_1^n dx_idy_i\bar\eta(x_1)\eta(y_1)...\eta(y_n)\sum_{\pi\in S_n}\prod_{i=1}^n\text{sign}(\pi)G_F(x_2,y_{\pi(i)}). \tag{12.15}$$

Comparing with (12.13) and using the fact that the sources are arbitrary, proves the Wick theorem (12.12).

Finally we turn to the fermionic thermal Green's functions. As we have already seen in in quantum mechanics, the transition to the Euclidean sector is made by replacing $t \to -i\tau$ such that

$$\partial_0 \to i\partial_0, \quad A_0 \to iA_0, \quad A^0 \to -iA^0, \quad \gamma_0 \to i\gamma_0, \quad \gamma^0 \to -i\gamma^0 \tag{12.16}$$

(and keeping the other quantities fixed) or equivalently by replacing $x^j \to ix^j$ such that

$$\partial_j \to -i\partial_j, \quad A_j \to -iA_j, \quad A^j \to iA^j, \quad \gamma_j \to -i\gamma_j, \quad \gamma^j \to i\gamma^j. \tag{12.17}$$

Since we prefer to use a Minkowskian metric with signature $(+,-,-,-)$ we continue according to (12.17) rather than (12.16). In the case of Dirac fermions the exponent in (12.5) becomes then

$$\begin{aligned} iS + i\int(\bar\eta\psi + \bar\psi\eta) &\longrightarrow -S_E + \int(\bar\eta\psi + \bar\psi\eta), \\ S_E = \int \mathcal{L}_E, \quad \mathcal{L}_E &= -i\bar\psi \not{D}\psi + m\bar\psi\psi. \end{aligned} \tag{12.18}$$

When calculating the partition function $Z(\beta)$ at finite temperature we must choose antiperiodic boundary conditions for the fields, in contrast to to the bosonic case (see (8.22)). The reason is that the fermionic Green's functions are β-periodic in imaginary time [48]. This is taken into account if antiperiodic boundary conditions in the path integral are chosen and then the partition function becomes

$$Z(\beta) = \text{const} \cdot \int\limits_{a.p.} \mathcal{D}\psi\mathcal{D}\bar\psi\; e^{-S_E[\bar\psi,\psi]}, \tag{12.19}$$

where $a.p.$ should indicate that we integrate over anti-periodic fields $\psi(\hbar\beta, \boldsymbol{x}) = -\psi(0, \boldsymbol{x})$ and analog for $\bar\psi$. In analogy to (12.5) the generating functional for the thermal Green's functions reads

$$Z[\beta,\bar\eta,\eta] = \int\limits_{a.p.} \mathcal{D}\psi\mathcal{D}\bar\psi\; \exp\Big(-S[\bar\psi,\psi] + \int\limits_0^{\beta\hbar} d^dx\,[\bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)]\Big) \tag{12.20}$$

and the thermal correlation functions are obtained by differentiation with respect to the external current

$$T\langle 0|\psi_{\alpha_1}(x_1)\bar\psi_{\beta_1}(y_1)...\bar\psi_{\beta_n}(y_n)|0\rangle_\beta = \frac{(-)^n}{Z[0]}\frac{\delta^{2n}}{\delta\eta^{\beta_n}(y_n)...\delta^{\alpha_1}\bar\eta(x_1)}Z[\bar\eta,\eta]|_{\bar\eta=\eta=0} \tag{12.21}$$

where T denotes the Euclidean time ordering. Note the presence of the factor $(-1)^n$ in contrast to (12.11). This is due to the Wick rotation to imaginary time.

Next we simplify (12.21). We could calculate a 'classical' path with antiperiodic boundary conditions, calculate the partition kernel and then integrate over the boundary conditions. This approach analogous to is rather involved in the present situation. Therefore we choose a somewhat different (and more formal) approach which can be applied for quadratic actions (and for simple boundary conditions). We just apply the Gauss integral formula to (12.20)

$$Z[\beta,\bar\eta,\eta]=\det(i\not{D}-m)\;e^{-\left(\bar\eta(x),G_\beta(x,y)\eta(y)\right)},\qquad G_\beta(x,y)=\langle x|\frac{1}{i\not{D}-m}|y\rangle. \tag{12.22}$$

$G_\beta(x,y)$ is the thermal Green's function of $(i\not{D}-m)$ (that is the Green's function on the space of the functions antiperiodic in β). The formula (12.22) then implies in particular

$$T\langle\psi(x)\bar\psi(y)\rangle_\beta=-\frac{1}{Z[\beta,0]}\frac{\delta^2}{\delta\eta(y)\delta\bar\eta(x)}Z[\beta,\bar\eta,\eta]|_{\bar\eta=\eta=0}=G_\beta(x,y). \tag{12.23}$$

