Chapter 10

Berezin Integral

In any field theory describing the elementary particles in nature there are bosonic *and* fermionic fields. The latter describe the propagation of electrons, muons, neutrinos, quarks and so on. In this chapter we introduce anticommuting Grassmann-variables and the Berezin integral [33]. These enter the path integral quantization of fermionic degrees of freedom.

10.1 Grassmann variables

So far we used the coordinate and momentum representations to formulate path integrals. For what follows it is more convenient to use the *Fock-space representation*, based on the creation and annihilation operators. In the particular case of the extensively discussed harmonic oscillator these operators are related to the position and momentum operators as follows,

$$a^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega m}\, q - \frac{i}{\sqrt{\omega m}} p\right) \qquad a = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega m}\, q + \frac{i}{\sqrt{\omega m}} p\right), \tag{10.1}$$

and they satisfy the commutation relation

$$[a, a^\dagger] = 1. \tag{10.2}$$

The creation and annihilation operators are represented on the *anti-holomorphic functions* $f(\bar{z})$ endowed with the scalar product

$$(f_1, f_2) \equiv \frac{1}{2\pi i}\int \bar{f}_1(z) f_2(\bar{z}) e^{-\bar{z}z}\, dz d\bar{z}, \quad z = x + iy,\ dz d\bar{z} = 2i dx dy. \tag{10.3}$$

The normalization is such that the constant function $f = 1$ has unit norm. The creation- and annihilation operators are represented as

$$(af)(\bar{z}) = \frac{\partial}{\partial \bar{z}} f(\bar{z}) \quad \text{and} \quad (a^\dagger f)(\bar{z}) = \bar{z} f(\bar{z}). \tag{10.4}$$

Using $H = \hbar\omega(a^\dagger a + 1/2)$ and that the anti-holomorphic functions $f_n(\bar z) = (n!)^{-1/2}\bar z^n$ form an orthonormal base in this space,

$$(f_m, f_n) = \frac{2}{n!}\,\delta_{mn}\int e^{-r^2} r^{2n+1} dr = \delta_{mn}, \quad z = re^{i\varphi},$$

we can calculate the matrix element $\langle z'|e^{-itH/\hbar}|z\rangle$ of the evolution operator. One subtle point in the bosonic case is the *normal ordering*. One starts with the normal ordered Hamiltonian, that is the Hamiltonian with zero-point energy subtracted

$$\mathbf{:}H\mathbf{:} = H - \langle\Omega|\,H|\Omega\rangle\,.$$

In order to replace the operators by classical variables, $H(a^\dagger, a) \to h(\bar z, z)$, one needs to normal order the evolution operator $\mathbf{:}e^{-itH/\hbar}\mathbf{:}$ and not only the Hamiltonian. However, in the continuum limit only the first order term $1 - i\epsilon\mathbf{:}H(a^\dagger, a)\mathbf{:}/\hbar$ in the series expansion for the normal ordered evolution operator contributes. But this term is assumed to be already normally ordered.

Now we turn to the fermions, that is we replace (10.2) by

$$\{a, a^\dagger\} = 1 \quad \text{and} \quad (a^\dagger)^2 = (a)^2 = 0. \tag{10.5}$$

These anti-commutation relations cannot be represented on functions of commuting variables as $\bar z$. But they can be represented on functions of anticommuting Grassmann-variables $\bar\alpha, \alpha$,

$$\{\bar\alpha, \alpha\} = 0 \quad \text{and} \quad \bar\alpha^2 = \alpha^2 = 0. \tag{10.6}$$

As representation space we can choose the *analytic functions* depending on $\bar\alpha$ only. Since $\bar\alpha^2 = 0$ such functions have a terminating series expansion

$$f(\bar\alpha) = f_0 + f_1\bar\alpha.$$

The Grassmann variables $(\bar\alpha, \alpha)$ generate the *Grassmann algebra*

$$G_2 \equiv C \oplus \Lambda_1(V) \oplus \Lambda_2(V)$$

and elements in G_2 have the form $f = f_{00} + f_{10}\alpha + f_{01}\bar\alpha + f_{11}\bar\alpha\alpha$. More generally, for n degrees of freedom (10.6) generalizes to

$$\{\alpha_i, \alpha_j\} = \{\bar\alpha_i, \alpha_j\} = \{\bar\alpha_i, \bar\alpha_j\} = 0, \quad i, j = 1, 2, \ldots, n. \tag{10.7}$$

Grassmann variables are nilpotent, $\alpha_i^2 = \bar\alpha_i^2 = 0$, and they generate the Grassmann algebra

$$G_n \equiv \oplus\Lambda_k(V), \quad k = 1, 2, \ldots, 2n,$$

where $\Lambda_1(V) = V$ has base $\{\alpha_i, \bar\alpha_i\}$ and the elements

$$\alpha_{i_1}\cdots\alpha_{i_p}\bar\alpha_{j_1}\cdots\bar\alpha_{j_q} \quad \text{with} \quad p + q = k \quad \text{and} \quad i_1 < ... < i_p,\, j_1 < ... < j_q$$

form a basis of $\Lambda_k(V)$. Actually Λ_k is isomorph to the exterior algebra of k-forms on an $2n$-dimensional manifold.

