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Übungen zu Symmetrien in der Physik

Blatt 7

Problem 25: Complex conjugate representation

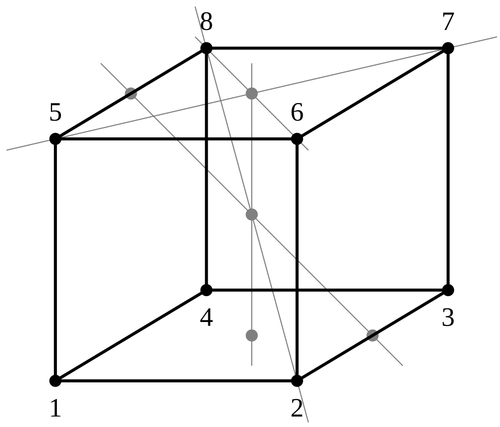
If $g \to D(g)$ is a irreducible representation of a group G, show that $g \to D^*(g)$ also forms a representation. If the two representations are equivalent, so that $D^*(g) = SD(g)S^{-1}$, show that $SS^* = \lambda\mathbb{1}$. If further D is unitary show that $SS^\dagger = \lambda'\mathbb{1}$. Show that in then S is either symmetric or antisymmetric.

Problem 26: Representations of SU(2)

Show that for $g \in \mathrm{SU}(2)$ one has $\sigma_2 g \sigma_2 = g^*$. Based on this observation show, that SU(2) is a pseudo-real group. This means, that every irreducible representation is equivalent to its complex conjugate representation.

Problem 27: Cubic group

The symmetry transformations of a cube form a finite subgroup of the rotation group. It is one of the platonic groups. There are three types of symmetry axis: the axes going through the centers of opposite faces of the cube, the axes going through the centers of opposite edges and the body diagonals as depicted in the figure. The group is isomorphic to the octahedral group or the permutation group of 4 elements.



- Show that the order of the group is $1 + F/2 \times 3 + E/2 \times 1 + V/2 \times 2 = 24$, where F, E and V denote the numbers of faces, edges and vertices of the cube.
- Show that the group contains the following five conjugacy classes:
  1. the trivial class with neutral element e,
  2. the class $\mathcal{C}_2$ with the π-rotations about the axes connecting opposite faces,
  3. the class $\mathcal{C}_3$ with the $2\pi/3$-rotations about the body diagonals,
  4. the class $\mathcal{C}_2'$ with the π-rotations about the axes connecting opposite edges,
  5. the class $\mathcal{C}_4$ of the $\pm\pi/2$-rotations about the axes connecting opposite faces.
- Hence the group has five irreducible representations. What are the dimensions of these representations?

Problem 28: Relation between O(4) and SU(2)×SU(2)

The set of matrices

$$x = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}$$

form a linear space $\mathbb{C}^2 \cong \mathbb{R}^4$ with scalar product

$$\langle x, y \rangle = \mathrm{tr}\left(x^\dagger y\right) .$$

The matrices with $|a|^2 + |b|^2 = 1$ belong to SU(2) and form the unit sphere in $\mathbb{R}^4$.

- Let g_1, g_2 be two matrices in SU(2) and x a matrix of the above form. Show that the map $R(g_1, g_2)$ defined by

  $$R(g_1, g_2)x = g_1 x g_2^{-1}$$

  is linear and preserves the scalar product,

  $$\langle R(g_1, g_2)x, R(g_1, g_2)y \rangle = \langle x, y \rangle .$$

  Hence $R(g_1, g_2)$ can be considered as a linear map $\mathbb{R}^4 \to \mathbb{R}^4$ which preserves the lengths of vectors.

- Show that this map defines a group homomorphism SU(2)×SU(2) → SO(4), i.e. that

  $$R(g_1', g_2')R(g_1, g_2) = R(g_1' g_1, g_2' g_2) .$$

- Is this homomorphism faithful? If not, can you identify SO(4) with SU(2)×SU(2)/Z for some discrete normal subgroup Z of SU(2)×SU(2) ?