More generally, the Wick-theorem (12.12) still holds if we drop the $(-i)^n$ and replace the Feynman propagator by the thermal Green's function (or Euclidean propagator) on the right hand side of (12.12). This concludes our proof of the equivalence between the functional integral approach and the canonical approach for fermionic systems. We have seen that formally there is a close analogy of fermionic path integrals with those in quantum mechanics. So far we haven't dealt with the inherent divergences of field theories, a feature with is not present in ordinary quantum mechanics. Finally we have seen the path integral formalism allows for a unified treatment of zero-temperature and finite temperature systems.

12.2 The index theorem for the Dirac operator

When solving the (Euclidean) Schwinger model we must calculate the partition Z in (12.19) or equivalently its logarithm, the effective action

$$\Gamma=\log Z=\log\det\not{D} \tag{12.24}$$

(see (12.22)), where we assume the fermions to be massless. As we shall see later, this determinant can be calculated explicitly in 2 dimensions by integrating the chiral anomaly. As a first step we now determine the number of zero modes of $\not{D}$. It will turn out that this number is a physically and mathematically interesting number.

We use the notation and convention as in (8.67-8.70) and assume that space-time is even dimensional so that we can introduce $\gamma_5=(-i)^{n(n-1)/2}\gamma^1\gamma^1\cdots\gamma^n$ (the factor is chosen such that $\gamma_5^2=Id$) which anti-commutes with all γ's

$$\{\gamma_5,\gamma^\mu\}=0\Longrightarrow\{\gamma_5,\not{D}\}=[\gamma_5,\not{D}^2]=0. \tag{12.25}$$

In Euclidean space-time we may take $\gamma^1, \dots, \gamma^n$ to be hermitean so that $i\not{D}$ is selfadjoint and we shall assume that its spectrum is discrete. Since γ_5 anticommutes with the Dirac operator all 'excited' eigenfunction of $\not{D}$ come in pairs,

$$i\not{D}\chi = \langle\chi \Longrightarrow i\not{D}(\gamma_5\chi) = -\gamma_5(i\not{D}\chi) = -\langle(\gamma_5\chi) \tag{12.26}$$

i.e. the γ_5-transform of an eigenmode has the opposite eigenvalue (note that $\gamma_5\chi$ has the same norm as χ and hence cannot be zero). This implies that all excited states of $-\not{D}^2$ are (at least) double degenerate, more precisely to each left-handed eigenmode $\gamma_5\psi_L = \psi_L$ there is a right-handed partner $\gamma_5\psi_R = -\psi_R$ with the same eigenvalue $E = \lambda^2$. In terms of the eigenfunctions of $i\not{D}$ they read $\psi_L = \frac{1}{2}(1+\gamma_5)\chi$ and $\psi_R = \frac{1}{2}(1-\gamma_5)\chi$. This pairing need not and generally does not occur for the zero-energy states. The ground states of $-\not{D}^2$ are also eigenstates of $i\not{D}$ with eigenvalue zero (this is not true for the excited states) and thus have fixed chirality. Now we define the index of the Dirac operator as the number of left-handed minus the number of right-handed zero modes of $-\not{D}^2$ or $i\not{D}$:

$$\text{index}\,(i\not{D}) = n_+ - n_-. \tag{12.27}$$

This index can be computed quite differently. For that we note that the (super) trace tr $\gamma_5 \exp(\beta\not{D}^2)$ can be computed via path integrals similarly to the partition function in (8.3) and (8.4). Using the eigenfunction of $-\not{D}^2$ in evaluating the trace we find

$$\text{tr}\,\gamma_5 e^{\beta\not{D}^2} = \sum_n \left(e^{-\beta E_{L,n}} - e^{-\beta E_{R,n}}\right) = n_+ - n_- = \text{index}\,(i\not{D}), \tag{12.28}$$

where we have used that due to the pairing of the excited states only the zero-modes contribute to the sum. Note in particular that the super-trace is β-independent.

The supertrace can now be calculated by using the density (8.65) of the partition function. This way we find for the index

$$n_+ - n_- = \text{tr}\,\gamma_5 e^{\beta\not{D}^2} = \int d^n x\ \text{tr}\left(\gamma_5 Z(\beta, x)\right) \tag{12.29}$$

where the last trace is over spin- and internal color indices and $Z(\beta, x)$ possesses the path integral representation (8.74). Since the super-trace is independent of β we may assume β to be very small and use the high temperature expansion (8.78).