Due to the anticommutation property there exist two types of derivatives. The *left derivative* and the *right derivative*. We shall always use the former. To compute the left-derivative ∂_i of a monomial in the Grassmann variables one first brings α_i to the left (using the anti-commutation rules) and then drops this variable. For example,

$$\partial_i(\alpha_k\alpha_\ell) = \delta_{ik}\alpha_\ell - \delta_{i\ell}\alpha_k. \tag{10.8}$$

Then the derivative is extended to polynomials and hence to all functions of the Grassmann variables $\{\alpha_i, \bar\alpha_i\}$.

The fermionic creation and annihilation operators in (10.2) are represented by differential operators acting on analytic functions $f(\bar\alpha) = f(\bar\alpha_1, \dots, \bar\alpha_n)$ as follows,

$$(a_i f)(\bar\alpha) = \frac{\partial}{\partial\bar\alpha_i} f(\bar\alpha) \quad \text{and} \quad (a_i^\dagger f)(\bar\alpha) = \bar\alpha_i f(\bar\alpha) \Longrightarrow [a_i, a_j^\dagger] = \delta_{ij}\mathbb{1}. \tag{10.9}$$

We also would like to introduce a *scalar product* on the space of analytic functions $f(\bar\alpha)$. For that aim we introduce an integration over Grassmann variables. Such integrals have been introduced by Berezin and they are defined by the following linear functional [33, 34]:

$$\int d\alpha_i\,\alpha_j = \int d\bar\alpha_i \bar\alpha_j = \delta_{ij} \quad \text{and} \quad \int d\alpha_i = \int d\bar\alpha_i = 0. \tag{10.10}$$

To integrate a monomial with respect to α_i one first brings α_i in the monomial to the left (using the anti-commutation rules) and then drops this variable. For example,

$$\int d\alpha_i\,\alpha_j\alpha_k = \delta_{ij}\alpha_k - \delta_{ik}\alpha_j, \tag{10.11}$$

and similarly for higher monomials. We see that the Berezin integral $\int d\alpha_i$ is equivalent to left derivative with respect to ∂_{α_i}. For the integral over *all Grassmann variables* we choose the sign convention such that

$$\int \mathcal{D}\alpha\mathcal{D}\bar\alpha\,\prod_1^n(\bar\alpha_i\alpha_i) = 1, \quad \text{where} \quad \mathcal{D}\alpha\mathcal{D}\bar\alpha \propto \prod_1^n d\alpha_i \prod_1^n d\bar\alpha_i, \tag{10.12}$$

and it is supposed that the $d\alpha_i$ and $d\bar\alpha_j$ anticommute with each other and with α_i and $\bar\alpha_j$. The integral over Grassmann variables which are permutations of the α's and $\bar\alpha$'s in (10.12) is then given by the anti-commutation rules. The integral of less then $2n$ variables is always zero,

$$\int \mathcal{D}\alpha\mathcal{D}\bar\alpha\,\prod_1^p \alpha_i \prod_1^q \bar\alpha_j = 0 \quad \text{for} \quad p+q < 2n. \tag{10.13}$$

From this property it follows that under a shift of the integration variables by Grassmann variables the Berezin integral is not changed,

$$\int \mathcal{D}\alpha\mathcal{D}\bar\alpha\, f(\alpha+\eta, \bar\alpha+\bar\eta) = \int \mathcal{D}\alpha\mathcal{D}\bar\alpha\, f(\alpha, \bar\alpha). \tag{10.14}$$

Actually, to prove this translational invariance one also uses $(\alpha+\eta)^2=\alpha\eta+\eta\alpha=0$. Let us now see how the Berezin integral changes under linear transformations

$$\beta_i=\sum_j U_{ij}\alpha_j \quad \text{and} \quad \bar\beta_i=\sum_j V_{ij}\bar\alpha_j \tag{10.15}$$

of the integration variables in (10.12). One finds

$$\int \mathcal{D}\alpha\mathcal{D}\bar\alpha\,\prod_1^n(\beta_i\bar\beta_i)=\sum_{\{j_i,k_\ell\}}\prod_{i,\ell}U_{ij_i}V_{\ell k_\ell}\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,(\alpha_{j_i}\bar\alpha_{k_\ell}).$$