In two dimensions $\Sigma F = \gamma_5 F_{01}$ and we find

$$\int \text{tr}\,\gamma_5 Z(\beta, x) = \frac{1}{4\pi}\int \text{tr}\,(\gamma_5^2)F_{01} + O(\beta). \tag{12.30}$$

With our sign convention for γ_5 we obtain in four dimensions tr $(\gamma_5\Sigma_{\mu\nu}\Sigma_{\alpha\beta}) = \epsilon_{\mu\nu\alpha\beta}$ and thus

$$\int \text{tr}\,\gamma_5 Z(\beta, x) = \frac{1}{(4\pi)^2}\int \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + O(\beta). \tag{12.31}$$

The higher orders in β must be identically zero and we conclude

$$\text{index}\,(i\not{D}) = \frac{1}{2\pi}\int d^2x \text{tr}\, F_{01} \qquad n=2 \tag{12.32}$$

and

$$\text{index}\,(i\not{D}) = \frac{1}{32\pi^2}\int d^4x \epsilon_{\mu\nu\alpha\beta}\text{tr}\,(F_{\mu\nu}F_{\alpha\beta}) = \frac{1}{16\pi^2}\text{tr}\ F^*F \qquad n=4. \tag{12.33}$$

These identities and their analog in higher dimensions relate the index of $i\not{D}$ to certain flux-integrals (Chern-densities). In particular we conclude that these fluxes are always integers, at least if the spectrum of the Dirac operator is discrete. The spectrum is certainly discrete if the Euclidean space-time is bounded, for example a sphere or a torus. On unbounded spaces or spaces with boundaries the index-theorem is modified [49] (since the fluxes are not integers in general).

12.3 The Schwinger model, Part I

As already mentioned earlier, the Schwinger model [42] is Quantum-electrodynamics for massless fermions in 2 dimensions and the corresponding action contains the fermion field coupled to the electromagnetic field, that is (12.18) with a vanishing mass, and the addition of the Maxwell term for the 'photons':

$$S[A,\bar{\psi},\psi] = \int \mathcal{L}_F + \mathcal{L}_B, \qquad \mathcal{L}_F = -i\bar{\psi}\not{D}\psi, \quad \mathcal{L}_B = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{gf}, \tag{12.34}$$

where $\mathcal{L}_{gf}$ are gauge terms due to the gauge fixing procedure (see below). We solve the Schwinger model at zero temperature, so that the integrals in (12.34) are over the whole Euclidean plane. The (Euclidean) generating functional may also contain a source term for the electromagnetic field, so that

$$Z[J,\bar{\eta},\eta] = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\Big(-S[A,\bar{\psi},\psi] + \int d^2x\,[A_\mu J^\mu + \bar{\eta}\psi + \bar{\psi}\eta]\Big) \tag{12.35}$$

In a first step we treat the fermionic part of the path integral only and thus may assume the photon field to be an external field. Integrating out the fermionic degrees of freedom according to (12.22) yields

$$Z[J,\bar{\eta},\eta] = \int \mathcal{D}A\ \det(i\not{D}_A)e^{-\int \mathcal{L}_B + i\int \bar{\eta}(x)G(x,y)\eta(y) + \int A_\mu J^\mu}. \tag{12.36}$$

The reason which allows the model to be solved exactly is that the electron propagator and the fermionic determinant in an arbitrary external field can be found explicitly, as has been observed by Schwinger. For that purpose we introduce

$$\Phi = \frac{1}{\Delta}\Big(\partial_\alpha A^\alpha + i\Sigma^{\alpha\beta}F_{\alpha\beta}\Big) \tag{12.37}$$

where the matrices $\Sigma^{\alpha\beta}$ have been introduced in (8.71). Note that under a gauge transformation $A \to A + d\Lambda$ this functions transforms as $\Phi \to \Phi + \Lambda$. Using the identity $2i\Sigma^{\alpha\beta} = \gamma^\alpha\gamma^\beta - \delta^{\alpha\beta}$ one sees that

$$\partial\!\!\!/ \Phi = \frac{1}{\Delta}\Big(\gamma^\mu\gamma^\alpha\gamma^\beta\partial_\mu\partial_\alpha A_\beta\Big) = \frac{1}{\Delta}\gamma^\beta\Delta A_\beta = \gamma^\beta A_\beta. \tag{12.38}$$

In particular in 2 dimensions $\Sigma F = \gamma_5 F_{01}$ and

$$\Phi = \frac{1}{\Delta}\Big(\partial A + i\gamma_5 F_{01}\Big). \tag{12.39}$$

Taking into account that γ_5 anti-commutes with the Dirac operator we can rewrite the Dirac operator as

$$D\!\!\!\!/ = \partial\!\!\!/ - i\gamma^\mu A_\mu = \partial\!\!\!/ - i\partial\!\!\!/\Phi = e^{i\Phi^\dagger}\partial\!\!\!/ e^{-i\Phi}. \tag{12.40}$$

Note, that the last identity holds only in 2 dimensions since we have used that $\gamma^\mu\Phi = \Phi^\dagger\gamma^\mu$. Now it is clear that the exact propagator which obeys the equation

$$iD\!\!\!\!/ G(x,y,A) = \delta^2(x-y) \tag{12.41}$$

has the form

$$G(x,y,A) = e^{i\Phi(x)}G_0(x-y)e^{-i\Phi^\dagger(y)} \tag{12.42}$$

where G_0 denotes the free massless propagator

$$G_0(\xi) = -i\partial\!\!\!/\Delta_0(\xi) \quad \text{where} \quad \Delta_0(\xi) = -\frac{1}{4\pi}\log(\mu^2\xi^2). \tag{12.43}$$

(μ is an infrared cut-off which could be left out if we would quantize the model on a finite region instead of R^2).

To compute the fermionic determinant in (12.36) we employ the zeta-function method. We formally define $\det(iD\!\!\!\!/)$ as the square root of $\det(-D\!\!\!\!/^2)$. From and (6.93) and (6.94) we see that

$$\zeta_{-D\!\!\!\!/^2}(s) = \frac{1}{\Gamma(s)}\int dt t^{s-1}\mathrm{tr}\, e^{tD\!\!\!\!/^2} \tag{12.44}$$

and

$$\log\det(iD\!\!\!\!/) = \frac{1}{2}\log\det(-D\!\!\!\!/^2) = -\frac{1}{2}\frac{d}{ds}\zeta_{-D\!\!\!\!/^2}(s)|_{s=0}. \tag{12.45}$$

Let us now define a one parametric family of Dirac operators which interpolates between $D\!\!\!\!/$ and $\partial\!\!\!/$, namely

$$D\!\!\!\!/_\alpha = e^{i\alpha\Phi^\dagger}\partial\!\!\!/ e^{-i\alpha\Phi}, \tag{12.46}$$

such that

$$\delta \not{D}_\alpha = i(\Phi^\dagger \not{D}_\alpha - \not{D}_\alpha \Phi). \tag{12.47}$$

The variation of the zeta-function becomes then (we suppress the index α)

$$\begin{aligned}\delta\zeta_{-\not{D}^2}(s) &= \frac{1}{\Gamma(s)}\int t^{s-1}\mathrm{tr}\; e^{t\not{D}^2} t(\delta\not{D}\,\not{D} + \not{D}\,\delta\not{D}) = \frac{2i}{\Gamma(s)}\int t^s \mathrm{tr}\; e^{t\not{D}^2}\not{D}^2(\Phi^\dagger - \Phi)\\ &= \frac{4}{\Gamma(s)}\int t^s\mathrm{tr}\; e^{t\not{D}^2}\not{D}^2\gamma_5\frac{1}{\Delta}F_{01} = -\frac{4s}{\Gamma(s)}\int t^{s-1}\mathrm{tr}\; e^{t\not{D}^2}\gamma_5\frac{1}{\Delta}F_{01},\end{aligned}$$

where we have partially integrated to obtain the last equality. Since finally $s \to 0$ only the singular part (more precisely the single pole at $s = 0$) of the integral survives, because of the factor s. We may split the integration over t into an integration from 0 to ϵ and from ϵ to ∞. The second integral is finite for $s = 0$ (recall that the free heat kernel falls off like t^{-1}) and we need only consider the interval near 0. Here we may use the asymptotic expansion (8.78) for the heat kernel of $\not{D}_\alpha^2$ and find ($F_\alpha = \alpha F$):

$$\delta\zeta_{-\not{D}^2}(s) = -\frac{s}{\pi\Gamma(s)}\int_0^\epsilon dt\big[t^{s-2}\mathrm{tr}\,(\gamma_5\frac{1}{\Delta}F_{01}) + t^{s-1}\alpha\mathrm{tr}\,(\gamma_5 F_{01}\gamma_5\frac{1}{\Delta}F_{01}) + O(t^s)\big]. \tag{12.48}$$

Since the first term in (12.39) vanishes the integration over t yields

$$-\frac{2\alpha}{\pi\Gamma(s)}\epsilon^s\Big(F_{01}\frac{1}{\Delta}F_{01} + O(s)\Big).$$