Note that only those terms contribute for which $\{j_1,\dots,j_n\}$ and $\{k_1,\dots,k_n\}$ are permutations of $\{1,\dots,n\}$. These permutations are denoted by σ and $\tilde\sigma$. Thus we find

$$\int\dots=\sum_{\sigma,\tilde\sigma}\prod_{i,\ell}U_{i\sigma(i)}V_{\ell\tilde\sigma(\ell)}\,\mathrm{sgn}(\sigma)\mathrm{sgn}(\tilde\sigma)=\det U\cdot\det V. \tag{10.16}$$

For theories containing fermions the Gaussian Berezin integrals are as important as the ordinary Gaussian integrals are for theories containing bosons. With the help of (10.16) it is not difficult to compute the Gaussian integral

$$Z=\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,e^{-\bar\alpha A\alpha},\quad \text{where}\quad \bar\alpha A\alpha=\bar\alpha_iA_{ij}\alpha_j. \tag{10.17}$$

One just changes variables according to $\beta_i=A_{ij}\alpha_j$ (and leaves the $\bar\alpha$'s) so that

$$Z=\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,e^{-\bar\alpha_i\beta_i}=\frac{1}{n!}\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,(\beta_i\bar\alpha_i)^n=\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,\prod(\beta_i\bar\alpha_i)=\det(A).$$

We end up with the important formula

$$\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,e^{-\bar\alpha A\alpha}=\det(A),\quad \bar\alpha A\alpha=\bar\alpha_iA_{ij}\alpha_j. \tag{10.18}$$

This should be compared with the corresponding bosonic Gaussian integral for which one obtains the inverse square root of the determinant of A.

The *generating function* for Grassmann integrals can be computed by shifting the integration variables in (10.18) according to

$$\alpha\longrightarrow\alpha-A^{-1}\eta\quad\text{and}\quad\bar\alpha\longrightarrow\bar\alpha-\bar\eta A^{-1}.$$

Using the translational invariance of the Berezin integral, see (10.14), one arrives at

$$\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,e^{-\bar\alpha A\alpha+\bar\eta\alpha+\bar\alpha\eta}=\det(A)\,e^{\bar\eta A^{-1}\eta},\quad \bar\eta\alpha=\bar\eta_i\alpha_i. \tag{10.19}$$

Now we define the *scalar product* of two analytic (in $\bar\alpha$) functions, similarly as in the bosonic case, according to

$$(g,f)=\int\mathcal{D}\alpha\mathcal{D}\bar\alpha\,g^\dagger(\alpha)f(\bar\alpha)e^{-\bar\alpha\alpha}, \tag{10.20}$$

where the adjoint of a function $g = g_0 + g_i\bar{\alpha}_i + g_{ij}\bar{\alpha}_i\bar{\alpha}_j + \dots$ is given by $g^\dagger = \bar{g}_0 + \bar{g}_i\alpha_i + \bar{g}_{ij}\alpha_j\alpha_i + \dots$. Inserting the expansions for $g^\dagger$ and f yields

$$(g, f) = \sum_{p=0}^{n} \bar{g}_{i_1\dots i_p} f_{i_1\dots i_p} \tag{10.21}$$

for the scalar product of two functions $g(\bar{\alpha})$ and $f(\bar{\alpha})$. The last formula makes clear that the scalar product is indeed sesqui-linear and positive as required. The space of analytic functions $f(\bar{\alpha})$, endowed with this scalar product, forms the Hilbert space on which the linear operators are represented. One can show that the operators a and $a^\dagger$ are (formally) adjoint of each other on this Hilbert space. A basis of the Hilbert space is defined by the orthonormal set of Fock states $\prod a_i^\dagger|0\rangle$, where $|0\rangle$ is represented by the constant function 1.

Returning to $n = 2$ we consider a general *normal ordered* linear operator $\hat{A} = :\hat{A}:$,

$$\begin{aligned} \hat{A} &= K_{00} + K_{01}a + K_{10}a^\dagger + K_{11}a^\dagger a \\ &= K_{00} + K_{01}\frac{\partial}{\partial\bar{\alpha}} + K_{10}\,\bar{\alpha} + K_{11}\,\bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}. \end{aligned} \tag{10.22}$$