Finally, since $\Gamma(s) \sim 1/s$ for $s \to 0$ the s-derivative in (12.45) yields

$$\frac{d}{d\alpha}\log\det(i\not{D}) = \frac{\alpha}{\pi}\mathrm{tr}\,\Big(F_{01}\frac{1}{\Delta}F_{01}\Big). \tag{12.49}$$

Integrating α from 0 to 1 yields

$$\log\det(i\not{D}) - \log\det(i\not{\partial}) = \frac{e^2}{2\pi}\int F_{01}\frac{1}{\Delta}F_{01} = -\frac{e^2}{2\pi}\int A^\mu(\frac{\delta_{\mu\nu} - \partial\mu\partial\nu}{\Delta})A^\nu, \tag{12.50}$$

where we have reinserted the electric charge e. In what follows we may drop the (divergent) determinant of the free Dirac operator, since it is independent of the gauge potential and chancels in expectation values. Hence the effective action in (12.36) entering the integration over the remaining 'photon'-field is

$$\Gamma[A] = \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{2\pi}\int A^\mu(\delta_{\mu\nu} - \frac{\partial\mu\partial\nu}{\Delta})A^\nu. \tag{12.51}$$

This effective action belongs to a free particle with mass $e/\sqrt{\pi}$. We see that, due to the interaction of the photons with the electrons, the classically massless photons acquire a mass.

This so-called Schwinger mechanism happens without gauge symmetry breaking. We see that the statement that a mass-term for the photon field breaks the gauge symmetry is not true in general, in particular if we allow non-local interactions like in (12.51). Note, however, that in the Lorentz gauge $\partial A = 0$ the effective action becomes local in the gauge-potential. This observation will simplify the remaining path integral considerably.

Let us now discuss two consequences of (12.51). From

$$\det(i\not{D}) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{\int (i\bar{\psi}\not{\partial}\psi + A_\mu\bar{\psi}\gamma^\mu\psi)} \tag{12.52}$$

we see that

$$j^\mu = \langle\bar{\psi}\gamma^\mu\psi\rangle = \frac{1}{\det(i\not{D})}\frac{\delta}{\delta A_\mu}\det(i\not{D}) = -\frac{e^2}{\pi}(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Delta})A_\nu \tag{12.53}$$

and hence

$$\partial_\mu j^\mu = 0, \tag{12.54}$$

that is that the vector current is conserved. This is just a consequence of the gauge invariance of the effective action (or the gauge invariant zeta-function regularization). Using the identity $\gamma_5\gamma^\mu = i\epsilon_{\mu\nu}\gamma^\nu$, valid in 2 dimensions, we can also calculate the axial current:

$$j_5^\mu = \langle\bar{\psi}\gamma_5\gamma^\mu\psi\rangle = i\epsilon_{\mu\nu}j^\nu = -i\frac{e^2}{\pi}(\epsilon_{\mu\alpha} - \epsilon_{\mu\nu}\frac{\partial_\nu\partial_\alpha}{\Delta})A_\alpha. \tag{12.55}$$

Hence we find

$$\partial_\mu j_5^\mu = -i\hbar\frac{e^2}{\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu = -i\hbar\frac{e^2}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}, \tag{12.56}$$

and thus the axial current, contrary to the vector current, is not conserved. We have reinserted $\hbar$ in order to see that this non-conservation is a quantum effect. Classically the axial current is conserved since it is the Noether current belonging to the chiral transformation

$$\psi \longrightarrow e^{\alpha\gamma_5}\psi \quad \text{and} \quad \bar{\psi} \longrightarrow \bar{\psi}\, e^{\alpha\gamma_5} \tag{12.57}$$

which in any even dimension (for which γ_5 exists) leave the classical action invariant since γ_5 anti-commutes with the Dirac operator. What we have shown then is that a classically conserved current is not anymore conserved after quantization or that the classical axial symmetry is broken due to quantum effects. Such a phenomena is called an anomaly.

Let us now return to the problem of computing the correlation functions of the Schwinger model. We begin with the representation for the 2-point function

$$\langle\psi(x)\bar{\psi}(y)\rangle = \frac{1}{Z(0)}\int \mathcal{D}A\, e^{-\Gamma[A]}G(x,y) = \frac{1}{Z(0)}\int \mathcal{D}A\, e^{-\Gamma + i[\Phi(x)-\Phi(y)]}G_0(x,y) \tag{12.58}$$

which only involves a Gaussian integral over the photon field. The same is of course true for the higher correlation functions. To proceed we must first study how one evaluates path integrals over gauge potentials. Due to the gauge invariance of the action we have to extend the path integral to systems subject to constraints (coming from the gauge-invariance).