Applying this operator to an element of the Hilbertspace $f(\bar{\alpha}) = f_0 + f_1\bar{\alpha}$ we obtain

$$\begin{aligned} (\hat{A}f)(\bar{\alpha}) &= K_{00}(f_0 + f_1\bar{\alpha}) + K_{01}f_1 + K_{10}f_0\bar{\alpha} + K_{11}f_1\bar{\alpha} \\ &= \int A(\bar{\alpha}, \beta)e^{-\bar{\beta}\beta} f(\bar{\beta})d\bar{\beta}d\beta. \end{aligned} \tag{10.23}$$

where the kernel on the right hand side is given by

$$A(\bar{\alpha}, \beta) = e^{\bar{\alpha}\beta}A^N(\bar{\alpha}, \beta) \quad \text{with} \quad A^N(\bar{\alpha}, \beta) = K_{00} + K_{01}\beta + K_{10}\bar{\alpha} + K_{11}\bar{\alpha}\beta.$$

This generalizes in an obvious way to more than one degree of freedom: a normally ordered linear operator $\hat{A}$ has a kernel A which is obtained from $\hat{A}$ by replacing $a, a^\dagger$ by $\beta, \bar{\alpha}$ and multiplying the resulting expression with $\exp(\bar{\alpha}\beta)$. Similarly one can show that

$$(AB)(\bar{\alpha}, \alpha) = \int A(\bar{\alpha}, \beta)B(\bar{\beta}, \alpha)e^{-\bar{\beta}\beta}d\bar{\beta}d\beta.$$

With these formulas we can now derive the path integral representation for the *kernel* of the normal ordered evolution operator $K(t, \hat{a}, \hat{a}^\dagger)$. As in the bosonic case we divide the time interval $[0, t]$ into n time steps of equal length $\epsilon = t/n$ and obtain for the kernel

$$K(t, \bar{\alpha}_n, \alpha_0) = \int \prod_{i=1}^{n-1} d\bar{\alpha}_i d\alpha_i \prod_{i=1}^{n} K_\epsilon^N(\bar{\alpha}_i, \alpha_{i-1}) \exp\left(-\sum_1^{n-1} \bar{\alpha}_i\alpha_i + \sum_{i=1}^{n} \bar{\alpha}_i\alpha_{i-1}\right), \tag{10.24}$$

where the variables α_0 and $\bar{\alpha}_n$ at initial and final time are held fixed. In the continuum limit $n \to \infty$ or $\epsilon \to 0$ we may approximate

$$K_\epsilon^N(\bar{\alpha}, \alpha) \sim \exp\left(-\frac{i\epsilon}{\hbar} H^N(\bar{\alpha}, \alpha)\right)$$

and thus we can rewrite (10.24) as follows

$$\begin{aligned} K(t,\bar\alpha_n,\alpha_0) &= \lim_{n\to\infty}\int \mathcal{D}\alpha\mathcal{D}\bar\alpha\,\exp\left(\bar\alpha_n\alpha_n+\sum_{i=1}^{n}\Big[\bar\alpha_i(\alpha_{i-1}-\alpha_i)-i\epsilon H^N(\bar\alpha_i,\alpha_{i-1})\Big]\right)\\ &= \lim_{n\to\infty}\int \mathcal{D}\alpha\mathcal{D}\bar\alpha\,\exp\left(\bar\alpha_0\alpha_0+\sum_{i=0}^{n-1}\Big[(\bar\alpha_{i+1}-\bar\alpha_i)\alpha_i-i\epsilon H^N(\bar\alpha_{i+1},\alpha_i)\Big]\right), \end{aligned}$$

where one integrates over the Grassmann variables $\{\bar\alpha_i,\bar\alpha_i\}$ with $i=1,2,..,n-1$. The second form follows from the first by a 'partial integration' and shows, that the factors $\bar\alpha_n\alpha_n$ and $\bar\alpha_0\alpha_0$ are surface terms which can be neglected in the continuum limit. Thus in the continuum limit we end up with the following path integral

$$K(t,\bar\alpha_n,\alpha_0)=\int\limits_{\alpha(0)=\alpha_0}^{\bar\alpha(t_2)=\bar\alpha_n}\mathcal{D}\bar\alpha\mathcal{D}\alpha\,\exp\left(-\int\limits_{t_1}^{t_2}dt\,\Big[\bar\alpha\dot\alpha+iH^N(\bar\alpha,\alpha)\Big]\right). \tag{10.25}$$

Note that the function in the exponent is just the action corresponding to the (normal ordered) Hamiltonian H. This means that the path integral for fermionic degrees of freedom is formally the same as for bosonic systems. The crucial difference (which forbids a probabilistic interpretation) is the replacement of c-numbers by 'Grassmann numbers'. Before turning to the field-theoretical generalization we discuss an interesting application of (10.25) to supersymmetric quantum mechanics.